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A PROOF OF THE WEIGHTED PÓLYA–KNOPP INEQUALITY FOLLOWING ARIÑO–MUCKENHOUPT'S METHOD

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Abstract. We give a simple proof of the weighted Pólya-Knopp inequality following Ariño-Muckenhoupt's method.

1. Introduction

The aim of this article is to give a simple proof of the weighted Pólya-Knopp inequality following Ariño-Muckenhoupt's method employed in [1].

Let the Ariño-Muckenhoupt class B_p $(1 \leqslant p < \infty)$ be defined as the set of all nonnegative functions W for which a constant B exists such that the inequality

$$\int_r^\infty \left(\frac{r}{x}\right)^p W(x)\,dx \leqslant B \int_0^r W(x)\,dx \tag{1}$$

holds for every $r > 0$, and let the Ariño-Muckenhoupt class B_∞ be the union of all such B_p. We note that Riesz [4] tells us that $(\int_0^x f(t)^{1/p}\,dt/x)^p$ decreases and tends to $Gf(x) := \exp(\int_0^x \log f(t)\,dt/x)$ as p increases to ∞. Then, as a limiting case of Theorem (1.7) of [1], it is natural to state that a constant C exists such that the weighted Pólya-Knopp inequality

$$\int_0^\infty Gf(x)W(x)\,dx \leqslant C \int_0^\infty f(x)W(x)\,dx \tag{2}$$

holds for all positive, nonincreasing functions f on $[0,\infty)$ if and only if W belongs to the class B_∞. This limiting case is already proved in Sbordone-Wik [5]. The proof of the fact that (2) for all positive, nonincreasing f implies (1) for some p is easy. However, even in their paper [5], the proof of the converse is rather difficult.

We now give the proof of the converse, analogously to that of §3 of [1]. The key point of our proof is that the basic Lemma (2.1) of [1], stating that if (1) holds for p, then for some $\delta > 0$ it holds for $p - \delta$ as well, is not needed. We hope that this article will be a kind of supplement to the fundamental paper [1] and the subsequent paper [5].

The organization of the article is as follows. In section 2, the proof promised immediately above is given. That is, we aim at proving the fact that the B_∞-condition for weight functions W implies the inequality (2) for all nonincreasing functions f. In section 3, some comments are mentioned.

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2. Proof

We employ an analogous method as that of §3 of [1]. However, the basic Lemma (2.1) of [1] is not needed here.

Having fixed an f, we define sequences $\{a_n\}$ and $\{b_n\}$ inductively as follows. Let $b_0 = 0$. Given b_{n-1}, we take a_n to be the infimum of all $x > b_{n-1}$ such that $f(x)/Gf(x)$ is less than or equal to $\varepsilon/10$, where ε is a positive number to be determined later. By this definition we have

$$Gf(x) \leqslant \frac{10}{\varepsilon} f(x), \quad b_{n-1} < x \leqslant a_n, \tag{3}$$

and, since we may assume without loss of generality that the function f is continuous,

$$Gf(a_n) = \frac{10}{\varepsilon} f(a_n). \tag{4}$$

Given a_n, define b_n to be the infimum of all $x > a_n$ such that $f(x)/Gf(x)$ is greater than ε. Then

$$Gf(x) \geqslant \frac{1}{\varepsilon} f(x), \quad a_n < x \leqslant b_n, \tag{5}$$

and

$$Gf(b_n) = \frac{1}{\varepsilon} f(b_n). \tag{6}$$

Since f is nonincreasing and $b_n \leqslant a_{n+1}$, $Gf(a_{n+1}) \leqslant Gf(b_n)$; from (4) and (6) we see that $10f(a_{n+1}) \leqslant f(b_n)$. It follows that

$$10f(a_{n+1}) \leqslant f(a_n). \tag{7}$$

If $a_n < t \leqslant b_n$ we have by (5) that $(\int_0^t \log f(u)\,du)/t \geqslant \log(f(t)/\varepsilon)$, i.e.,

$$\frac{d}{dt}\left(\frac{1}{t}\int_0^t \log f(u)\,du\right) \leqslant \frac{\log\varepsilon}{t}.$$

Integrating both sides with respect to t from a_n to x $(a_n < x \leqslant b_n)$,

$$\frac{1}{x}\int_0^x \log f(u)\,du - \frac{1}{a_n}\int_0^{a_n} \log f(u)\,du \leqslant \log\varepsilon \left(\log\frac{x}{a_n}\right),$$

that is,

$$Gf(x) \leqslant \left(\frac{a_n}{x}\right)^{-\log\varepsilon} Gf(a_n), \quad a_n < x \leqslant b_n. \tag{8}$$

Now to prove (2) for $W \in B_p$ with some p, write the left side of (2) as

$$\sum_{n=1}^{\infty}\int_{b_{n-1}}^{a_n} Gf(x)W(x)dx + \sum_{n=1}^{\infty}\int_{a_n}^{b_n} Gf(x)W(x)dx.$$

By (3) we see that the first term is bounded by the right side of (2) with $C = 10/\varepsilon$. For the second term, use (8) to get the bound

$$\sum_{n=1}^{\infty}\left[\int_{a_n}^{b_n}\left(\frac{a_n}{x}\right)^{-\log\varepsilon} W(x)\,dx\right] Gf(a_n).$$

Since $W \in B_p$ and (4) holds true, if $\varepsilon := e^{-p}$ then this is bounded by

$$\sum_{n=1}^{\infty} B\left[\int_0^{a_n} W(x)\,dx\right]\left[\frac{10}{\varepsilon} f(a_n)\right],$$

which can be bounded by

$$B\left(\frac{10}{\varepsilon}\right)\int_0^{\infty}\left[\sum_{a_n \geqslant x} f(a_n)\right] W(x)\,dx.$$

(Note that as p gets larger, ε, by definition, gets smaller, which corresponds exactly to a weaker B_p-condition on W.) From (7) and the fact that f is nonincreasing we get the bound

$$B\left(\frac{10}{\varepsilon}\right)\left(\frac{10}{9}\right)\int_0^{\infty} f(x)W(x)\,dx,$$

completing the proof of the inequality (2) with $W \in B_p$.

3. Comments

Carleson [2] gave a proof for power-weighted and integral version of Carleman's inequality. That is, he proved that the inequality (2) with $W(x) = x^{\alpha}$ $(\alpha > -1)$ holds true for all nonincreasing functions f. Lemma 3 of Sbordone-Wik [5] is corresponding to Lemma (2.1) of Ariño-Muckenhoupt [1], and the definition of B_{∞} in [5] is given by using the doubling conditoin. Theorem 6 of [5] states that the inequality (2) holds true for all nonincreasing functions f if and only if $W \in B_{\infty}$, and its proof does not rely on [1]. Therefore, it can safely be said that the essence of Ariño-Muckenhoupt's proof method is highlighted through our argument given in the previous section.

We finally note that Persson-Stepanov [3] completes a characterization, given $0 < p, q < \infty$, of V and W for which a constant C exists such that

$$\left[\int_0^{\infty} Gf(x)^q W(x)\,dx\right]^{1/q} \leqslant C\left(\int_0^{\infty} f(x)^p V(x)\,dx\right)^{1/p} \tag{9}$$

holds for all positive functions f on $[0,\infty)$. Thus, the problem to be considered in another publication should be a characterization of V and W for which (9) holds for all positive, *nonincreasing* functions f.

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