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STRICT SINGULARITY AND CLOSED RANGE OF VOLTERRA INTEGRATION OPERATOR ON DIRICHLET–MORREY TYPE SPACES

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Abstract. The strict singularity and closed range property of the Volterra integral operator T_g with the symbol g on the Dirichlet-Morrey type space $\mathscr{D}_{\lambda,k}$ were investigate in this paper.

1. Introduction

Let $\mathbb{D}$ represent the open unit disk within the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ denote the class of functions that are analytic in $\mathbb{D}$. Let $H^\infty = H^\infty(\mathbb{D})$ denote the space of all bounded analytic functions on $\mathbb{D}$. The norm in this space is given by $\|f\|_\infty = \sup_{w\in\mathbb{D}}|f(w)|$. The Bloch space, denoted by $\mathscr{B}(\mathbb{D}) = \mathscr{B}$, consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{w\in\mathbb{D}}(1-|w|^2)|f'(w)| < \infty.$$

The space $\mathscr{B}$ is a Banach space with respect to the norm $\|\cdot\|_{\mathscr{B}}$, meaning it is complete under this norm. It can be shown that $H^\infty \subset \mathscr{B}$. The little Bloch space, denoted by $\mathscr{B}_0$, contains all $f \in H(\mathbb{D})$ such that $\lim_{|w|\to 1}(1-|w|^2)|f'(w)| = 0$.

Let $0 < p < \infty$ and $\alpha > -1$. The weighted Bergman space A^p_α consists of all analytic functions f in $\mathbb{D}$, i.e., $f \in H(\mathbb{D})$, for which the following norm is finite:

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(w)|^p(1-|w|^2)^\alpha dA(w) < \infty.$$

In the above expression, dA represents the normalized area measure on $\mathbb{D}$. A function $f \in H(\mathbb{D})$ is said to be in the weighted Dirichlet space $\mathscr{D}^p_\alpha(\mathbb{D}) = \mathscr{D}^p_\alpha$ if

$$\|f\|^p_{\mathscr{D}^p_\alpha} = |f(0)|^p + \int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^\alpha dA(w) < \infty.$$

It is worth to note that when $\alpha = 1$ and $p = 2$, the weighted Dirichlet space $\mathscr{D}^p_\alpha$ is identical to the classical Hardy space H^2.

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Paper JMI-19-46

Let $0 < p < \infty$, $-2 < q < \infty$, $0 \leqslant s < \infty$. A function $g \in H(\mathbb{D})$ is in $F(p,q,s)$ if

$$\|g\|_{F(p,q,s)}^p = |g(0)|^p + \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |g'(w)|^p (1-|w|^2)^q (1-|\sigma_b(w)|^2)^s dA(w) < \infty.$$

Here $\sigma_b(w) = \frac{b-w}{1-\bar{b}w}$ denotes the automorphism of $\mathbb{D}$ exchanging 0 for b. A function g belongs to the space $F_0(p,q,s)$ if

$$\lim_{|b| \to 1} \int_{\mathbb{D}} |g'(w)|^p (1-|w|^2)^q (1-|\sigma_b(w)|^2)^s dA(w) = 0.$$

Zhao introduced $F(p,q,s)$ in [41]. From [41], when $q = p-2$, $F(p,p-2,s) = \mathscr{B}$ if $s > 1$, $F(p,p-2,0) = B_p$. When $p = 2$, $F(p,p-2,s) = Q_s$. In particular, $F(2,0,1) = BMOA$, the set of all analytic functions of bounded mean oscillation. For a generalization of the space on the unit ball and some operators acting from or into it see, e.g., [14, 32, 33] and the references therein.

Let $K : [0,\infty) \to [0,\infty)$ be a non-decreasing, right-continuous function. Assume K is not identically zero, with $K(0) = 0$, $K(t) > 0$ for every $t > 0$, and $K(t) = K(1)$ for all $t \geqslant 1$. For $-1 < \lambda < 0$, as Hu and Liu [8] defined, a class of Dirichlet-Morrey type space $\mathscr{D}_{\lambda,K}$ consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{D}_{\lambda,K}} = |f(0)| + \sup_{b \in \mathbb{D}} \frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} \|f \circ \sigma_b - f(b)\|_{\mathscr{D}_{\lambda}^1} < \infty.$$

We denote $\|f\|_{\mathscr{D}_{\lambda,K}} - |f(0)|$ by $\|f\|_{*,\mathscr{D}_{\lambda,K}}$. Analogously, an analytic function f is said to belong to the little Dirichlet-Morrey space $\mathscr{D}_{\lambda,K}^0$ if

$$\lim_{|b| \to 1} \frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} \|f \circ \sigma_b - f(b)\|_{\mathscr{D}_{\lambda}^1} = 0.$$

See [26, 35, 36, 38–40, 42, 44, 45] for various analytic Morrey type spaces, including Hardy-Morrey spaces, Bergman-Morrey spaces, Dirichlet-Morrey spaces and Besov-Morrey spaces.

Let $g \in H(\mathbb{D})$. The Volterra integral operator T_g is defined as follows:

$$T_g f(z) = \int_0^z f(w) g'(w) dw, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Pommerenke [22] was probably the first who studied the operator. He showed that T_g is bounded on the Hardy space H^2 if and only if $g \in BMOA$-the space of bounded mean oscillation. Up to the present day, the operator T_g along with its generalizations, like generalized Cesàro or Volterra operators or their products with composition operators, have attracted significant attention within the mathematical community. Extensive research has been carried out on the boundedness, compactness, norm, and essential norm of T_g over various function spaces, yielding rich results. For a comprehensive overview of additional findings regarding this operator and its various generalizations, one may refer to [1, 2, 4–9, 11–16, 20, 21, 23–36, 38–40, 42, 44, 45].

Recently, the study of the strict singularity of the integral operator T_g in various function spaces has piqued interest. Let us recall that for a bounded linear operator $T : X \to Y$ between Banach spaces, as introduced by Kato in [10], it is said to be strictly singular if for any infinite-dimensional closed subspace $M \subset X$, the restriction $T|_M$ is not an isomorphism onto its image. Precisely, there does not exist a positive constant c such that $\|T(y)\| \geqslant c\|y\|$ for all $y \in M$. Evidently, every compact operator is strictly singular. A well-known example of strictly singular yet non-compact operators is the inclusion mapping $i_{p,q} : \ell^p \to \ell^q$, where $1 \leqslant p < q < \infty$. In 2017, Miihkinen, in the work [20], demonstrated that for a non-compact operator $T_g : H^p \to H^p$ with $1 \leqslant p < \infty$, it contains an isomorphic copy of l^p within H^p. As a consequence, for the operator T_g on H^p, the properties of compactness and strict singularity are equivalent. In the context of Bergman spaces A^p for $1 \leqslant p < \infty$, the strict singularity of the operator T_g is equivalent to its compactness. In [7], Chen established that for the operator $T_g : F(p, p-2, s) \to F(p, p-2, s)$ with $s, p > 0$ satisfying $s + p > 1$, the compactness and strict singularity are identical. See [17] for the study of the strict singularity of another integral operator S_g on H^p.

Meanwhile, studying the closed range of the operator has also attracted a lot of attention. Using the closed graph theorem, we see that T has closed range if and only if it is bounded below when T is a one-to-one bounded linear operator on X. Hence, to investigate the closed range of the operator T_g, we only need to consider the boundedness below of the operator T_g. Recall that a linear operator T on a quasi-Banach space $(X, \|\cdot\|)$ is said to be bounded below if there exists $C > 0$ such that

$$\|Tx\| \geqslant C\|x\|$$

for all $x \in X$. Anderson [3] showed that T_g can be bounded below on weighted Bergman spaces. In 2014, Anderson, Jovovic, and Smith showed that T_g is never bounded below on the Hardy space H^2, the Bloch space $\mathscr{B}$, and the space $BMOA$ [4]. Chen [7] proved that T_g also has no lower bound on $F(p, pa-2, s)$.

Motivated by the above-mentioned works, in this paper, we study the strictly singular property and closed range property of the integration operator T_g on the Dirichlet-Morrey type space $\mathscr{D}_{\lambda,k}$. We show that when $g \in F(1,-1,\lambda+1) \setminus F_0(1,-1,\lambda+1)$, T_g is not strictly singular. Also, T_g has no closed range on $\mathscr{D}_{\lambda,k}$. Moreover, we also investigate some properties of $\mathscr{D}^0_{\lambda,K}$.

Throughout the paper, we write $A \lesssim B$ (or $B \gtrsim A$) to denote that there is some inessential constant C such that $A \leqslant CB$. If $A \lesssim B \lesssim A$, then we write $A \asymp B$.

2. Vanishing K-Carleson measure and embedding of $\mathscr{D}^0_{\lambda,K}$

In this section, we investigate some basic properties of $\mathscr{D}^0_{\lambda,K}$. We require certain properties of K. Throughout the remainder of this paper, we consistently assume that the subsequent condition regarding K is satisfied (refer to [37]):

$$\int_1^\infty \frac{\varphi_K(x)}{x^{1+\delta}} dx < \infty, \ \delta > 0, \tag{2.1}$$

where

$$\varphi_K(x) = \sup_{0<s\leqslant 1} \frac{K(sx)}{K(s)}, \ 0 < x < \infty.$$

Obviously, $K(x) = x^p$ satisfies inequality (2.1) for $0 < p < \delta$.

LEMMA 2.1. ([37], Theorem 3.7) *If K satisfies condition (2.1) for some $\delta > 0$, then there exists a weight function K_1 such that*

$$\lim_{t\to 0^+} \frac{K_1(t)}{t^\delta} = \infty.$$

Furthermore, K_1 still satisfies all standing assumptions on weights, K_1 is comparable with K on $(0,1)$.

For arc I on the unit circle $\partial\mathbb{D}$, $|I|$ is the normalized arc length ($|\partial\mathbb{D}| = 1$). Let

$$S(I) = \{w = re^{i\theta} : 1 - |I| < |w| < 1, e^{i\theta} \in I\}$$

denote the Carleson box based on I. A positive Borel measure μ on $\mathbb{D}$ is called a K-Carleson measure if (see [35])

$$\|\mu\|_K = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{K(|I|)} < \infty.$$

When $K(t) = t^s, 0 < s < \infty$, the K-Carleson measure is an s-Carleson measure. When $K(t) = t$, μ is the classical Carleson measure. The following lemma gives a characterization for K-Carleson measure.

LEMMA 2.2. ([35]) *Suppose K satisfies (2.1) for some $\delta \in (0,2)$. Let μ be a positive Borel measure on $\mathbb{D}$. The measure μ is a K-Carleson measure if and only if*

$$\sup_{b\in\mathbb{D}} \frac{1}{K(1-|b|^2)} \int_{\mathbb{D}} \left(\frac{1-|b|^2}{|1-\bar{b}w|}\right)^t d\mu(w) < \infty, \ \ \delta \leqslant t < \infty.$$

Similarly, we call μ a vanishing K-Carleson measure if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{K(|I|)} = 0.$$

Combining the proofs of Theorem 1 in [19] and Lemma 2.2, or Theorem 2.1 in [35], we can get the following lemma. Since the proof is routine, we omit the detail.

LEMMA 2.3. *Suppose K satisfies (2.1) for some $\delta \in (0,2)$. Let μ be is a positive Borel measure on $\mathbb{D}$. The measure μ is a vanishing K-Carleson measure if and only if*

$$\lim_{|b|\to 1} \frac{1}{K(1-|b|^2)} \int_{\mathbb{D}} \left(\frac{1-|b|^2}{|1-\bar{b}w|}\right)^t d\mu(w) = 0, \delta \leqslant t < \infty.$$

From the definition of $\mathscr{D}^0_{\lambda,K}$ and Lemma 2.3 we get the following characterization for the space $\mathscr{D}^0_{\lambda,K}$.

PROPOSITION 2.4. *Let $-1<\lambda<0$ and K satisfy (2.1) for some $\delta\in(0,\lambda+1]$. Let $f\in H(\mathbb{D})$ and $d\mu_f(w)=|f'(w)|(1-|w|^2)^{\lambda}dA(w)$. Then the following statements are equivalent:*

(i) $f\in\mathscr{D}^0_{\lambda,K}$;

(ii)

$$\lim_{|b|\to1}\frac{(1-|b|^2)^{1+\lambda}}{K(1-|b|)}\int_{\mathbb{D}}|f'(w)|(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{\lambda+1}dA(w)=0;$$

(iii) μ is a vanishing K-Carleson measure.

The following integral estimate is of great importance in our proof.

LEMMA 2.5. [43, Lemma 3.10] *Suppose $z\in\mathbb{D}$, c is real, $t>-1$, and*

$$I_{c,t}(z)=\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^t}{|1-z\bar{w}|^{2+t+c}}dA(w).$$

(a) If $c<0$, then as a function of $z, I_{c,t}(z)$ is bounded from above and bounded from below on $\mathbb{D}$.

(b) If $c>0$, then

$$I_{c,t}(z)\asymp\frac{1}{\left(1-|z|^2\right)^c},\ |z|\to1^-.$$

The following two propositions illustrate the relationship between polynomials and $\mathscr{D}^0_{\lambda,K}$. We show that $\mathscr{D}^0_{\lambda,K}$ is the closure of all polynomials in $\mathscr{D}_{\lambda,K}$.

PROPOSITION 2.6. *Let $-1<\lambda<0$ and K satisfy (2.1) for some $\delta\in(0,\lambda+1]$. Then $\mathscr{D}^0_{\lambda,K}$ (and so $\mathscr{D}_{\lambda,K}$) contains all polynomials.*

Proof. Since K satisfies (2.1) for some $\delta\in(0,\lambda+1]$, by Lemma 2.1, there exists a weight function K_1 such that

$$t^{\delta}/K_1(t)\to0,$$

as $t\to0$. Because $\delta\leqslant\lambda+1$, we have

$$\frac{(1-|b|^2)^{\lambda+1}}{K_1(1-|b|^2)}\leqslant\frac{(1-|b|^2)^{\delta}}{K_1(1-|b|^2)}.$$

Furthermore, K_1 still satisfies all standing assumptions on weights, K_1 is comparable with K on $(0,1)$. Hence

$$\lim_{|b|\to1}\frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)}=0. \tag{2.2}$$

If f is a polynomial, then $\sup_{w\in\mathbb{D}}|f'(w)| \leqslant M < \infty$. By Lemma 2.5,

$$\begin{aligned}
&\frac{(1-|b|^2)^{1+\lambda}}{K(1-|b|^2)}\int_{\mathbb{D}}|f'(w)|(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{1+\lambda}dA(w)\\
\leqslant& M\frac{(1-|b|^2)^{1+\lambda}}{K(1-|b|^2)}\int_{\mathbb{D}}(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{1+\lambda}dA(w)\\
\lesssim& M\frac{(1-|b|^2)^{1+\lambda}}{K(1-|b|^2)}\int_{\mathbb{D}}\frac{(1-|w|^2)^{\lambda}}{|1-\bar{b}w|^{2+2\lambda}}dA(w)\\
\lesssim& M\frac{(1-|b|^2)^{1+\lambda}}{K(1-|b|^2)}.
\end{aligned}$$

By (2.2), we obtain that

$$\lim_{|b|\to 1}\frac{(1-|b|^2)^{1+\lambda}}{K(1-|b|^2)}\int_{\mathbb{D}}|f'(w)|(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{1+\lambda}dA(w) = 0.$$

Thus $f\in\mathscr{D}^0_{\lambda,K}$. □

Similarly to the proof of Proposition 2.15 in [41], we get the following proposition.

PROPOSITION 2.7. *Let* $-1<\lambda<0$ *and* K *satisfy* (2.1) *for some* $\delta\in(0,\lambda+1]$ *and* $g\in\mathscr{D}^0_{\lambda,K}$. *Then*

$$\lim_{r\to 1}\|g_r - g\|_{\mathscr{D}_{\lambda,K}} = 0,$$

where $g_r(w) = g(rw)$. *In particular,* $\mathscr{D}^0_{\lambda,K}$ *is the closure of all polynomials in* $\mathscr{D}_{\lambda,K}$.

Let μ be a positive Borel measure on $\mathbb{D}$. The space $\mathscr{T}_K(\mu)$ consists of all measurable functions f that satisfy

$$\|f\|_{\mathscr{T}_K(\mu)} = \sup_{I\subset\partial\mathbb{D}}\frac{1}{K(|I|)}\int_{S(I)}|f(w)|d\mu(w) < \infty.$$

Let $\mathscr{T}_{K,0}(\mu)$ denote the space of all measure functions f such that

$$\lim_{|I|\to 0}\frac{1}{K(|I|)}\int_{S(I)}|f(w)|d\mu(w) = 0.$$

PROPOSITION 2.8. *Let* $-1<\lambda<0$ *and* K *satisfy* (2.1) *for some* $\delta\in(0,\lambda+1]$. *Let* μ *be a positive Borel measure on* $\mathbb{D}$. *Then the identity operator* $I_d:\mathscr{D}^0_{\lambda,K}\to\mathscr{T}_{K,0}(\mu)$ *is bounded if and only if* μ *is a* $(\lambda+1)$*-Carleson measure.*

Proof. Necessity. Suppose $I_d:\mathscr{D}^0_{\lambda,K}\to\mathscr{T}_{K,0}(\mu)$ is bounded. Let $I\subset\partial\mathbb{D}$, $e^{i\theta}$ be the center of I and $a=(1-|I|)e^{i\theta}$. It is easy to see that

$$|1-\bar{a}w|\approx 1-|a|^2\approx|I|,\quad w\in S(I).$$

Let

$$f_a(w) = \frac{(1-|a|^2)^{1+\lambda}K(1-|a|^2)}{(1-\bar{a}w)^{2\lambda+2}}, \quad w \in \mathbb{D}.$$

Clearly, we have that $f_a \in \mathscr{D}^0_{\lambda,K}$. Thus

$$\begin{aligned}&\frac{\mu(S(I))}{|I|^{1+\lambda}} = \frac{K(|I|)}{|I|^{1+\lambda}}\frac{1}{K(|I|)}\int_{S(I)} d\mu \\ \approx&\frac{1}{K(|I|)}\int_{S(I)}|f_a(w)|d\mu(w) \lesssim \|f_a\|_{\mathscr{T}_K(\mu)} \leqslant \|f_a\|_{\mathscr{D}_{\lambda,K}} < \infty,\end{aligned}$$

which implies that μ is a $(\lambda+1)$-Carleson measure.

Sufficiency. Suppose μ is a $(\lambda+1)$-Carleson measure. Carefully check the proof of Theorem 3.2 in [8], we see that the identity operator $I_d : \mathscr{D}_{\lambda,K} \to \mathscr{T}_K(\mu)$ is bounded. By Proposition 2.7, for any $\varepsilon > 0$, there exists a polynomial $P(z)$ such that

$$\|f-P\|_{\mathscr{D}_{\lambda,K}} < \varepsilon \tag{2.3}$$

for any $f \in \mathscr{D}^0_{\lambda,K}$. Thus,

$$\begin{aligned}\frac{1}{K(|I|)}\int_{S(I)}|f(w)|d\mu(w) \lesssim& \frac{1}{K(|I|)}\int_{S(I)}|f(w)-P(w)|d\mu(w) + \frac{1}{K(|I|)}\int_{S(I)}|P(w)|d\mu(w) \\ \lesssim& \|f-P\|_{\mathscr{T}_K(\mu)} + \|P\|_{H^\infty}\frac{\mu(S(I))}{|I|^{1+\lambda}}\frac{|I|^{1+\lambda}}{K(|I|)}.\end{aligned}$$

By Lemma 2.1, for $\delta \in (0, 1+\lambda]$, we have that

$$\lim_{|I|\to 0}\frac{|I|^{1+\lambda}}{K(I)} = 0. \tag{2.4}$$

Combining (2.3) and (2.4), we get that

$$\lim_{|I|\to 0}\frac{1}{K(|I|)}\int_{S(I)}|f(w)|d\mu(w) = 0,$$

which implies $f \in \mathscr{T}_{K,0}(\mu)$. This completes the proof of this proposition. □

3. Strict singularity

In this section, we mainly show that a non-compact operator T_g is not strictly singular, i.e., there is a subspace $M \subset \mathscr{D}_{\lambda,K}$ such that $T_g|_M$ is bounded below on M which yields the compactness and strict singularity are equivalent for T_g on $\mathscr{D}_{\lambda,K}$. In order to prove the main result in this section, we need the following lemma, see [8, Theorem 1 and Corollary 1].

LEMMA 3.1. *Let* $g \in H(\mathbb{D})$, $-1 < \lambda < 0$ *and* K *satisfy* (2.1) *for some* $\delta \in (0, \lambda + 1]$. *Then the following statements hold.*

1. T_g *is bounded on* $\mathscr{D}_{\lambda,k}$ *if and only if* $g \in F(1,-1,\lambda+1)$;

2. T_g *is compact on* $\mathscr{D}_{\lambda,k}$ *if and only if* $g \in F_0(1,-1,\lambda+1)$.

COROLLARY 3.2. *Let* $g \in H(\mathbb{D})$, $-1 < \lambda < 0$ *and* K *satisfy* (2.1) *for some* $\delta \in (0,\lambda+1]$. *Then* T_g *is bounded on* $\mathscr{D}^0_{\lambda,K}$ *if and only if* $g \in F(1,-1,\lambda+1)$.

Proof. Since $g \in F(1,-1,\lambda+1)$ if and only if $d\mu_g = |g'(w)|(1-|w|^2)^\lambda$ is a $(\lambda+1)$-Carleson measure, by Proposition 2.8, we can get this corollary. $\square$

Using Theorem 9 in [18], we immediately obtain the following result.

LEMMA 3.3. *Let* $-1 < \lambda < 0$ *and* μ *be a positive Borel measure on* $\mathbb{D}$. *Then* μ *is a* $(\lambda+1)$*-Carleson measure if and only if for all functions* $f \in \mathscr{D}^1_\lambda$,

$$\int_{\mathbb{D}} |f(w)| d\mu(w) \lesssim \|f\|_{\mathscr{D}^1_\lambda}.$$

COROLLARY 3.4. *Let* $-1 < \lambda < 0$. *Let* $g \in F(1,-1,\lambda+1)$. *Then* T_g *is bounded on* $\mathscr{D}^1_\lambda$.

Proof. By the assumption we see that $d\mu_g(w) := |g'(w)|(1-|w|^2)^\lambda dA(w)$ is a $(\lambda+1)$-Carleson measure. By Lemma 3.3, $I_d : \mathscr{D}^1_\lambda \to L(\mathbb{D}, d\mu_g)$ is bounded. Hence, for any $f \in \mathscr{D}^1_\lambda$,

$$\|T_g f\|_{\mathscr{D}^1_\lambda} = \int_{\mathbb{D}} |f(w)||g'(w)|(1-|w|^2)^\lambda dA(w) \lesssim \|f\|_{\mathscr{D}^1_\lambda},$$

as desired. $\square$

To prove the main result in this section, one of important steps is to construct the operator $Q : \mathscr{C}_0 \to \mathscr{D}^0_{\lambda,K}$ such that Q is bounded and is an isomorphism onto its range. Here $\mathscr{C}_0$ denote the Banach space of complex sequences converging to zero endowed with the supremum norm $\|\cdot\|_{l^\infty}$. We have the following result.

PROPOSITION 3.5. *Let* $-1 < \lambda < 0$ *and* K *satisfy* (2.1) *for some* $\delta \in (0,1+\lambda]$. *Let* $\{f_n\}_{n=1}^\infty$ *be a sequence in* $\mathscr{D}^0_{\lambda,K}$ *with* $\|f_n\|_{*,\mathscr{D}_{\lambda,K}} \approx 1$ *and* $\lim_{n\to\infty} \|f_n\|_{\mathscr{D}^1_\lambda} = 0$. *Then there exists a subsequence* $\{f_{n_k}\} \subseteq \{f_n\}_{n=1}^\infty$ *such that the operator* $Q : \mathscr{C}_0 \to \mathscr{D}^0_{\lambda,K}$ *defined as follows*

$$Q(\eta) = \sum_{k=1}^\infty \eta_k f_{n_k}, \quad \eta = \{\eta_k\} \subset \mathscr{C}_0,$$

is an isomorphism onto its range.

Proof. We denote

$$\Phi(f_n,b)=\frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)}\int_{\mathbb{D}}|f_n'(w)|(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{\lambda+1}dA(w).$$

Since $f_n\in\mathscr{D}^0_{\lambda,K}$, for each $n\geqslant 1$, we have that

$$\lim_{|b|\to 1}\Phi(f_n,b)=0. \tag{3.1}$$

If $|b|\leqslant t_0$ for some $t_0\in(0,1)$, according to the monotonicity of K and the fact that $\lim_{n\to\infty}\|f_n\|_{\mathscr{D}^1_\lambda}=0$, we get

$$\Phi(f_n,b)\lesssim\frac{1}{K(1-t_0^2)}\int_{\mathbb{D}}|f_n'(w)|(1-|w|^2)^{\lambda}dA(w)\lesssim\frac{1}{K(1-t_0^2)}\|f_n\|_{\mathscr{D}^1_\lambda}\to 0,$$

as $n\to\infty$. Therefore, for any $t\in(0,1)$,

$$\lim_{n\to\infty}\sup_{|b|\leqslant t}\Phi(f_n,b)=0. \tag{3.2}$$

By the assumed condition $\|f_n\|_{*,\mathscr{D}_{\lambda,K}}\asymp 1$, there are $S_1,S_2>0$ such that,

$$S_1\leqslant\|f_n\|_{*,\mathscr{D}_{\lambda,K}}\leqslant S_2 \tag{3.3}$$

for any $n\geqslant 1$.

Combining (3.1), (3.2) with $\lim_{n\to\infty}\|f_n\|_{\mathscr{D}^1_\lambda}=0$, there exits a subsequence $\{f_{n_k}\}$ and an increasing sequence $\{t_k\}$ with $t_k\in(0,1)$ such that, for any $k\geqslant 1$,

$$\sup_{|b|\leqslant t_k}\Phi(f_{n_k},b)<3^{-k-1}S_1,$$

and

$$\sup_{|b|>t_{k+1}}\Phi(f_{n_k},b)<3^{-k-1}S_1$$

and

$$\|f_{n_k}\|_{\mathscr{D}^1_\lambda}<3^{-k-1}S_1.$$

Since b belongs to at most one circular region $t_k<|b|\leqslant t_{k+1}$ within $\mathbb{D}$, for any $b\in\mathbb{D}$, the inequality

$$\Phi(f_{n_k},b)<3^{-k-1}S_1$$

holds for all but at most one index k. For the exceptional index k, $t_k<|b|\leqslant t_{k+1}$, the inequality $\Phi(f_{n_k},b)\leqslant S_2$ is satisfied. In addition, from (3.3), for any $k\geqslant 1$, there is $b_k\in\mathbb{D}$ such that

$$\Phi(f_{n_k},b_k)\geqslant\frac{2S_1}{3}.$$

By the triangle inequality, for any $\eta = \{\eta_k\} \in \mathscr{C}_0$, we obtain that

$$\begin{aligned}\|Q(\eta)\|_{\mathscr{D}_{\lambda,K}} &= \Big|\sum_{k=1}^{\infty} \eta_k f_{n_k}(0)\Big| + \sup_{b\in\mathbb{D}} \Phi(Q(\eta),b) \\ &\leqslant \|\eta\|_{l^\infty} \sum_{k=1}^{\infty} \|f_{n_k}\|_{\mathscr{D}^1_\lambda} + \sup_{b\in\mathbb{D}} \sum_{k=1}^{\infty} |\eta_k| \Phi(f_{n_k},b) \\ &< \|\eta\|_{l^\infty} \sum_{k=1}^{\infty} 3^{-k-1} S_1 + \|\eta\|_{l^\infty} \left(\sum_{k=1}^{\infty} 3^{-k-1} S_1 + S_2\right) \\ &< (S_1+S_2)\|\eta\|_{l^\infty},\end{aligned}$$

which implies that $Q(\eta) \in \mathscr{D}_{\lambda,K}$ and $Q : \mathscr{C}_0 \to \mathscr{D}_{\lambda,K}$ is bounded.

Next, we prove that $Q(\eta) \in \mathscr{D}^0_{\lambda,K}$. In fact, for any $b \in \mathbb{D}$ and $N > 1$,

$$\begin{aligned}\Phi(Q(\eta),b) &\leqslant \sum_{k=1}^{\infty} |\eta_k| \Phi(f_{n_k},b) \leqslant \|\eta\|_{l^\infty} \sum_{k=1}^{N} \Phi(f_{n_k},b) + \sup_{k>N} |\eta_k| \sum_{k=N+1}^{\infty} \Phi(f_{n_k},b) \\ &\lesssim \|\eta\|_{l^\infty} \sum_{k=1}^{N} \Phi(f_{n_k},b) + \sup_{k>N} |\eta_k|.\end{aligned}$$

Because $f_{n_k} \in \mathscr{D}^0_{\lambda,K}$ for each $k \geqslant 1$, we get

$$\overline{\lim_{|b|\to 1}}\, \Phi(Q(\eta),b) \lesssim \sup_{k>N} |\eta_k|.$$

Letting $N \to \infty$, we have that

$$\overline{\lim_{|b|\to 1}}\, \Phi(Q(\eta),b) = 0$$

and hence $Q(\eta) \in \mathscr{D}^0_{\lambda,K}$, which implies that $Q : \mathscr{C}_0 \to \mathscr{D}^0_{\lambda,K}$ is bounded.

Finally we will prove that $Q : \mathscr{C}_0 \to \mathscr{D}^0_{\lambda,K}$ is bounded below. For any $\eta = \{\eta_k\}$ and $j \geqslant 1$, by triangle inequality we obtain

$$\begin{aligned}\|Q(\eta)\|_{\mathscr{D}_{\lambda,K}} \geqslant \Phi(Q(\eta),b_j) &\geqslant |\eta_j| \Phi(f_{n_j},b_j) - \sum_{k\neq j} |\eta_k| \Phi(f_{n_k},b_j) \\ &\geqslant \frac{2S_1}{3}|\eta_j| - \sum_{k\neq j} |\eta_k| 3^{-k-1} S_1 \\ &\geqslant \frac{2S_1}{3}|\eta_j| - \frac{S_1}{6}\|\eta\|_{l^\infty}.\end{aligned}$$

The arbitrariness of j gives that

$$\|Q(\eta)\|_{\mathscr{D}_{\lambda,K}} \gtrsim \frac{S_1}{2}\|\eta\|_{l^\infty}.$$

Thus $Q : \mathscr{C}_0 \to \mathscr{D}^0_{\lambda,K}$ is an isomorphism onto its range. □

THEOREM 3.6. *Let* $-1<\lambda<0$ *and* K *satisfy* (2.1) *for some* $\delta\in(0,1+\lambda]$. *Suppose* $g\in F(1,-1,\lambda+1)\setminus F_0(1,-1,\lambda+1)$. *Then there is a subspace* $M\subset\mathscr{D}_{\lambda,K}$ *isomorphic to* $\mathscr{C}_0$ *such that* $T_g|_M: M\to T_g(M)$ *is an isomorphism. In particular,* T_g *is not strictly singular.*

Proof. Let

$$\Upsilon := \limsup_{|I|\to 0}\frac{1}{|I|^{1+\lambda}}\int_{S(I)}|g'(w)|(1-|w|^2)^{\lambda}dA(w).$$

Because $g\in F(1,-1,\lambda+1)\setminus F_0(1,-1,\lambda+1)$, we get that $\Upsilon>0$ and there exists $\{I_n\}\subset\partial\mathbb{D}$ with $\lim_{n\to\infty}|I_n|=0$ such that

$$\Upsilon=\lim_{n\to\infty}\frac{1}{|I_n|^{1+\lambda}}\int_{S(I_n)}|g'(w)|(1-|w|^2)^{\lambda}dA(w)>0. \tag{3.4}$$

For each $n\geqslant 1$, set $\gamma_n=(1-|I_n|)\eta_n$, where η_n is the center of I_n. It is known that

$$|1-\bar{\gamma}_n w|\approx 1-|\gamma_n|^2\approx |I_n|,\qquad w\in S(I_n). \tag{3.5}$$

Take

$$f_n(w)=\frac{(1-|\gamma_n|^2)^{\lambda+1}K(1-|\gamma_n|^2)}{(1-\overline{\gamma}_n w)^{2\lambda+2}},\ \ w\in\mathbb{D}.$$

It is easy to check that $f_n\in\mathscr{D}^0_{\lambda,K}$ and $\|f_n\|_{\mathscr{D}_{\lambda,K}}\lesssim 1$ for any $n\geqslant 1$. Form (3.5), it is easy to see that

$$|f_n(w)|\gtrsim\frac{K(|I_n|)}{|I_n|^{1+\lambda}},\ \ w\in S(I_n). \tag{3.6}$$

Then by using (3.4) and (3.6), there exist some constant $\beta>0$ such that

$$\frac{1}{K(|I_n|)}\int_{S(I_n)}|f_n(w)||g'(w)|(1-|w|^2)^{\lambda}dA(w)\geqslant\frac{\Upsilon\beta}{3}$$

for n larger than some fixed positive number $N\geqslant 1$.

Since $g\in F(1,-1,\lambda+1)$, T_g is bounded on $\mathscr{D}_{\lambda,K}$, which implies that $T_gf_n\in\mathscr{D}_{\lambda,K}$ and

$$\|T_gf_n\|_{\mathscr{D}_{\lambda,K}}\lesssim 1.$$

Moreover, by Corollary 3.2, T_g is bounded also on $\mathscr{D}^0_{\lambda,K}$ and then $T_gf_n\in\mathscr{D}^0_{\lambda,K}$. Thus, passing to a further subsequence if necessary, we assume that

$$\frac{1}{K(|I_{n+1}|)}\int_{S(I_{n+1})}|f_n(w)||g'(w)|(1-|w|^2)^{\lambda}dA(w)\leqslant\frac{\Upsilon\beta}{4}$$

for any $n\geqslant N$.

Set

$$H_n=f_{n+1}-f_n, n\geqslant N.$$

Then $H_n \in \mathscr{D}^0_{\lambda,K}$ and $\|H_n\|_{\mathscr{D}_{\lambda,K}} \lesssim 1$. According to the above facts, we have that

$$\begin{aligned}
&\frac{1}{K(|I_{n+1}|)}\int_{S(I_{n+1})}|H_n(z)||g'(w)|(1-|w|^2)^\lambda dA(w)\\
=&\frac{1}{K(|I_{n+1}|)}\int_{S(I_{n+1})}|f_{n+1}(w)-f_n(w)||g'(w)|(1-|w|^2)dA(w)\\
\geqslant&\frac{1}{K(|I_{n+1}|)}\int_{S(I_{n+1})}|f_{n+1}(w)||g'(w)|(1-|w|^2)dA(w)\\
&-\frac{1}{K(|I_{n+1}|)}\int_{S(I_{n+1})}|f_n(w)||g'(w)|(1-|w|^2)dA(w)\\
\geqslant&\frac{\Upsilon\beta}{12}.
\end{aligned}$$

Therefore, for any $n \geqslant N$,

$$\begin{aligned}
\frac{\Upsilon\beta}{12} \leqslant&\frac{1}{K(|I_{n+1}|)}\int_{S(I_{n+1})}|H_n(w)||g'(w)|(1-|w|^2)^\lambda dA(w)\\
\lesssim&\|T_gH_n\|_{*,\mathscr{D}_{\lambda,K}} \lesssim \|H_n\|_{*,\mathscr{D}_{\lambda,K}} \lesssim 1.
\end{aligned}$$

Thus $T_gH_n \in \mathscr{D}^0_{\lambda,K}$ and

$$\|T_gH_n\|_{*,\mathscr{D}_{\lambda,K}} \asymp \|H_n\|_{*,\mathscr{D}_{\lambda,K}} \asymp 1$$

for any $n \geqslant N$.

Since $H_n \in \mathscr{D}^0_{\lambda,K}$, we have that $H_n \in \mathscr{D}^1_\lambda$. Set $P_n^\lambda = (1-|\gamma_n|^2)^{\lambda+1}K(1-|\gamma_n|^2)$.

$$\begin{aligned}
&\int_{\mathbb{D}}|H_n'(w)|(1-|w|^2)^\lambda dA(w) = \int_{\mathbb{D}}|f_{n+1}'(w)-f_n'(w)|(1-|w|^2)^\lambda dA(w)\\
=&(2\lambda+2)\int_{\mathbb{D}}\left|\frac{P_{n+1}^\lambda\gamma_{n+1}}{(1-\overline{\gamma_{n+1}}w)^{2\lambda+3}}-\frac{P_n^\lambda\gamma_n}{(1-\overline{\gamma_n}w)^{2\lambda+3}}\right|(1-|w|^2)^\lambda dA(w)\\
\lesssim&\int_{\mathbb{D}}\frac{\left|P_{n+1}^\lambda\gamma_{n+1}(1-\overline{\gamma_n}w)^{2\lambda+3}-P_n^\lambda\gamma_n(1-\overline{\gamma_{n+1}}w)^{2\lambda+3}\right|}{|1-\overline{\gamma_{n+1}}w|^{2\lambda+3}|1-\overline{\gamma_n}w|^{2\lambda+3}}(1-|w|^2)^\lambda dA(w)\\
\lesssim&\int_{\mathbb{D}}\frac{P_{n+1}^\lambda\left|(1-\overline{\gamma_n}w)^{2\lambda+3}-(1-\overline{\gamma_{n+1}}w)^{2\lambda+3}\right|}{|1-\overline{\gamma_{n+1}}w|^{2\lambda+3}|1-\overline{\gamma_n}w|^{2\lambda+3}}(1-|w|^2)^\lambda dA(w)\\
&+\int_{\mathbb{D}}\frac{\left|P_{n+1}^\lambda\gamma_{n+1}-P_n^\lambda\gamma_n\right||1-\overline{\gamma_{n+1}}w|^{2\lambda+3}}{|1-\overline{\gamma_{n+1}}w|^{2\lambda+3}|1-\overline{\gamma_n}w|^{2\lambda+3}}(1-|w|^2)^\lambda dA(w)\\
\lesssim&\int_{\mathbb{D}}\frac{P_{n+1}^\lambda(1-|w|^2)^\lambda dA(w)}{|1-\overline{\gamma_{n+1}}w|^{2\lambda+3}}+\int_{\mathbb{D}}\frac{P_{n+1}^\lambda(1-|w|^2)^\lambda dA(w)}{|1-\overline{\gamma_n}w|^{2\lambda+3}}+\int_{\mathbb{D}}\frac{P_n^\lambda(1-|w|^2)^\lambda dA(w)}{|1-\overline{\gamma_n}w|^{2\lambda+3}}.
\end{aligned} \tag{3.7}$$

By Lemma 2.5,

$$\begin{aligned}&\int_{\mathbb{D}} \frac{P_n^{\lambda}(1-|w|^2)^{\lambda} dA(w)}{|1-\overline{\gamma_n}w|^{2\lambda+3}} \\ =&(1-|\gamma_n|^2)^{\lambda+1} K(1-|\gamma_n|^2) \int_{\mathbb{D}} \frac{(1-|w|^2)^{\lambda} dA(w)}{|1-\overline{\gamma_n}w|^{2\lambda+3}} \lesssim K(1-|\gamma_n|^2) \to 0\end{aligned} \tag{3.8}$$

as $n \to \infty$. Similarly,

$$\int_{\mathbb{D}} \frac{P_{n+1}^{\lambda}(1-|w|^2)^{\lambda} dA(w)}{|1-\overline{\gamma_{n+1}}w|^{2\lambda+3}} \to 0 \quad \text{and} \quad \int_{\mathbb{D}} \frac{P_{n+1}^{\lambda}(1-|w|^2)^{\lambda} dA(w)}{|1-\overline{\gamma_n}w|^{2\lambda+3}} \to 0 \tag{3.9}$$

as $n \to \infty$. Combining (3.7), (3.8) and (3.9), we obtain that

$$\lim_{n\to\infty} \|H_n\|_{\mathscr{D}_{\lambda}^1} = 0.$$

Since T_g is bounded on $\mathscr{D}_{\lambda}^1$ by Corollary 3.4, we get

$$\lim_{n\to\infty} \|T_g H_n\|_{\mathscr{D}_{\lambda}^1} = 0.$$

Using Proposition 3.5, we can construct a subsequence $\{H_{n_k}\}$ of $\{H_n\}$ such that $Q : \mathscr{C}_0 \to \mathscr{D}_{\lambda,K}^0$ and $V : \mathscr{C}_0 \to \mathscr{D}_{\lambda,K}^0$ are both isomorphism onto their respective ranges, where

$$Q(\eta) = \sum_{k=1}^{\infty} \eta_k H_{n_k}, \quad \eta = \{\eta_k\} \subset \mathscr{C}_0,$$

and

$$V(\eta) = \sum_{k=1}^{\infty} \eta_k T_g H_{n_k}, \quad \eta = \{\eta_k\} \subset \mathscr{C}_0.$$

Let M be the closure of span $\{H_{n_k}\}$ in $\mathscr{D}_{\lambda,K}^0$. Then M is isomorphic to $\mathscr{C}_0$ and $T_g|_M : M \to T_g(M)$ is an isomorphism. This complete the proof of this theorem. □

4. Closed range of T_g

In this section, we will study the closed range property of T_g on $\mathscr{D}_{\lambda,k}$ and show that $T_g : \mathscr{D}_{\lambda,k} \to \mathscr{D}_{\lambda,k}$ does not have closed range.

THEOREM 4.1. *Let $-1 < \lambda < 0$, $g \in F(1,-1,\lambda+1)$ and K satisfy (2.1) for some $\delta \in (0,1+\lambda]$. Then $T_g : \mathscr{D}_{\lambda,k} \to \mathscr{D}_{\lambda,k}$ does not have closed range, or equivalently to say that T_g is not bounded below on $\mathscr{D}_{\lambda,k}$.*

Proof. If $g \in F_0(1,-1,1)$, by Lemma 3.1, we have that T_g is compact on $\mathscr{D}_{\lambda,k}$ and then T_g is not bounded below on $\mathscr{D}_{\lambda,k}$. Thus we just need to prove that the conclusion is correct when $g \in F(1,-1,\lambda+1) \setminus F_0(1,-1,\lambda+1)$ happens.

From the proof of Theorem 2.9, for any $\varepsilon > 0$, there is a positive constant $\rho_0 \in (0,1)$ such that

$$\frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} < \varepsilon \tag{4.1}$$

for any $\rho_0 < |b| < 1$.

Let $f_n(w) = w^n$. Clearly, $f_n \in \mathscr{D}_{\lambda,k}$ for any $n \geqslant 1$. Then, for any $n \geqslant 1$, by (4.1) we get that

$$\begin{aligned} &\sup_{\rho_0<|b|<1} \frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} \int_{\mathbb{D}} |w^n||g'(w)|(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{2\lambda+2} dA(w) \\ \leqslant &\|g\|_{F(1,-1,\lambda+1)} \sup_{\rho_0<|b|<1} \frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} \lesssim \varepsilon. \end{aligned} \tag{4.2}$$

Now let us prove the case of $|b| \leqslant \rho_0$. Since $g \in F(1,-1,\lambda+1)$, we see that $g \in \mathscr{D}^1_\lambda$. Thus there exists $r \in (0,1)$ such that

$$\int_{\mathbb{D}\setminus r\mathbb{D}} |g'(w)|(1-|w|^2)^\lambda dA(w) < \varepsilon.$$

Thus, by the monotonicity of K we have that

$$\begin{aligned} &\sup_{|b|\leqslant\rho_0} \frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} \int_{\mathbb{D}\setminus r\mathbb{D}} |w^n||g'(w)|(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{\lambda+1} dA(w) \\ = &\sup_{|b|\leqslant\rho_0} \frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} \int_{\mathbb{D}\setminus r\mathbb{D}} |w^n||g'(w)|(1-|w|^2)^\lambda \frac{1}{|1-\bar{b}w|^{2\lambda+2}} dA(w) \\ \lesssim &\sup_{|b|\leqslant\rho_0} \frac{1}{K(1-|b|^2)} \int_{\mathbb{D}\setminus r\mathbb{D}} |g'(w)|(1-|w|^2)^\lambda dA(w) \\ \lesssim &\frac{\varepsilon}{K(1-|\rho_0|^2)}. \end{aligned} \tag{4.3}$$

In addition, we can choose a sufficiently large N such that $r^n < \varepsilon$ whenever $n > N$. Then

$$\begin{aligned} &\sup_{|b|\leqslant\rho_0} \frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} \int_{\overline{r\mathbb{D}}} |w^n||g'(w)|(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{\lambda+1} dA(w) \\ \lesssim &\frac{\varepsilon\|g\|_{F(1,-1,\lambda+1)}}{K(1-|\rho_0|^2)}. \end{aligned} \tag{4.4}$$

Combing (4.3) with (4.4) we get

$$\begin{aligned} &\sup_{|b|\leqslant\rho_0} \frac{(1-|b|^2)^{\lambda+1}}{K(1-|b|^2)} \int_{\mathbb{D}} |w^n||g'(w)|(1-|w|^2)^{-1}(1-|\sigma_b(w)|^2)^{\lambda+1} dA(w) \\ \lesssim &\frac{\varepsilon\|g\|_{F(1,-1,\lambda+1)}}{K(1-|\rho_0|^2)} \lesssim \varepsilon. \end{aligned} \tag{4.5}$$

By (4.2), (4.5) and the fact that $\varepsilon > 0$ is arbitrary, we conclude that T_g is not bounded below on $\mathscr{D}_{\lambda,k}$. □

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