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A NOTE ON INEQUALITIES FOR GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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Abstract. We establish new inequalities for Gaussian hypergeometric functions that are not zero-balanced. In the literature, such inequalities are referred to as Landen-type inequalities, as they generalize the classical Landen identity for the complete elliptic integral of the first kind. We also discuss comparisons, including numerical simulations, between our results and existing similar ones in the literature.

1. Introduction

For any real numbers a,b and c where $c \neq 0,-1,-2,\cdots$, the Gaussian hypergeometric function $F(a,b;c;x)$ is defined as

$$F(a,b;c;x) := {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n, \qquad x \in (-1,1). \tag{1}$$

where $(a)_n$ is the Pochhammer's symbol given by

$$(a)_0 = 1 \text{ when } a \neq 0 \quad \text{and} \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1). \tag{2}$$

When $c = a+b$ in (1), the hypergeometric function $F(a,b;c;x)$ is called *zero-balanced*.

We begin with the following two identities:

$$F\left(1,\frac{1}{2};1;x\right) = \left(1-\tfrac{1}{2}x\right)^{-1} F\left(1,\frac{1}{2};1;\left(\frac{x}{2-x}\right)^2\right), \quad x \in (0,1), \tag{3}$$

which comes from [5, (15.8.13)] by setting $a = 1$ and $b = \frac{1}{2}$; and

$$F(a,a;1;x) = \left(1+\sqrt{x}\right)^{-2a} F\left(a,\frac{1}{2};1;\frac{4\sqrt{x}}{(1+\sqrt{x})^2}\right), \quad x \in (0,1), \tag{4}$$

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derived from [5, (15.8.21)] by setting $a=b$. Note that, when $a=\frac{1}{2}$, $F\left(\frac{1}{2},\frac{1}{2};1;x\right)$ is a zero-balanced hypergeometric function. The above identity (4) is indeed the Landen identity (cf. [1])

$$\mathscr{K}\left(\frac{2\sqrt{k}}{1+k}\right)=(1+k)\mathscr{K}(k), \tag{5}$$

where $\mathscr{K}(k)$ is the complete elliptic integral of the first kind

$$\mathscr{K}(k)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;k^2\right)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\quad k\in(0,1).$$

There is another Landen identity which is equivalent to (5):

$$\mathscr{K}\left(\frac{1-k}{1+k}\right)=\frac{1+k}{2}\mathscr{K}(\sqrt{1-k^2}). \tag{6}$$

For general zero-balanced hypergeometric functions $F(a,b;a+b;x)$, the Landen identities (5) and (6) no longer hold. For instance, Simić and Vuorinen [8] showed that, for $a,b>0$ with $ab\leqslant\frac{1}{4}$,

$$F\left(a,b;a+b;\frac{4x}{(1+x)^2}\right)\leqslant(1+x)F(a,b;a+b;x^2),\quad x\in(0,1); \tag{7}$$

and for $a,b>0$ with $1/a+1/b\leqslant 4$,

$$F\left(a,b;a+b;\frac{4x}{(1+x)^2}\right)\geqslant(1+x)F(a,b;a+b;x^2),\quad x\in(0,1). \tag{8}$$

Since these inequalities share a similar structure with (5) and (6), they are referred to as the *Landen inequalities*. The Landen inequalities have attracted a lot of research interests in the literature, with various modifications and improvements being achieved. For example, see [6, 7, 8, 9, 10, 11].

Based on the Landen identities (5) and (6), Baricz [2] investigated Landen-type inequalities for hypergeometric functions $F(a,b;c;x)$ that are not zero-balanced. He obtained the following result ([2, Theorem 1]): let $a,b,c\in\mathbb{R}$ such that c is not a negative integer or zero, then

a. If $a+b\geqslant c$ and $4ab\geqslant\max\{1,c\}$, then

$$F\left(a,b;c;\frac{4x}{(1+x)^2}\right)\geqslant(1+x)F(a,b;c;x^2),\quad x\in(0,1), \tag{9}$$

$$F\left(a,b;c;\frac{(1-x)^2}{(1+x)^2}\right)\leqslant\frac{1+x}{2}F(a,b;c;1-x^2),\quad x\in(0,1). \tag{10}$$

b. If $a+b\leqslant c$ and $4ab\leqslant\min\{1,c\}$, then for $x\in(0,1)$

$$F\left(a,b;c;\frac{4x}{(1+x)^2}\right)\leqslant(1+x)F(a,b;c;x^2),\quad x\in(0,1), \tag{11}$$

$$F\left(a,b;c;\frac{(1-x)^2}{(1+x)^2}\right)\geqslant\frac{1+x}{2}F(a,b;c;1-x^2),\quad x\in(0,1). \tag{12}$$

Note that by setting $c=a+b$, the inequalities (9) and (11) reduce to (7) and (8). Therefore, Baricz's results can be viewed as a generalization of the Landen inequalities. It is also worth mentioning that when $a=b=\frac{1}{2}$ and $c=1$, all four of the above inequalities hold, yielding the Landen identities (5) and (6). Other Landen-type inequalities for hypergeometric functions $F(a,b;c;x)$ that are not zero-balanced can also be found in [4].

In this paper, we establish new inequalities for general hypergeometric functions using the identities (3) and (4). Similar to the inequalities (9)–(12), our results apply to general hypergeometric functions that are not zero-balanced. Furthermore, we address a minor oversight in the inequalities (9)–(12), where the condition $a,b,c\in\mathbb{R}$ should be replaced by $a,b,c>0$. We clarify this condition in the proof below and provide a numerical illustration in Sec. 3.

2. Main results

Like most proofs in the literature concerning Landen-type inequalities, we require the following lemma from Biernacki and Krzyż [3], which provides a tool for comparing two power series.

LEMMA 2.1. *Consider the power series* $f(x)=\sum_{n=0}^{\infty}a_nx^n$ *and* $g(x)=\sum_{n=0}^{\infty}b_nx^n$, *where* $a_n\in\mathbb{R}$ *and* $b_n>0$ *for all* $n\in\{0,1,\ldots\}$, *and suppose that both series converge on* $(-r,r)$, $r>0$. *If the sequence* $\{a_n/b_n\}_{n\geqslant 0}$ *is increasing (decreasing), then the function* $x\mapsto f(x)/g(x)$ *is increasing (decreasing) too on* $(0,r)$.

Then, we list our main results in the following theorem.

THEOREM 2.2. *With* $a,b,c>0$, *we have the following inequalities*

a. *If* $a\geqslant\max\{\frac{c}{2b},c+\frac{1}{2}-b\}$, *then*

$$F(a,b;c;x)\geqslant\left(1-\frac{x}{2}\right)^{-1}F\left(a,b;c;\frac{x^2}{(2-x)^2}\right),\quad x\in(0,1). \tag{13}$$

b. *If* $a\leqslant\min\{\frac{c}{2b},c+\frac{1}{2}-b\}$, *then*

$$F(a,b;c;x)\leqslant\left(1-\frac{x}{2}\right)^{-1}F\left(a,b;c;\frac{x^2}{(2-x)^2}\right),\quad x\in(0,1). \tag{14}$$

c. *If* $b\geqslant\max\{a+c-1,ac\}$ *and* $a\geqslant\frac{1}{2}$, *then*

$$F(a,b;c;x^2)\leqslant(1+x)^{-2a}F\left(a,b;c;\frac{4x}{(1+x)^2}\right),\quad x\in(0,1). \tag{15}$$

d. *If* $b \leqslant \min\{a+c-1, ac\}$ *and* $a \leqslant \frac{1}{2}$, *then*

$$F(a,b;c;x^2) \geqslant (1+x)^{-2a} F\left(a,b;c;\frac{4x}{(1+x)^2}\right), \quad x \in (0,1). \tag{16}$$

REMARK 2.3. Note that there is a unique choice for the parameters a, b and c such that the conditions for both inequalities (15) and (16) are satisfied. Specifically, these values are $a = b = \frac{1}{2}$ and $c = 1$. In this case, inequalities (15) and (16) reduce to the Landen identity (5).

REMARK 2.4. To the best of our knowledge, the inequalities (13) and (14) are new in the literature. The inequalities (15) and (16) are similar to those in [2, Theorem 4].

We note that there is an oversight in [2, Theorem 4]: the parameter c should be positive rather than $c \in \mathbb{R}$. We will present figures in Sec. 3 to demonstrate that the condition $c > 0$ is necessary.

Proof. We begin by proving parts **a** and **b**. Consider the function $Q : (0,1) \to (0,\infty)$

$$Q(x) := \frac{F(a,b;c;x)}{F(1,\frac{1}{2};1;x)} = \frac{\sum\limits_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!}}{\sum\limits_{n=0}^{\infty} \frac{(1)_n (\frac{1}{2})_n}{(1)_n} \cdot \frac{x^n}{n!}}. \tag{17}$$

According to Lemma 2.1, the monotonicity of the function Q depends on the monotonicity of the sequence $\{\alpha_n\}_{n\geqslant 0}$

$$\alpha_n := \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{(1)_n}{(1)_n (\frac{1}{2})_n} = \frac{(a)_n (b)_n}{(c)_n (\frac{1}{2})_n}. \tag{18}$$

This sequence is increasing (decreasing) if

$$\frac{\alpha_{n+1}}{\alpha_n} \geqslant (\leqslant) 1, \qquad \text{for all } n \geqslant 0.$$

Substituting (18) into the above formula, we find that $a, b, c > 0$* and

$$\left(a+b-c-\frac{1}{2}\right) n + ab - \frac{c}{2} \geqslant (\leqslant) 0, \qquad \text{for all } n \geqslant 0.$$

The above inequality holds when $a+b-c-\frac{1}{2}$ and $ab-\frac{c}{2}$ are nonnegative (nonpositive), which implies that $a \geqslant (\leqslant) c + \frac{1}{2} - b$ and $a \geqslant (\leqslant) \frac{c}{2b}$. Consequently, by

*If the condition $a, b, c > 0$ is replaced by $a, b, c \in \mathbb{R}$, we can still obtain $\frac{\alpha_{n+1}}{\alpha_n} \geqslant (\leqslant) 1$ for sufficiently large n. However, this condition may fail for $n = 0, 1, 2, \cdots$. This oversight appears in [2, Theorem 1]. Additionally, some figures in Sec. 3 demonstrate that if we only require $a, b, c \in \mathbb{R}$, the inequalities in (9)–(12) may not hold.

using Lemma 2.1, the function Q is increasing (or decreasing) on the interval $(0,1)$. Therefore, we have $Q(y_1) \leqslant (\geqslant) Q(y_2)$ if $0 < y_1 < y_2 < 1$. Next, we choose $y_1 = x$ and $y_2 = \frac{x^2}{(2-x)^2}$, where $1 > x > \frac{x^2}{(2-x)^2} > 0$ for $x \in (0,1)$. Recalling the definition of Q in (17), we obtain:

$$\frac{F(a,b;c;x)}{F(1,\frac{1}{2};1;x)} \geqslant (\leqslant) \frac{F\left(a,b;c;\frac{x^2}{(2-x)^2}\right)}{F\left(1,\frac{1}{2};1;\frac{x^2}{(2-x)^2}\right)}.$$

Rearranging this inequality gives us

$$F(a,b;c;x) \geqslant (\leqslant) \frac{F\left(1,\frac{1}{2};1;x\right)}{F\left(1,\frac{1}{2};1;\frac{x^2}{(2-x)^2}\right)} F\left(a,b;c;\frac{x^2}{(2-x)^2}\right).$$

By applying (3), we obtain the desired results in parts **a** and **b**.

Next, we prove parts **c** and **d** in a similar way. Consider the function $R : (0,1) \to (0,\infty)$

$$R(x) := \frac{F(a,b;c;x)}{F(a,a;1;x)} = \frac{\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!}}{\sum_{n=0}^{\infty} \frac{(a)_n (a)_n}{(1)_n} \cdot \frac{x^n}{n!}}. \tag{19}$$

The monotonicity of R depends on the monotonicity of the series $\{\beta_n\}_{n \geqslant 0}$

$$\beta_n := \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{(1)_n}{(a)_n (a)_n} = \frac{(1)_n (b)_n}{(c)_n (a)_n}. \tag{20}$$

This sequence is increasing (decreasing) if

$$\frac{\beta_{n+1}}{\beta_n} \geqslant (\leqslant) 1, \qquad \text{for all } n \geqslant 0.$$

Substituting (20) into the above formula, we have $a, c > 0$ and

$$(b-c-a+1)n + b - ca \geqslant (\leqslant) 0, \qquad \text{for all } n \geqslant 0.$$

To ensure the validity of the above inequalities, we require that $b-c-a+1$ and $b-ca$ are nonnegative (or non-positive). This leads to the first condition in parts **c** and **d**, specifically $b \geqslant \max\{a+c-1, ac\}$ and $b \leqslant \min\{a+c-1, ac\}$. By Lemma 2.1, the function R is increasing (or decreasing) on the interval $(0,1)$, i.e., $R(y_1) \leqslant (\geqslant) R(y_2)$ if $0 < y_1 < y_2 < 1$. Next, we choose $y_1 = x^2$ and $y_2 = \frac{4x}{(1+x)^2}$, where $0 < x^2 < \frac{4x}{(1+x)^2} < 1$ for $x \in (0,1)$. With the definition of R in (19), we obtain:

$$F(a,b;c;x^2) \leqslant (\geqslant) \frac{F(a,a;1;x^2)}{F\left(a,a;1;\frac{4x}{(1+x)^2}\right)} F\left(a,b;c;\frac{4x}{(1+x)^2}\right).$$

Using the identity (4), the above inequalities yield

$$F(a,b;c;x^2) \leqslant (\geqslant) \frac{F\left(a,\frac{1}{2};1;\frac{4x}{(1+x)^2}\right)}{F\left(a,a;1;\frac{4x}{(1+x)^2}\right)} \cdot (1+x)^{-2a} F\left(a,b;c;\frac{4x}{(1+x)^2}\right). \tag{21}$$

When $b \geqslant \max\{a+c-1, ac\}$, the above inequality takes the "$\leqslant$" sign. Together with the fact that, when $a \geqslant \frac{1}{2}$,

$$F\left(a,\frac{1}{2};1;\frac{4x}{(1+x)^2}\right) \leqslant F\left(a,a;1;\frac{4x}{(1+x)^2}\right), \qquad x \in (0,1),$$

we obtain (15), which proves part **c**. Similarly, when $b \leqslant \min\{a+c-1, ac\}$ and $a \leqslant \frac{1}{2}$, we get the inequality (16).

This concludes the proof of our theorem. □

3. Discussions and numerical simulations

We first present the graphs in Figure 1 to demonstrate that the condition $a,b,c>0$ is necessary for the validity of inequalities (9) and (10). When we set all of a,b,c to be negative (specifically, $a=-1,\ b=-1,\ c=-2.5$), the left graph in Figure 1 indicates that inequality (9) is reversed. When we choose only c to be negative (with $a=1,\ b=1,\ c=-1.2$), inequality (10) fails to hold for all $x \in (0,1)$. When $a=1,\ b=1,\ c=-1.2$, there exists a value $x_0 \approx 0.953066$ such that the quantity $F\left(a,b;c;\frac{(1-x)^2}{(1+x)^2}\right) - \frac{1+x}{2} F(a,b;c;1-x^2)$ changes sign in the neighborhood of x_0.

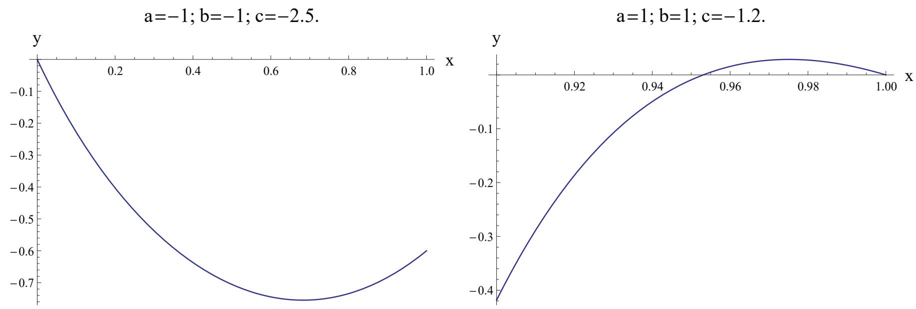

Figure 1: *The left graph for* $F\left(a,b;c;\frac{4x}{(1+x)^2}\right) - (1+x)F(a,b;c;x^2)$ *and the right graph for* $F\left(a,b;c;\frac{(1-x)^2}{(1+x)^2}\right) - \frac{1+x}{2} F(a,b;c;1-x^2)$.

Figure 2 illustrates the validity of our inequality (13) using the example $a=3$, $b=0.5$ and $c=2$. If we reduce a to $\max\{\frac{c}{2b}, c+\frac{1}{2}-b\}$, specifically by choosing $a=c$ and $b=0.5$, the inequalities (13) and (14) reduce to the following one

$$F\left(a,\frac{1}{2};a;x\right) = \left(1-\frac{x}{2}\right)^{-1} F\left(a,\frac{1}{2};a;\frac{x^2}{(2-x)^2}\right), \quad x \in (0,1). \tag{22}$$

The above identity is trivial because $F\left(a,\frac{1}{2};a;x\right)=(1-x)^{-\frac{1}{2}}$; see [5, (15.4.6)].

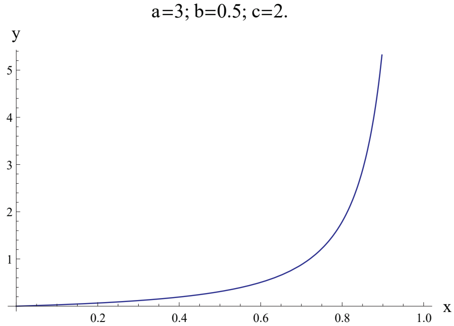

Figure 2: *The graph for* $F(a,b;c;x)-\left(1-\frac{x}{2}\right)^{-1}F\left(a,b;c;\frac{x^2}{(2-x)^2}\right)$.

Figure 3 shows that the condition $c>0$ is necessary in [2, Theorem 4]. When c is negative (for instance, with $a=1$, $b=0.5$, $c=-1.5$), the function $F\left(a,b;c;\frac{4x}{(1+x)^2}\right)$ may not be greater than $(1+x)^{2a}F(a,b;c;x^2)$ for all $x\in(0,1)$, as asserted in [2, Theorem 4]. When $a=1$, $b=0.5$, $c=-1.5$, there exists a value $x_0\approx 0.0736607$ such that the quantity $F\left(a,b;c;\frac{4x}{(1+x)^2}\right)-(1+x)^{2a}F(a,b;c;x^2)$ changes sign in the neighborhood of x_0.

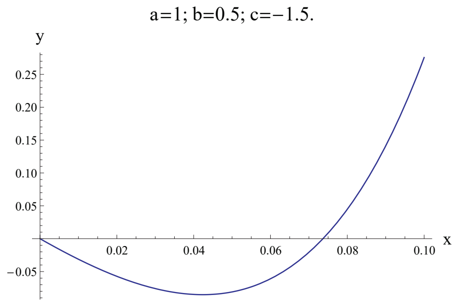

Figure 3: *The graph for* $F\left(a,b;c;\frac{4x}{(1+x)^2}\right)-(1+x)^{2a}F(a,b;c;x^2)$.

Note that the conditions in parts **c** and **d** of Theorem 2.2 differ from those in [2, Theorem 4]. Both sets of conditions serve as sufficient criteria for the validity of inequality (15). In the left graph of Figure 4, the values $a=1$, $b=3$ and $c=2$ satisfy the condition in Theorem 2.2 but violate that in [2, Theorem 4]. Conversely, in the right graph of Figure 4, the values $a=2$, $b=1$, and $c=1$ satisfy the condition in [2, Theorem 4] but violate the condition in Theorem 2.2. This discrepancy arises because, while Lemma 2.1 is useful for deriving Landen-type inequalities, it only provides a sufficient condition. Improving the results in parts **c** and **d** of Theorem 2.2 and [2, Theorem 4] by establishing necessary and sufficient conditions for the validity of inequalities (15) and (16) would be very interesting. Achieving such results may require new ideas.

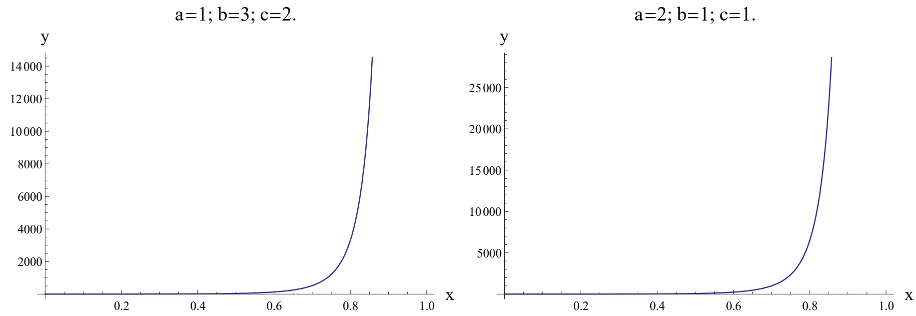

Figure 4: *The graph for* $F\left(a,b;c;\frac{4x}{(1+x)^2}\right)-(1+x)^{2a}F(a,b;c;x^2)$.

We would like to make a final remark about the validity conditions for Landen-type inequalities. For zero-balanced hypergeometric functions, as illustrated in (7) and (8), it is possible to divide the first quadrant of the (a,b)-plane into several regions. This allows us to clearly identify the regions where the inequalities (7) and (8) hold, while in other regions, the inequalities may not be valid for all $x \in (0,1)$; for example, see such kind of treatments in [7, 8, 9]. However, since we are considering general hypergeometric functions that are not zero-balanced, we have three free parameters: a,b and c. Consequently, if we attempt to divide the first octant of the (a,b,c)-space into several solids to discuss the validity of inequalities in different regions, the resulting picture may not be as clear as in the two-dimensional case. Therefore, we did not adopt this method in the present paper.

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