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A NEW PROOF OF THE SHARP WEIGHTED POWER MEAN BOUNDS FOR THE SCHWAB–BORCHARDT MEAN

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Abstract. Let $SB(x,y)$ be Schwab-Borchardt mean of two positive numbers x and y. In this paper, by using hypergeometric function theory and some new analytical techniques, we provide a new proof of the sharp weighted power mean bounds for the Schwab-Borchardt mean, that is, the double inequality

$$\left(\frac{1}{3}x^p+\frac{2}{3}y^p\right)^{1/p}<SB(x,y)<\left(\frac{1}{3}x^q+\frac{2}{3}y^q\right)^{1/q}$$

holds for $0<x<y$ if and only if $p\leqslant 4/5$ and $q\geqslant \log_{\pi/2}(3/2)=0.8978\cdots$, and it holds for $x>y>0$ if and only if $p\leqslant 0$ and $q\geqslant 4/5$.

1. Introduction

The Schwab-Borchardt mean of two positive numbers x and y is defined as follows

$$SB(x,y)\equiv SB=\begin{cases}\frac{\sqrt{y^2-x^2}}{\arccos(x/y)}, & x<y,\\ \frac{\sqrt{x^2-y^2}}{\operatorname{arccosh}(x/y)}, & y<x\end{cases} \tag{1.1}$$

(cf. [3, (2.3)], [11, (1.1)]). It is apparent from (1.1) that the Schwab-Borchardt mean is a homogeneous function of degree one in its variables and $SB(x,y)\neq SB(y,x)$ for $x\neq y$. Using elementary identities for the inverse trigonometric (hyperbolic) function, one can rewrite the formula (1.1) as

$$SB(x,y)=\frac{\sqrt{y^2-x^2}}{\arcsin\left(\sqrt{1-(x/y)^2}\right)}=\frac{\sqrt{y^2-x^2}}{\arctan\left(\sqrt{(y/x)^2-1}\right)} \tag{1.2}$$

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for $0<x<y$ and

$$SB(x,y)=\frac{\sqrt{x^2-y^2}}{\operatorname{arcsinh}\left(\sqrt{(x/y)^2-1}\right)}=\frac{\sqrt{x^2-y^2}}{\operatorname{arctanh}\left(\sqrt{1-(y/x)^2}\right)} \tag{1.3}$$

for $0<y<x$.

It is well known that many classical homogenous means are the particular cases of the Schwab-Borchardt mean. For $x,y>0$ with $x\neq y$, if we let H,G,A,Q and C stand, respectively, for the harmonic, geometric, arithmetic, quadratic and contra-harmonic means of x and y, that is,

$$H=H(x,y)=\frac{2xy}{x+y},\quad G=G(x,y)=\sqrt{xy},\quad A=A(x,y)=\frac{x+y}{2},$$

$$Q=Q(x,y)=\sqrt{\frac{x^2+y^2}{2}},\quad C=C(x,y)=\frac{x^2+y^2}{x+y}.$$

Then two Seiffert means

$$P=P(x,y)=\frac{x-y}{2\arcsin\left(\frac{x-y}{x+y}\right)}$$

(cf. [16]) and

$$T=T(x,y)=\frac{x-y}{2\arctan\left(\frac{x-y}{x+y}\right)}$$

(cf. [17]), the logarithmic mean

$$L=L(x,y)=\frac{x-y}{\log x-\log y}=\frac{x-y}{2\operatorname{arctanh}\left(\frac{x-y}{x+y}\right)}$$

(cf. [5]) and the Neuman-Sándor mean

$$NS=NS(x,y)=\frac{x-y}{2\operatorname{arcsinh}\left(\frac{x-y}{x+y}\right)}$$

(cf. [11, (2.6)]) can be expressed by

$$P=SB(G,A),\quad T=SB(A,Q),\quad L=SB(A,G),\quad NS=SB(Q,A),$$

respectively (cf. [11, (2.8)]).

An alternative family of bivariate means derived from the Schwab-Borchardt mean are defined by

$$S_{AH}=SB(A,H),\quad S_{HA}=SB(H,A),\quad S_{CA}=SB(C,A),$$
$$S_{AC}=SB(A,C),\quad S_{CH}=SB(C,H),\quad S_{HC}=SB(H,C)$$

(cf. [8, (8)]). Generally speaking, a pair of bivariate means X and Y generates a new mean $SB(X,Y)$ by the Schwab-Borchardt mean, which is called a Schwab-Borchardt type mean with the generating means X and Y.

In the past twenty years, optimal bounds for the Schwab-Borchardt type means in terms of their generating means have been studied in [4, 6, 7, 8, 9, 10, 15]. As applications, a lot of equivalent sharp inequalities for the trigonometric functions and hyperbolic functions have also been derived. For example, the authors [4, 7, 15] proved

$$G^{1/3}A^{2/3} < P < \frac{1}{3}G + \frac{2}{3}A, \tag{1.4}$$

$$A^{1/3}G^{2/3} < L < \frac{1}{3}A + \frac{2}{3}G, \tag{1.5}$$

for all $x,y > 0$ with $x \neq y$, and therefore obtained

$$\cos^{1/3} t < \frac{\sin t}{t} < \frac{2}{3} + \frac{1}{3}\cos t, \qquad t \in \left(0, \frac{\pi}{2}\right),$$

$$(\cosh x)^{1/3} < \frac{\sinh t}{t} < \frac{2}{3} + \frac{1}{3}\cosh x, \quad t \in (0,\infty)$$

by letting $(x,y) = (1+\sin t, 1-\sin t)$ in (1.4) for $t \in (0,\pi/2)$ and $(x,y) = (1+\tanh t, 1-\tanh t)$ in (1.5) for $t \in (0,\infty)$. These inequalities are known as Cusa-Huygens and Mitrinović-Adamović inequalities, and their hyperbolic counterpart.

It is worthy noting that inequalities (1.4) and (1.5) can be unified by

$$M_0(x,y;1/3) = x^{1/3}y^{2/3} < SB(x,y) < \frac{x+2y}{3} = M_1(x,y;1/3) \tag{1.6}$$

for $x,y > 0$ with $x \neq y$, where

$$M_p(x,y;w) = \begin{cases} \left[wx^p + (1-w)y^p\right]^{1/p}, & p \neq 0, \\ x^w y^{1-w}, & p = 0 \end{cases} \tag{1.7}$$

is the weighted power mean of positive numbers x and y with weight $w \in (0,1)$. We remark that $p \mapsto M_p(x,y;w)$ is strictly increasing for fixed $x,y > 0$ with $x \neq y$ and $w \in (0,1)$, and $M_p(x,y;w)$ for $w \neq 1/2$ is also a non-symmetric mean of x and y.

To strengthen the double inequality (1.4), Yang [23] had proved that the double inequality

$$\left(\frac{1}{3}G^p + \frac{2}{3}A^p\right)^{1/p} < P < \left(\frac{1}{3}G^q + \frac{2}{3}A^q\right)^{1/q}$$

for $x,y > 0$ if and only if $p \leqslant 4/5$ and $q \geqslant \log_{\pi/2}(3/2) = 0.8978\cdots$, which is, by Seiffert and Arc sine transformation, equivalent to the double inequality

$$M_p\left(\cos t, 1; \frac{1}{3}\right) = \left(\frac{1}{3}\cos^p t + \frac{2}{3}\right)^{1/p} < \frac{\sin t}{t}$$

$$< \left(\frac{1}{3}\cos^q t + \frac{2}{3}\right)^{1/q} = M_q\left(\cos t, 1; \frac{1}{3}\right) \tag{1.8}$$

holds for all $t \in (0, \pi/2)$ if and only if $p \leqslant 4/5$ and $q \geqslant \log_{\pi/2}(3/2) = 0.8978\cdots$, where the second inequality was first proved in [26], and the first one also appeared in [18, Proposition 5]. For its hyperbolic counterpart, it also had been proved in [27, Theorem 1.1] that the double inequality

$$M_p\left(\cosh t, 1; \frac{1}{3}\right) = \left(\frac{1}{3}\cosh^p t + \frac{2}{3}\right)^{1/p} < \frac{\sinh t}{t}$$
$$< \left(\frac{1}{3}\cosh^q t + \frac{2}{3}\right)^{1/q} = M_q\left(\cosh t, 1; \frac{1}{3}\right) \tag{1.9}$$

is valid for $t \in (0, \infty)$ if $p < 0$ and $q \geqslant 4/5$.

According to (1.1), it is not difficult to verify that the inequalities (1.8) and (1.9) can be rewritten as inequalities (1.10) and (1.11) in the following theorem, which improve the inequality (1.6). However, the proofs of two inequalities (1.8) and (1.9) are based on derivatives and elementary trigonometric (hyperbolic) functions, which is not easy to extended to the case of generalized trigonometric (hyperbolic) functions. The main goal of this paper is provide a new proof of Theorem 1.1 through the methods of hypergeometric function theory and the unimodal monotonicity rule.

THEOREM 1.1. *Let $p, q \in \mathbb{R}$. Then we have the following conclusions.*
(i) The inequality

$$M_p(x, y; 1/3) < SB(x, y) < M_q(x, y; 1/3) \tag{1.10}$$

holds for all $0 < x < y$ if and only if $p \leqslant 4/5$ and $q \geqslant \tau = \log_{\pi/2}(3/2) = 0.8978\cdots$.
(ii) The inequality

$$M_p(x, y; 1/3) < SB(x, y) < M_q(x, y; 1/3) \tag{1.11}$$

holds for all $0 < y < x$ if and only if $p \leqslant 0$ and $q \geqslant 4/5$.

The rest of this paper is as follows. In the following section, we shall recall the definition and basic facts of Gaussian hypergeometric function, introduce two technical tools and then establish several lemmas. In Section 3, we prove two monotonicity theorems involving the inverse (hyperbolic) sine function and complete the proof of Theorem 1.1.

2. Preliminaries

2.1. Basic knowledge

Given $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by

$$F(a, b; c; x) := {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad \text{for} \quad |x| < 1,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the *shifted factorial function* or the *Pochhammer symbol* for $n \in \mathbb{N}$. Here $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ the classical Euler gamma function (cf. [13]).

Recall that the hypergeometric function $F(a,b;c;x)$ has a simple derivative formula

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x).$$

Further, the behavior of hypergeometric function $F(a,b;c;x)$ near $x = 1$ satisfies the following properties:

(1) $c > a+b$ (cf. [14, p. 49])

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2.1}$$

(2) $c = a+b$ (cf. [1, 15.3.10]) the Ramanujan's asymptotic formula $(x \to 1)$

$$B(a,b)F(a,b;c;x) + \log(1-x) = R(a,b) + O[(1-x)\log(1-x)], \tag{2.2}$$

(3) $c < a+b$ (cf. [12, (1.2)]), as $x \to 1$,

$$\begin{aligned} F(a,b;c;x) &= (1-x)^{c-a-b}F(c-a,c-b;c;x) \\ &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}[1+o(1)], \end{aligned} \tag{2.3}$$

where $B(a,b) = [\Gamma(a)\Gamma(b)]/\Gamma(a+b)$, $R(a,b) = -2\gamma - \psi(a) - \psi(b)$, $\psi(x) = \Gamma'(x)/\Gamma(x)$ and γ are the beta function, the Ramanujan constant, the psi function and the Euler-Mascheroni constant.

As a special case, the inverse sine and hyperbolic tangent functions can be presented [1, (15.1.4) and (15.1.6)], in terms of hypergeometric functions, by

$$\arcsin x = xF(1/2,1/2;3/2;x^2) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\sqrt{\pi}(2n+1)n!}x^{2n+1}, \tag{2.4}$$

$$\operatorname{arctanh} x = xF(1/2,1;3/2;x^2) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}. \tag{2.5}$$

2.2. Tools and Lemmas

In this subsection, we introduce two tools to used to prove our results. The first tool is the monotonic rule, which plays an important role in dealing with the monotonicity of the ratio of power series.

Before stating this monotone rule, we need to introduce the so-called H-function $H_{f,g}$; see [18, 21, 22] for more properties. Let f and g be differentiable on (a,b) and $g' \neq 0$ on (a,b) for $-\infty \leqslant a < b \leqslant \infty$. Then the function $H_{f,g}$ is defined by

$$H_{f,g} := \frac{f'}{g'}g - f.$$

PROPOSITION 2.1. ([24]) *Let* $f(t)=\sum_{n=0}^{\infty}a_nt^n$ *and* $g(t)=\sum_{n=0}^{\infty}b_nt^n$ *be two real power series converging on* $(-r,r)$ *and* $b_n>0$ *for all non-negative integer* n*. Then the following statements hold true:*

(1) *If the non-constant sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is increasing (decreasing) for all* $n\geqslant 0$*, then the function* f/g *is strictly increasing (decreasing) on* $(0,r)$*;*

(2) *Suppose that for certain* $m\in\mathbb{N}$*, the sequence* $\{a_k/b_k\}_{0\leqslant k\leqslant m}$ *and* $\{a_k/b_k\}_{k\geqslant m}$ *are both non-constant, and they are increasing (decreasing) and decreasing (increasing), respectively. Then the function* f/g *is strictly increasing (decreasing) on* $(0,r)$ *if and only if* $H_{f,g}(r^-)\geqslant(\leqslant)0$*. If* $H_{f,g}(r^-)<(>)0$*, then there exists* $t_0\in(0,r)$ *such that* f/g *is strictly increasing (decreasing) on* $(0,t_0)$ *and strictly decreasing (increasing) on* (t_0,r)*.*

REMARK 2.2. The first part of Proposition 2.1 is first established by Biernacki and Krzyz [2], while the second part comes from Yang et al. [20, Theorem 2.1]. But we cite the latest version of the second part [24, Lemma 2], where the authors have corrected a bug in the previous version [20, Theorem 2.1].

The second tool is to give a recurrence relation of maclaurin's coefficients for the product of power function and hypergeometric function, which has been proved by Yang in [19] that the coefficients of the function $x\mapsto(1-\theta x)^pF(a,b;c;x)$ satisfy a 3-order recurrence relation for $\theta\in[-1,1]$, and in particular they satisfy a 2-order recurrence relation for $\theta=1$.

As a special case of [19, Corollary 2], we state it in the following proposition.

PROPOSITION 2.3. *For* $a,p\in(0,\infty)$*, we define the function* $r\mapsto\phi_p(a,r)$ *on* $(0,1)$ *by*

$$\phi_p(a,r)=(1-r)^{-p/2}F(a,1;5/2;r)=\sum_{n=0}^{\infty}u_nr^n. \tag{2.6}$$

Then $u_0=1$*,* $u_1=(4a+5p)/10$ *and the coefficients* u_n *satisfy*

$$u_{n+1}=\alpha_nu_n-\beta_nu_{n-1} \tag{2.7}$$

for $n\geqslant 1$*, where*

$$\alpha_n=\frac{8n^2+2(2a+2p+5)n+4a+5p}{2(n+1)(2n+5)},\quad \beta_n=\frac{(2n+p)(2n+2a+p-2)}{2(n+1)(2n+5)}.$$

Moreover, $u_n>0$ *for all* $n\geqslant 0$*.*

Proof. To prove $u_n>0$ for all $n\geqslant 0$, it suffices to express u_n as, by the Cauchy product formula,

$$u_n=\sum_{k=0}^{n}\frac{(p/2)_{n-k}}{(n-k)!}\cdot\frac{(a)_k}{(5/2)_k},$$

which are all positive for $n\geqslant 0$. □

LEMMA 2.4. *For $a \in (0,2)$ and $p \in (3/5,1)$, we define the function*

$$f_p(a,r) = \frac{\phi_p(a,r)}{\varphi(a,r)}$$

on $(0,1)$, where $\phi_p(a,r)$ is defined by (2.6) and $\varphi(a,r) = F(a+1,1;5/2;r)$.

Then we have the following conclusions:

(i) In the case of $a \in (0,2/3]$,

- *if $p \in [4/5,1)$, then $f_p(a,r)$ is strictly increasing on $(0,1)$;*
- *if $p \in (3/5,4/5)$, then there exists $\delta_1 \in (0,1)$ such that $f_p(a,r)$ is strictly decreasing on $(0,\delta_1)$ and is strictly increasing on $(\delta_1,1)$.*

(ii) In the case of $a \in [1,2)$,

- *if $p \in (3/5,4/5]$, then $f_p(a,r)$ is strictly decreasing on $(0,1)$;*
- *if $p \in (4/5,1)$, then there exists $\delta_2 \in (0,1)$ such that $f_p(a,r)$ is strictly increasing on $(0,\delta_2)$ and is strictly decreasing on $(\delta_2,1)$.*

Proof. In terms of power series expansion, by (1.1) and Proposition 2.3, it can be rewritten as

$$f_p(a,r) = \frac{\phi_p(a,r)}{\varphi(a,r)} = \frac{\sum_{n=0}^{\infty} u_n r^n}{\sum_{n=0}^{\infty} v_n r^n},$$

where u_n is defined as in (2.7) with $u_0 = 1$, $u_1 = (4a+5p)/10$ and

$$v_n = \frac{(a+1)_n}{(5/2)_n}.$$

In order to study the monotonicity of $f_p(a,r)$, by Proposition 2.1, it suffices to consider the monotonicity of the sequence $\{u_n/v_n\}_{n=0}^{\infty}$, equivalently, the sign of

$$D_n = u_{n+1} - \frac{v_{n+1}}{v_n} u_n = \hat{\alpha}_n u_n - \beta_n u_{n-1} \tag{2.8}$$

due to (2.7) and $u_n > 0$, where

$$\hat{\alpha}_n = \alpha_n - \frac{v_{n+1}}{v_n} = \frac{4n^2 + 2(2p+1)n + 5p - 4}{2(n+1)(2n+5)}.$$

Moreover, from (2.7) and (2.8), we see that D_n satisfies the following recurrence relation

$$D_{n+1} = \hat{\alpha}_{n+1} D_n + [\hat{\alpha}_{n+1}(\alpha_n - \hat{\alpha}_n) - \beta_{n+1}]\, u_n. \tag{2.9}$$

It is easy to see that α_n, β_n and $\hat{\alpha}_n$ are all positive for $n \geqslant 1$ when $a, p \in (0,\infty)$.

We divide into two cases to complete the proof.

Case 2.4.1. $a\in(0,2/3]$. In this case, for $p\in(3/5,1)$, it follows that

$$\hat{\alpha}_{n+1}(\alpha_n-\hat{\alpha}_n)-\beta_{n+1}=\frac{(2-p)[2-8a+5p+2(1-2a+p)n]}{2(n+2)(2n+5)(2n+7)}$$
$$\geqslant\frac{(2-p)[(4-6a)+(7p-6a)]}{2(n+2)(2n+5)(2n+7)}>0\quad(n\geqslant 1). \tag{2.10}$$

We divide the proof into two cases.

- If $p\in[4/5,1)$, then by (2.6) we see that $D_0=(5p-4)/10\geqslant 0$ and

$$D_1=\frac{-16(2-p)a-10p+35p^2}{280}>\frac{(5p-4)(7p+6)}{280}\geqslant 0.$$

  Assume that $D_n>0$ for $n\geqslant 1$, it follows from (2.9) and (2.10) together with $\hat{\alpha}_{n+1}>0$ and $u_n>0$ that $D_{n+1}>0$ for $n\geqslant 1$. This shows that $D_n\geqslant 0$ for $n\geqslant 0$ by the induction. Hence, by Proposition 2.1(1), we see that $f_p(a,r)$ is strictly increasing on $(0,1)$.

- If $p\in(3/5,4/5)$, then we see that $D_0=(5p-4)/10<0$. First we prove that there exists some $j\geqslant 1$ such that $D_j>0$. Otherwise, $D_n\leqslant 0$ for all $n\geqslant 0$. According to this with Proposition 2.1(1) yields that $\phi_p(a,r)/\varphi(a,r)$ is decreasing on $(0,1)$. Moreover, by (1.2), $\lim_{r\to 1^-}(1-r)^{p/2}F(a+1,1;5/2;r)=0$ if $a\in(0,1/2]$; $\lim_{r\to 1^-}(1-r)^{p/2}F(a+1,1;5/2;r)=\lim_{r\to 1^-}(1-r)^{(p+1)/2-a}F(3/2-a,3/2;5/2;r)=0$ if $a\in(1/2,2/3]$. Combining this with the monotonicity of $\phi_p(a,r)/\varphi(a,r)$, it follows that

$$1=\frac{\phi_p(a,0^+)}{\varphi(a,0^+)}\geqslant\frac{\phi_p(a,1^-)}{\varphi(a,1^-)}=\lim_{r\to 1^-}\frac{F(a,1;5/2;r)}{(1-r)^{p/2}F(a+1,1;5/2;r)}=\infty,$$

  which is a contradiction.

  Suppose that D_{j_*} is the first positive term, that is to say, $D_n\leqslant 0$ for $0\leqslant n\leqslant j_*-1$. As proved in the above, if $D_n\geqslant 0$ for some $n\geqslant 1$, then $D_{n+1}>0$ by (2.9) and (2.10). According to this with $D_{j_*}>0$, it follows that $D_n>0$ for $n\geqslant j_*$ by the induction. In other words, it has been proved that the sequence $\{u_n/v_n\}$ is decreasing for $0\leqslant n\leqslant j_*-1$ and is increasing for $n\geqslant j_*$. On the other hand, we can compute

$$\begin{aligned}H_{\phi_p,\varphi}(r)&=\frac{\phi_p'(a,r)}{\varphi'(a,r)}\varphi(a,r)-\phi_p(a,r)\\&=\frac{5p(1-r)^{-p/2-1}F(a,1;5/2;r)+4a(1-r)^{-p/2}F(a+1,2;7/2;r)}{4(a+1)F(a+2,2;7/2;r)}\\&\quad\times F(a+1,1;5/2;r)-(1-r)^{-p/2}F(a,1;5/2;r)\\&=\frac{1}{(1-r)^{p/2}}\left[\begin{array}{c}\dfrac{5pF(a,1;5/2;r)+4a(1-r)F(a+1,2;7/2;r)}{4(a+1)F(3/2-a,3/2;7/2;r)}\\ \times(1-r)^{a-1/2}F(a+1,1;5/2;r)-F(a,1;5/2;r)\end{array}\right],\end{aligned}$$

which together with (2.1) gives

$$H_{\phi_p,\varphi}(1^-)=\left\{\begin{matrix}\infty, & a\in(0,1/2],\\ \mathrm{sgn}\left(\frac{3(p+1-2a)}{(2a-1)(3-2a)}\right)\infty, & a\in(1/2,2/3]\end{matrix}\right\}=\infty.$$

According to this with the monotonicity of $\{u_n/v_n\}$ and Proposition 2.1(2), we conclude that there exists $\delta_1\in(0,1)$ such that $f_p(a,r)$ is strictly decreasing on $(0,\delta_1)$ and is strictly increasing on $(\delta_1,1)$.

Case 2.4.2. $a\in[1,2)$. In this case, it can be verified, for $p\in(3/5,1)$, that

$$\begin{aligned}\hat{\alpha}_{n+1}(\alpha_n-\hat{\alpha}_n)-\beta_{n+1}&=\frac{(2-p)[2-8a+5p+2(1-2a+p)n]}{2(n+2)(2n+5)(2n+7)}\\ &\leqslant-\frac{(2-p)[4(a-1)+(8a-7p)]}{2(n+2)(2n+5)(2n+7)}<0\quad(n\geqslant 1).\end{aligned}\tag{2.11}$$

We divide into two cases to complete the proof.

- If $p\in(3/5,4/5]$, then it is easy to see that $D_0=(5p-4)/10\leqslant 0$ and

$$D_1=\frac{-16(2-p)a-10p+35p^2}{280}<-\frac{24+(4-5p)(34+35p)}{1400}<0.$$

  Similar to Case 2.4.1, assume that $D_n<0$ for $n\geqslant 1$, it is easy to obtain $D_{n+1}<0$ for $n\geqslant 1$ by (2.9) and (2.11). This enables us to obtain $D_n<0$ for $n\geqslant 0$ by the induction. Hence, $f_p(a,r)$ is strictly decreasing on $(0,1)$ by Proposition 2.1(1).

- If $p\in(4/5,1)$, then we see that $D_0>0$. If $D_n\geqslant 0$ for all $n\geqslant 0$, by Proposition 2.1(1), then $f_p(a,r)$ is increasing on $(0,1)$. According to this with (2.3), it follows that

$$\begin{aligned}1=\frac{\phi_p(a,0^+)}{\varphi(a,0^+)}&\leqslant\frac{\phi_p(a,1^-)}{\varphi(a,1^-)}\\ &\leqslant\begin{cases}\lim_{r\to 1^-}\frac{(1-r)^{a-(p+1)/2}F(a,1;5/2;r)}{F(3/2-a,3/2;5/2;r)}=0, & a\in[1,3/2],\\ \lim_{r\to 1^-}\frac{(1-r)^{1-p/2}F(5/2-a,3/2;5/2;r)}{F(3/2-a,3/2;5/2;r)}=0, & a\in(3/2,2).\end{cases}\end{aligned}$$

  This is obviously impossible. So there exists $D_k<0$ for $k\geqslant 1$ and let D_{k_*} be the first negative term, namely, $D_n\geqslant 0$ for $1\leqslant n\leqslant k_*-1$. As shown in Case 2.4.1, it can be proven that $D_n<0$ for $n\geqslant k_*$ by the induction. That is to say, the sequence $\{u_n/v_n\}$ is increasing for $0\leqslant n\leqslant k_*-1$ and decreasing for $n\geqslant k_*$.

According to this with the fact that

$$H_{\phi_p,\varphi}(r)=\frac{\phi_p'(a,r)}{\varphi'(a,r)}\varphi(a,r)-\phi_p(a,r)$$
$$=\begin{cases}\frac{1}{(1-r)^{p/2}}\left[\begin{array}{c}\frac{5pF(a,1;5/2;r)+4a(1-r)^{3/2-a}F(5/2-a,3/2;7/2;r)}{4(a+1)F(3/2-a,3/2;7/2;r)}\\ \times F(3/2-a,3/2;5/2;r)-F(a,1;5/2;r)\end{array}\right],\\ \frac{1}{(1-r)^{p/2+a-3/2}}\left[\begin{array}{c}\frac{5pF(5/2-a,3/2;5/2;r)+4aF(5/2-a,3/2;7/2;r)}{4(a+1)F(3/2-a,3/2;7/2;r)}\\ \times F(3/2-a,3/2;5/2;r)-F(5/2-a,3/2;5/2;r)\end{array}\right],\end{cases}$$
$$=\left.\begin{cases}-\mathrm{sgn}\left(\frac{3(2a-p-1)}{(2a-1)(3-2a)}\right)\infty, & a\in[1,3/2),\\ -\infty, & a=3/2,\\ -\mathrm{sgn}\left(\frac{3(2-p)\sqrt{\pi}\Gamma(a-3/2)}{4(2a-1)\Gamma(a)}\right)\infty, & a\in(3/2,2)\end{cases}\right\}=-\infty \quad \text{as} \quad (r\to 1^-),$$

Proposition 2.1 leads to the conclusion that there exists $\delta_2\in(0,1)$ such that $f_p(a,r)$ is strictly increasing on $(0,\delta_2)$ and is strictly decreasing on $(\delta_2,1)$.

This completes the proof. □

We will provide two identities for general hypergeometric functions in the following lemma, although we only use its special case in this paper.

LEMMA 2.5. *For an integer $m\geqslant 0$, it holds that*

$$F(a,m+1;c;r)-1=\frac{ar}{c}\sum_{k=0}^{m}F(a+1,k+1;c+1;r),$$
$$1-(1-r)F(a,m+1;c;r)=\frac{r}{c}\left[\begin{array}{l}(c-a)F(a,m+1;c+1;r)\\ \quad -a\sum\limits_{k=0}^{m-1}F(a+1;k+1;c+1;r)\end{array}\right]$$

for $r\in(0,1)$. In particular, we say $\sum\limits_{k=0}^{m-1}(\,\cdot\,)=0$ if $m=0$.

Proof. Let us define $A_k=(n+k)!/n!=[k!(k+1)_n]/n!$ and $B_k=(n+k+1)!/(n+1)!=[k!(k+1)_{n+1}]/(n+1)!$ for $k\geqslant 0$.

We first prove the following identity

$$\frac{B_m}{m!}=\sum_{k=0}^{m}\frac{A_k}{k!}. \tag{2.12}$$

It is easy to see that (2.12) holds for $m=0$ and $m=1$. Assume that the identity (2.12) holds for $m\geqslant 1$, then a simple calculation gives

$$\frac{B_{m+1}-A_{m+1}}{(m+1)!}=\frac{1}{(m+1)!}\left[\frac{(n+m+2)!}{(n+1)!}-\frac{(n+m+1)!}{n!}\right]=\frac{B_m}{m!}=\sum_{k=0}^{m}\frac{A_k}{k!},$$

equivalently,

$$\frac{B_{m+1}}{(m+1)!} = \sum_{k=0}^{m+1} \frac{A_k}{k!},$$

which shows (2.12) by induction. By applying (2.12), it follows that

$$\frac{(m+1)_{n+1}}{n+1} = \frac{n!B_m}{m!} = \sum_{k=0}^{m} \frac{n!A_k}{k!} = \sum_{k=0}^{m} (k+1)_n. \tag{2.13}$$

For $m \geqslant 0$, by (2.13), it can be obtained that

$$\begin{aligned} F(a,m+1;c;r) - 1 &= \sum_{n=1}^{\infty} \frac{(a)_n(m+1)_n}{(c)_n} \frac{r^n}{n!} = \frac{ar}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n(m+1)_{n+1}}{(c+1)_n(n+1)} \frac{r^n}{n!} \\ &= \frac{ar}{c} \sum_{k=0}^{m} \sum_{n=0}^{\infty} \frac{(a+1)_n(k+1)_n}{(c+1)_n} \frac{r^n}{n!} = \frac{ar}{c} \sum_{k=0}^{m} F(a+1,k+1;c+1;r) \end{aligned}$$

and

$$\begin{aligned} 1-(1-r)F(a,m+1;c;r) &= rF(a,m+1;c,r) - \frac{ar}{c} \sum_{k=0}^{m} F(a+1,k+1;c+1;r) \\ &= \frac{r}{c} \begin{bmatrix} cF(a,m+1;c;r) - aF(a+1,m+1;c+1;r) \\ -a \sum_{k=0}^{m-1} F(a+1,k+1;c+1;r) \end{bmatrix} \\ &= \frac{r}{c} \begin{bmatrix} (c-a)F(a,m+1;c+1;r) \\ -a \sum_{k=0}^{m-1} F(a+1;k+1;c+1;r) \end{bmatrix}. \end{aligned}$$

This gives the proof of Lemma 2.5. □

Taking $m=0$ into Lemma 2.5, we obtain the following corollary which had been proved in [25, (3.5) and (3.6)].

COROLLARY 2.6. *For $a,c>0$, the following identities*

$$\begin{aligned} F(a,1;c;r) - 1 &= \frac{ar}{c} F(a+1,1;c+1;r), \\ 1-(1-r)F(a,1;c;r) &= \frac{(c-a)r}{c} F(a,1;c+1;r) \end{aligned}$$

holds for $r \in (0,1)$.

LEMMA 2.7. *For $w \in (0,1)$, the function $r \mapsto M_p(\sqrt{1-r},1;w)/M_q(\sqrt{1-r},1;w)$ is increasing on $(0,1)$ if and only if $p \geqslant q$, and is decreasing on $(0,1)$ if and only if $p \leqslant q$.*

Proof. For $w\in(0,1)$, it is easy to see that

$$\frac{d}{dr}\left[\log\left(\frac{M_p(\sqrt{1-r},1;w)}{M_q(\sqrt{1-r},1;w)}\right)\right]=\frac{w(1-w)\left[(1-r)^{q/2}-(1-r)^{p/2}\right]}{2(1-r)\left[M_p(\sqrt{1-r},1;w)\right]^p\left[M_q(\sqrt{1-r},1;w)\right]^q}\geqslant 0,$$

$$\frac{d}{dr}\left[\log\left(\frac{M_p(\sqrt{1-r},1;w)}{M_0(\sqrt{1-r},1;w)}\right)\right]=\frac{w(1-w)\left[1-(1-r)^{p/2}\right]}{2(1-r)\left[M_p(\sqrt{1-r},1;w)\right]^p}\geqslant 0$$

hold for $x\in(0,1)$ if and only if $p\geqslant q\neq 0$ and $p\geqslant 0$, which gives the desired result of Lemma 2.7. □

3. Proof of Theorem 1.1

Before proving Theorem 1.1, we first show two monotonicity theorems.

THEOREM 3.1. *Let $p\in(0,1)$ and the function $r\mapsto G_p(r)$ be defined by*

$$G_p(r)=\frac{\sqrt{r}}{\arcsin\sqrt{r}\left[\frac{1}{3}(1-r)^{p/2}+\frac{2}{3}\right]^{1/p}}.$$

Then the following statements are true:

(i) If $p\in(0,4/5]$, then $G_p(r)$ is strictly increasing from $(0,1)$ onto $(1,1/\sigma_p)$, and therefore, the double inequality

$$\sigma_p\left[\frac{1}{3}(1-x^2)^{p/2}+\frac{2}{3}\right]^{-1/p}<\frac{\arcsin x}{x}<\left[\frac{1}{3}(1-x^2)^{p/2}+\frac{2}{3}\right]^{-1/p} \tag{3.1}$$

holds for $x\in(0,1)$, where $\sigma_p=\pi(2/3)^{1/p}/2$ is the best constant.

(ii) If $p\in(4/5,1)$, then there exists $r_1\in(0,1)$ such that $G_p(r)$ is decreasing on $(0,r_1)$ and is increasing on $(r_1,1)$ and thereby the inequality

$$\min\{\sigma_p,1\}\left[\frac{1}{3}(1-x^2)^{p/2}+\frac{2}{3}\right]^{-1/p}<\frac{\arcsin x}{x} \tag{3.2}$$

holds for $x\in(0,1)$. In particular, the inequality

$$\left[\frac{1}{3}(1-x^2)^{p/2}+\frac{2}{3}\right]^{-1/p}<\frac{\arcsin x}{x} \tag{3.3}$$

holds for $x\in(0,1)$ when $\tau\leqslant p<1$, where τ is given in Theorem 1.1 (i).

Proof. Logarithmic derivative of $G_p(r)$ together with (2.3) and (2.4) gives rise to

$$\begin{aligned}\frac{G_p'(x)}{G_p(x)}&=\frac{1}{2r}\left[\frac{2(1-r)+(1-r)^{p/2}}{(1-r)(2+(1-r)^{p/2})}-\frac{\sqrt{r}}{\sqrt{1-r}\arcsin\sqrt{r}}\right]\\&=\frac{1}{2r}\left[\frac{2(1-r)+(1-r)^{p/2}}{(1-r)(2+(1-r)^{p/2})}-\frac{\sqrt{1-r}}{(1-r)F(1/2,1/2;3/2;r)}\right]\\&=\frac{g_p(r)}{2r(1-r)[2+(1-r)^{p/2}]F(1,1;3/2;r)},\end{aligned} \tag{3.4}$$

where

$$g_p(r) = (1-r)^{p/2}\Big[F(1,1;3/2;r)-1\Big] - 2\Big[1-(1-r)F(1,1;3/2;r)\Big].$$

By Corollary 2.6, we can simplify $g_p(r)$ as

$$\begin{aligned} g_p(r) &= \frac{2r}{3}(1-r)^{p/2}F(2,1;5/2;r) - \frac{2r}{3}F(1,1;5/2;r) \\ &= \frac{2r}{3}(1-r)^{p/2}F(2,1;5/2;r)\Big[1-f_p(1,r)\Big], \end{aligned} \tag{3.5}$$

where $f_p(a,r)$ is defined in Lemma 2.4.

We divide the proof into two cases:

Case 3.1.1. $p \in (0,4/5]$. For $p \in (3/5,4/5]$, by Lemma 2.4(ii), $f_p(1,r)$ is decreasing on $(0,1)$ and thereby, $f_p(1,r) \leqslant f_p(1,0^+) = 1$ for $r \in (0,1)$. This together with (3.5) yields $g_p(r) \geqslant 0$ for $r \in (0,1)$ and by (3.4), which in turn implies $G_p(r)$ is strictly increasing on $(0,1)$. In particular, $G_{4/5}(r)$ is strictly increasing on $(0,1)$. In this case, we rewrite $G_p(x)$ as

$$G_p(x) = G_{4/5}(x) \cdot \frac{M_{4/5}(\sqrt{1-r},1;1/3)}{M_p(\sqrt{1-r},1;1/3)}$$

which, by Lemma 2.7, is the product of two positive and increasing functions on $(0,1)$ and so is $G_p(x)$. Therefore, we conclude that, $r \in (0,1)$,

$$1 = G_p(0^+) < G_p(r) < G_p(1^-) = \frac{2(3/2)^{1/p}}{\pi},$$

that is, by making a change of variable $x = \sqrt{r}$,

$$\frac{\pi(2/3)^{1/p}}{2}\left[\frac{1}{3}(1-x^2)^{p/2}+\frac{2}{3}\right]^{-1/p} < \frac{\arcsin x}{x} < \left[\frac{1}{3}(1-x^2)^{p/2}+\frac{2}{3}\right]^{-1/p}$$

for $x \in (0,1)$, which gives (3.1).

Case 3.1.2. $p \in (4/5,1)$. In this case, Lemma 2.4(ii) enables us to know that there exists $\hat{r}_1 \in (0,1)$ such that $f_p(1,r)$ is increasing on $(0,\hat{r}_1)$ and is decreasing on $(\hat{r}_1,1)$. This together with $f_p(1,0^+) = 1$ and $f_p(1,1^-) = 0$ implies that there exists $\tilde{r}_1 \in (\hat{r}_1,1)$ such that $f_p(1,r) > 1$ for $r \in (0,\tilde{r}_1)$ and $f_p(1,r) < 1$ for $r \in (\tilde{r}_1,1)$. In other words, $g_p(r) < 0$ for $r \in (0,\tilde{r}_1)$ and $g_p(r) > 0$ for $r \in (\tilde{r}_1,1)$ by (3.5) and so is $G_p'(r)$ by (3.4). Consequently, we obtain

$$G_p(r) < \max\{G_p(0^+),G_p(1^-)\} = \max\left\{1,\frac{2(3/2)^{1/p}}{\pi}\right\}$$

for $r \in (0,1)$, which implies (3.2) by changing the variable $x = \sqrt{r}$. If $\tau \leqslant p < 1$, that is, $2(3/2)^{1/p}/\pi < 2(3/2)^{1/\tau}/\pi = 1$, the inequality $G_p(r) < 1$, namely,

$$\left[\frac{1}{3}(1-x^2)^{p/2}+\frac{2}{3}\right]^{-1/p} < \frac{\arcsin x}{x}$$

holds for $r\in(0,1)$. This completes the proof. □

THEOREM 3.2. *Let $p\in(3/5,\infty)$ and the function $r\mapsto \widetilde{G}_p(r)$ be defined by*

$$\widetilde{G}_p(r)=\frac{\sqrt{r}}{\operatorname{arctanh}\sqrt{r}\left[\frac{1}{3}+\frac{2}{3}(1-r)^{p/2}\right]^{1/p}}.$$

Then the following statements hold true:

(i) If $p\in[4/5,\infty)$, then $\widetilde{G}_p(r)$ is strictly decreasing from $(0,1)$ onto $(0,1)$, and therefore, the inequality

$$\left[\frac{1}{3}+\frac{2}{3}(1-x^2)^{p/2}\right]^{-1/p}<\frac{\operatorname{arctanh}x}{x} \tag{3.6}$$

holds for $x\in(0,1)$.

(ii) If $p\in(3/5,4/5)$, then there exits $r\in(0,1)$ such that $\widetilde{G}_p(r)$ is strictly increasing on $(0,r_2)$ and strictly decreasing on $(r_2,1)$ with $\widetilde{G}_p(0^+)=1$ and $\widetilde{G}_p(1^-)=0$.

Proof. Taking the logarithmic differentiation for $\widetilde{G}_p(r)$ with (2.5) yields

$$\begin{aligned}\frac{\widetilde{G}_p'(r)}{\widetilde{G}_p(r)}&=\frac{1}{2r(1-r)}\left[\frac{1-r+2(1-r)^{p/2}}{1+2(1-r)^{p/2}}-\frac{\sqrt{r}}{\operatorname{arctanh}\sqrt{r}}\right]\\&=\frac{1}{2r(1-r)}\left[\frac{1-r+2(1-r)^{p/2}}{1+2(1-r)^{p/2}}-\frac{1}{F(1/2,1;3/2;r)}\right]\\&=\frac{\tilde{g}_p(r)}{2r(1-r)[1+2(1-r)^{p/2}]F(1/2,1;3/2;r)},\end{aligned} \tag{3.7}$$

where

$$\tilde{g}_p(r)=\left[1-r+2(1-r)^{p/2}\right]F(1/2,1;3/2;r)-1-2(1-r)^{p/2}.$$

Similarly, due to Corollary 2.6, we can rewrite $\widetilde{g}_p(r)$ as

$$\begin{aligned}\tilde{g}_p(r)&=2(1-r)^{p/2}\left[F(1/2,1;3/2;r)-1\right]-\left[1-(1-r)F(1/2,1;3/2;r)\right]\\&=\frac{2r}{3}(1-r)^{p/2}F(3/2,1;5/2;r)-\frac{2r}{3}F(1/2,1;5/2;r)\\&=\frac{2r}{3}(1-r)^{p/2}F(3/2,1;5/2;r)\left[1-f_p(1/2,r)\right],\end{aligned} \tag{3.8}$$

where $f_p(a,r)$ is defined by Lemma 2.4.

We divide the proof into two cases.

Case 3.2.1. $p\in[4/5,\infty)$. For $p\in[4/5,1)$, by Lemma 2.4(i), $f_p(1/2,r)$ is increasing on $(0,1)$ and thereby, $f_p(1/2,r)\geqslant f_p(1/2,0^+)=1$ for $r\in(0,1)$. This together with (3.8) yields $\widetilde{g}_p(r) \leqslant 0$ for $r \in (0,1)$ and by (3.7), which in turn implies $\widetilde{G}_p(r)$ is strictly decreasing on $(0,1)$. In particular, $\widetilde{G}_{4/5}(r)$ is strictly decreasing on $(0,1)$. According to this with Lemma 2.7, it can be seen that for $p \in [4/5, \infty)$,

$$\widetilde{G}_p(x) = \widetilde{G}_{4/5}(x) \cdot \frac{M_{4/5}(\sqrt{1-r}, 1; 2/3)}{M_p(\sqrt{1-r}, 1; 2/3)}$$

is the product of two positive and decreasing functions on $(0,1)$ and so is $\widetilde{G}_p(x)$. Therefore, we conclude that, $r \in (0,1)$,

$$1 = \widetilde{G}_p(0^+) > \widetilde{G}_p(r) > \widetilde{G}_p(1^-) = 0,$$

that is, by making a change of variable $x = \sqrt{r}$,

$$\left[\frac{1}{3} + \frac{2}{3}(1-x^2)^{p/2}\right]^{-1/p} < \frac{\operatorname{arctanh} x}{x}$$

for $x \in (0,1)$, which gives (3.6).

Case 3.2.2. $p \in (3/5, 4/5)$. In this case, Lemma 2.4(i) leads to the conclusion that there exists $\hat{r}_2 \in (0,1)$ such that $f_p(1/2, r)$ is decreasing on $(0, \hat{r}_2)$ and is increasing on $(\hat{r}_2, 1)$. Combining this with $f_p(1/2, 0^+) = 1$ and $f_p(1/2, 1^-) = \infty$, it follows from (3.8) that there exists $\tilde{r}_2 \in (\hat{r}_2, 1)$ such that $\tilde{g}_p(r) > 0$ for $r \in (0, \tilde{r}_2)$ and $\tilde{g}_p(r) < 0$ for $r \in (\tilde{r}_2, 1)$. This, by (3.7), gives the desired monotonicity result of Theorem 3.2(ii). Obviously, $\widetilde{G}_p(0^+) = 1$ and $\widetilde{G}_p(1^-) = 0$. The proof is completed. $\square$

Now we are in a position to prove Theorem 1.1.

Proof. We divide into two cases $0 < x < y$ and $0 < y < x$ to complete the proof.

Case 1.1. $0 < x < y$. Let $r = 1 - (x/y)^2 \in (0,1)$. Then it can be easily seen from (1.2) that

$$SB(x,y) = y\frac{\sqrt{1-(x/y)^2}}{\arcsin\sqrt{1-(x/y)^2}} = y\frac{\sqrt{r}}{\arcsin\sqrt{r}} \tag{3.9}$$

and

$$M_p(x,y;1/3) = y\left[\frac{1}{3}(x/y)^p + \frac{2}{3}\right]^{1/p} = y\left[\frac{1}{3}(1-r)^{p/2} + \frac{2}{3}\right]^{1/p}. \tag{3.10}$$

Necessity. The necessary condition for inequality (1.10), by (3.9) and (3.10), requires to satisfy

$$\begin{aligned}
&\lim_{r\to 0^+} \frac{1}{r^2}\left\{\frac{\arcsin\sqrt{r}}{\sqrt{r}} - \left[\frac{1}{3}(1-r)^{p/2} + \frac{2}{3}\right]^{-1/p}\right\} \\
&= \lim_{r\to 0^+} \frac{1}{r^2}\left\{1 + \frac{r}{6} + \frac{3r^2}{40} + o(r^2) - \left[1 + \frac{r}{6} + \frac{(7-2p)r^2}{72} + o(r^2)\right]\right\} = \frac{5p-4}{180} \leqslant 0,
\end{aligned}$$

which gives $p \leqslant 4/5$ and

$$\lim_{r\to 1^-}\left\{\frac{\arcsin\sqrt{r}}{\sqrt{r}}-\left[\frac{1}{3}(1-r)^{p/2}+\frac{2}{3}\right]^{-1/p}\right\}=\frac{\pi}{2}-\left(\frac{3}{2}\right)^{1/p}\geqslant 0,$$

which implies $p \geqslant \tau = \log_{\pi/2}(3/2)$.

Sufficiency. As mentioned in the introduction, it has been known that $p \mapsto M_p(x,y;1/3)$ is strictly increasing on $(-\infty,\infty)$ for fixed $x,y>0$. Hence, it suffices to prove that the inequality (1.10) holds for $0<x<y$ when $p=4/5$ and $q=\tau$, which follows easily from (3.9) and (3.10) together with the right-side of (3.1) and (3.3).

Case 1.2. $0<y<x$. Let $r=1-(y/x)^2\in(0,1)$. Then it can be easily seen from (1.3) that

$$SB(x,y)=x\frac{\sqrt{1-(y/x)^2}}{\operatorname{arctanh}\sqrt{1-(y/x)^2}}=x\frac{\sqrt{r}}{\operatorname{arctanh}\sqrt{r}} \tag{3.11}$$

and

$$M_p(x,y;1/3)=x\left[\frac{1}{3}+\frac{2}{3}(y/x)^p\right]^{1/p}=x\left[\frac{1}{3}+\frac{2}{3}(1-r)^{p/2}\right]^{1/p}. \tag{3.12}$$

We divide into three cases to complete the proof.

- If $p\in[4/5,\infty)$, then the inequality

$$SB(x,y)<M_p(x,y;1/3) \tag{3.13}$$

  for all $0<y<x$ follows from (3.11) and (3.12) together with (3.6).

- If $p\in(0,4/5)$, then by Taylor series expansion,

$$\widetilde{G}_p(r)=1+\frac{4-5p}{180}r^2+o(r^2)$$

  gives $\widetilde{G}_p(r)>1$ for $r\in(0,\varepsilon_1)$ with sufficient small $\varepsilon_1>0$. This together with (3.11) and (3.12) implies that

$$SB(x,y)>M_p(x,y;1/3) \tag{3.14}$$

  for $\sqrt{1-\varepsilon_1}<y/x<1$. On the other hand, the continuity with $\widetilde{G}_p(1^-)=0$ shows that there exists sufficient small $\varepsilon_2>0$ such that $\widetilde{G}_p(r)<1$ for $r\in(1-\varepsilon_2,1)$, equivalently, by (3.11) and (3.12),

$$SB(x,y)<M_p(x,y;1/3) \tag{3.15}$$

  for $0<y/x<\sqrt{\varepsilon_2}$.

- If $p \in (-\infty, 0]$, then it is easy to obtain

$$M_p(x,y;1/3) \leqslant M_0(x,y;1/3) < SB(x,y) \tag{3.16}$$

for all $0 < y < x$, due to the monotonicity of $p \mapsto M_p(x,y;1/3)$ and $SB(x,y) > x^{1/3}y^{2/3}$.

The desired result of Theorem 1.1(ii) can be obtained from (3.13)–(3.16). □

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