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A MULTIDIMENSIONAL HARDY–HILBERT'S INTEGRAL INEQUALITY INVOLVING ONE DERIVATIVE FUNCTION OF HIGHER–ORDER

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Abstract. By means of the weight functions, the idea of introduced parameters and using the techniques of real analysis, a multidimensional Hardy-Hilbert's integral inequality involving one derivative function of higher-order is obtained. As applications, the equivalent statements of the best possible constant factor in the new inequality related to several parameters are considered. Some corollaries are obtained.

1. Introduction

Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,\ b_n \geqslant 0$, $0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$. We have the following Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [5], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

Setting $f(x),\ g(y)\geqslant 0,\ 0<\int_0^\infty f^p(x)dx<\infty$ and $0<\int_0^\infty g^q(y)dy<\infty$, we have the integral analogue of (1) as follows (cf. [5], Theorem 316):

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

In 2006, by means of the Euler-Maclaorin summation formula and the techniques of real analysis, [15] gave the following extension of (1):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda}<B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right)^{\frac{1}{q}}. \tag{3}$$

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where, $\lambda_1,\ \lambda_2 \in (0,2],\ \lambda_1+\lambda_2=\lambda\in(0,4]$, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible, and

$$B(u,v)=\int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}dt,\ \ u,v>0 \tag{4}$$

is the beta function.

In 2019, by means of the result of (1) and Abel's partial summation formula, Adiyasuren et al. [1] deduced an extension of (1) involving two partial sums. In 2020, Mo et al. [21] gave an extension of (2) involving two upper limit functions. Inequalities (1)–(2) with their extensions played an important role in analysis and its applications (cf. [2, 3, 6, 14, 16, 22, 24–28]).

In 2016–2017, Hong et al. [8, 10] considered some equivalent statements of the extensions of (1) and (2) with a few parameters. Some further results were provided by [4, 7, 9]. In 2023, Hong et al. [13] gave a more accurate half-discrete multidimensional Hilbert-type inequality involving one derivative function of m-order. Some further results were provided by [11–13, 19, 20, 23, 29].

In this paper, following the way of [20], by means of the weight functions, the idea of introduced parameters and the techniques of real analysis. A multidimensional Hardy-Hilbert's integral inequality with the new kernel as $\frac{1}{(x+||y||_\beta^\alpha)^\lambda}$ involving one derivative function of higher-order is obtained. As applications, the equivalent statements of the best possible constant factor in the new inequality related to several parameters are considered. Some corollaries are obtained.

2. Some formula and lemmas

In what follows, we suppose that $m\in\mathbf{N}_0=\{0,1,\cdots\}$, $n\in\mathbf{N}=\{1,2,\cdots\}$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1,\ \alpha,\beta\in\mathbf{R}_+,\ \lambda>0,\ \lambda_1,\lambda_2\in(0,\lambda),\ \widehat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\ \widehat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, $f(x)(\geqslant 0)$ is a differentiable function of m-order, and $f^{(m)}(x)(\geqslant 0)$ is a continuous function unless at finite points in $\mathbf{R}_+=(0,\infty)$, for $m\in\mathbf{N}$,

$$f^{(k-1)}(x)=o(e^{tx})\ \ (t>0;\ x\to\infty),\quad f^{(k-1)}(0^+)=0\ \ (k=1,\cdots,m),$$

and for $f^{(m)}(x),\ g(y)\geqslant 0\ \ (x\in\mathbf{R}_+,\ y\in\mathbf{R}_+^n,\ m\in\mathbf{N}_0)$, we have

$$0<\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx<\infty \quad\text{and}$$
$$0<\int_{\mathbf{R}_+^n}||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n}g^q(y)dy<\infty.$$

If $M>0,\ \psi(u)\ \ (u>0)$ is a nonnegative measurable function, then we have the following transfer formula (cf. [27], (9.1.5)):

$$\int\cdots\int_{\{y\in\mathbf{R}_+^n;\ 0<\sum_{i=1}^n(\frac{y_i}{M})^\beta\leqslant 1\}}\psi\Big(\sum_{i=1}^n\Big(\frac{y_i}{M}\Big)^\beta\Big)dy_1\cdots dy_n$$
$$=\frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}\int_0^1\psi(u)u^{\frac{n}{\beta}-1}du. \tag{5}$$

In particular, (i) in view of $||y||_\beta = M[\sum_{i=1}^n (\frac{y_i}{M})^\beta]^{\frac{1}{\beta}}$, by (5), we have

$$\begin{aligned}
&\int_{\mathbf{R}_+^n} \varphi(||y||_\beta)dy \\
&= \lim_{M\to\infty} \int \cdots \int_{\{y\in\mathbf{R}_+^n;0<\sum_{i=1}^n(\frac{y_i}{M})^\beta\leqslant 1\}} \varphi\Big(M\Big[\sum_{i=1}^n \Big(\frac{y_i}{M}\Big)^\beta\Big]^{\frac{1}{\beta}}\Big)dy_1\cdots dy_n \\
&= \lim_{M\to\infty} \frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}\int_0^1 \varphi(Mu^{\frac{1}{\beta}})u^{\frac{n}{\beta}-1}du \\
&\overset{v=Mu^{\frac{1}{\beta}}}{=} \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\int_0^\infty \varphi(v)v^{n-1}dv; \qquad (6)
\end{aligned}$$

(ii) for $\psi(u) = \varphi(Mu^{\frac{1}{\beta}}) = 0,\ u < (\frac{b}{M})^\beta\ \ (b > 0)$, by (5), we have

$$\begin{aligned}
\int_{\{y\in\mathbf{R}_+^n;||y||_\beta\geqslant b\}} \varphi(||y||_\beta)dy &= \lim_{M\to\infty} \frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}\int_{(\frac{b}{M})^\beta}^1 \varphi(Mu^{\frac{1}{\beta}})u^{\frac{n}{\beta}-1}du \\
&\overset{v=Mu^{\frac{1}{\beta}}}{=} \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\int_b^\infty \varphi(v)v^{n-1}dv. \qquad (7)
\end{aligned}$$

REMARK 1. For $b = 1$, $c \in \mathbf{R}_+$, by (7), we have

$$\int_{\{y\in\mathbf{R}_+^n;||y||_\beta\geqslant 1\}} ||y||_\beta^{-\alpha c-n}dy = \frac{\Gamma^n(\frac{1}{\beta})}{\alpha c\beta^{n-1}\Gamma(\frac{n}{\beta})}. \qquad (8)$$

LEMMA 1. *For $s \in (0,\infty)$, $s_1, s_2 \in (0,s)$, define the following weight functions:*

$$\omega_s(s_1,y) := ||y||_\beta^{\alpha(s-s_1)}\int_0^\infty \frac{x^{s_1-1}}{(x+||y||_\beta^\alpha)^s}dx\ \ (y\in\mathbf{R}_+^n), \qquad (9)$$

$$\varpi_s(s_2,x) := x^{s-s_2}\int_{\mathbf{R}_+^n} \frac{||y||_\beta^{\alpha s_2-n}}{(x+||y||_\beta^\alpha)^s}dy\ \ (x\in\mathbf{R}_+). \qquad (10)$$

We have the following expressions:

$$\omega_s(s_1,y) = B(s_1,s-s_1)\ \ (y\in\mathbf{R}_+^n), \qquad (11)$$

$$\varpi_s(s_2,x) = \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}B(s_2,s-s_2)\ \ (x\in\mathbf{R}_+). \qquad (12)$$

Proof. In (9), setting $u = \frac{x}{||y||_\beta^\alpha}$, we find

$$\begin{aligned}
\omega_s(s_1,y) &= ||y||_\beta^{\alpha(s-s_1)}\int_0^\infty \frac{(u||y||_\beta^\alpha)^{s_1-1}||y||_\beta^\alpha}{(u||y||_\beta^\alpha+||y||_\beta^\alpha)^s}du \\
&= \int_0^\infty \frac{u^{s_1-1}}{(u+1)^s}du = B(s_1,s-s_1),
\end{aligned}$$

and then we have (11).

In (10), setting $\varphi(v) = \frac{v^{s_2-\frac{n}{\alpha}}}{(x+v)^s}$, by (6), we have

$$\begin{aligned}
\varpi_s(s_2,x) &= x^{s-s_2}\int_{\{y\in\mathbf{R}_+^n\}} \frac{(||y||_\beta^\alpha)^{s_2-\frac{n}{\alpha}}}{(x+||y||_\beta^\alpha)^s}dy \\
&= x^{s-s_2}\int_{\{y\in\mathbf{R}_+^n\}} \varphi\left(M^\alpha\left[\sum_{i=1}^n\left(\frac{y_i}{M}\right)^\beta\right]^{\frac{\alpha}{\beta}}\right)dy_1\cdots dy_n \\
&= \lim_{M\to\infty}\frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}x^{s-s_2}\int_0^\infty \varphi(M^\alpha u^{\frac{\alpha}{\beta}})u^{\frac{n}{\beta}-1}du \\
&\overset{v=M^\alpha u^{\frac{\alpha}{\beta}}}{=} \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}x^{s-s_2}\int_0^\infty \frac{v^{s_2-1}}{(x+v)^s}dv \\
&\overset{t=v/x}{=} \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\int_0^\infty \frac{t^{s_2-1}}{(1+t)^s}dt \\
&= \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}B(s_2,s-s_2),
\end{aligned}$$

and then we have (12).

The lemma is proved. □

LEMMA 2. *For $t>0$, we have the following expression:*

$$\int_0^\infty e^{-tx}f(x)dx = t^{-m}\int_0^\infty e^{-tx}f^{(m)}(x)dx. \tag{13}$$

Proof. For $m=0$, (13) is naturally valid; for $m\in\mathbf{N}$, since $f(0^+)=0,\ f(x)=o(e^{tx})\ \ (t>0;\ x\to\infty)$, we find

$$\begin{aligned}
\int_0^\infty e^{-tx}f'(x)dx &= \int_0^\infty e^{-tx}df(x) = e^{-tx}f(x)|_0^\infty \\
-\int_0^\infty f(x)de^{-tx} &= \lim_{x\to\infty} e^{-tx}f(x) + t\int_0^\infty e^{-tx}f(x)dx \\
&= t\int_0^\infty e^{-tx}f(x)dx.
\end{aligned}$$

Inductively, for $f^{(k)}(0^+)=0,\ f^{(k)}(x)=o(e^{tx})\ \ (t>0,\ k=1,\cdots m;\ x\to\infty)$, we still have

$$\int_0^\infty e^{-tx}f(x)dx = t^{-1}\int_0^\infty e^{-tx}f'(x)dx = \cdots = t^{-m}\int_0^\infty e^{-tx}f^{(m)}(x)dx.$$

Hence, (13) is valid.

The lemma is proved. □

LEMMA 3. *We have the following inequality:*

$$I_\lambda := \int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty \frac{f^{(m)}(x)g(y)}{(x+||y||_\beta^\alpha)^\lambda}dxdy$$
$$< \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$
$$\times\left[\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\int_{\{y\in\mathbf{R}_+^n\}}||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n}g^q(y)dy\right]^{\frac{1}{q}}. \tag{14}$$

In particular, for $\lambda=\lambda_1+\lambda_2$, *we have*

$$0<\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx<\infty,$$
$$0<\int_{\{y\in\mathbf{R}_+^n\}}||y||_\beta^{q(n-\alpha\lambda_2)-n}g^q(y)dy<\infty,$$

and the following inequality:

$$I_\lambda := \int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty \frac{f^{(m)}(x)g(y)}{(x+||y||_\beta^\alpha)^\lambda}dxdy$$
$$< \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B(\lambda_1,\lambda_2)$$
$$\times\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\int_{\{y\in\mathbf{R}_+^n\}}||y||_\beta^{q(n-\alpha\lambda_2)-n}g^q(y)dy\right]^{\frac{1}{q}}. \tag{15}$$

Proof. By Hölder's inequality (cf. [17]), we have

$$I_\lambda = \int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty \frac{1}{(x+||y||_\beta^\alpha)^\lambda}\left[\frac{x^{(1-\lambda_1)/q}f^{(m)}(x)}{||y||_\beta^{(n-\alpha\lambda_2)/p}}\right]\left[\frac{||y||_\beta^{(n-\alpha\lambda_2)/p}g(y)}{x^{(1-\lambda_1)/q}}\right]dxdy$$
$$\leqslant\left\{\int_0^\infty\int_{\{y\in\mathbf{R}_+^n\}}\frac{1}{(x+||y||_\beta^\alpha)^\lambda}\frac{x^{(1-\lambda_1)(p-1)}(f^{(m)}(x))^p}{||y||_\beta^{n-\alpha\lambda_2}}dydx\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty\frac{1}{(x+||y||_\beta^\alpha)^\lambda}\frac{||y||_\beta^{(n-\alpha\lambda_2)(q-1)}g^q(y)}{x^{1-\lambda_1}}dxdy\right\}^{\frac{1}{q}}$$
$$=\left\{\int_0^\infty\left[x^{\lambda-\lambda_2}\int_{\{y\in\mathbf{R}_+^n\}}\frac{||y||_\beta^{\alpha\lambda_2-n}dy}{(x+||y||_\beta^\alpha)^\lambda}\right]x^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1]}(f^{(m)}(x))^p dx\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{\{y\in\mathbf{R}_+^n\}}\left[||y||_\beta^{\alpha(\lambda-\lambda_2)}\int_0^\infty\frac{x^{\lambda_1-1}dx}{(x+||y||_\beta^\alpha)^\lambda}\right]||y||_\beta^{q[n-\alpha(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-n}g^q(y)dy\right\}^{\frac{1}{q}}$$

$$= \left[\int_0^\infty \varpi_\lambda(\lambda_2,x)x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}$$
$$\times \left[\int_{\{y\in\mathbf{R}_+^n\}} \omega_\lambda(\lambda_1,y)||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n} g^q(y)dy\right]^{\frac{1}{q}}. \tag{16}$$

If (16) keeps the form of equality, then there exist constants A and B, such that they are not both zero, satisfying

$$A\frac{x^{(1-\lambda_1)(p-1)}(f^{(m)}(x))^p}{||y||_\beta^{n-\alpha\lambda_2}} = B\frac{||y||_\beta^{(n-\alpha\lambda_2)(q-1)}g^q(y)}{x^{1-\lambda_1}} \quad \text{a.e. in} \quad \mathbf{R}_+\times\mathbf{R}_+^n.$$

Assuming that $A\neq 0,$ there exists a $y\in\mathbf{R}_+^n,$ such that

$$x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p = \left[\frac{B}{A}||y||_\beta^{q(n-\alpha\lambda_2)}g^q(y)\right]x^{-1-(\lambda-\lambda_1-\lambda_2)} \quad \text{a.e. in} \quad \mathbf{R}_+.$$

Since for any $\lambda-\lambda_1-\lambda_2\in\mathbf{R},$ $\int_0^\infty x^{-1-(\lambda-\lambda_1-\lambda_2)}dx=\infty$, the above expression contradicts the fact that $0<\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx<\infty$.

Therefore, by (11) and (12) $(s=\lambda,\ s_1=\lambda_1,\ s_2=\lambda_2)$, we have (14) and then for $\lambda=\lambda_1+\lambda_2$, we have (15).

The lemma is proved. □

3. Main results

THEOREM 1. *We have the following multidimensional Hardy-Hilbert's integral inequality involving one derivative function of higher-order:*

$$I := \int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty \frac{f(x)g(y)}{(x+||y||_\beta^\alpha)^{\lambda+m}}dxdy$$
$$< \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$
$$\times\left[\int_0^\infty x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\int_{\{y\in\mathbf{R}_+^n\}}||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n}g^q(y)dy\right]^{\frac{1}{q}}. \tag{17}$$

In particular, for $\lambda=\lambda_1+\lambda_2$, by (17), *we have*

$$I := \int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty \frac{f(x)g(y)}{(x+||y||_\beta^\alpha)^{\lambda+m}}dxdy$$
$$< \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B(\lambda_1,\lambda_2)$$
$$\times\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\int_{\{y\in\mathbf{R}_+^n\}}||y||_\beta^{q(n-\alpha\lambda_2)-n}g^q(y)dy\right]^{\frac{1}{q}}. \tag{18}$$

Proof. By the following expression of the Gamma function:

$$\frac{1}{(x+||y||_\beta^\alpha)^{\lambda+m}} = \frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}e^{-(x+||y||_\beta^\alpha)t}dt \quad (\lambda, x>0), \tag{19}$$

(13) and Fubini theorem (cf. [18]), we have

$$\begin{aligned}
I &= \frac{1}{\Gamma(\lambda+m)}\int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty f(x)g(y)\left[\int_0^\infty t^{\lambda+m-1}e^{-(x+||y||_\beta^\alpha)t}dt\right]dxdy \\
&= \frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}\Big(\int_0^\infty e^{-xt}f(x)dx\Big)\Big(\int_{\{y\in\mathbf{R}_+^n\}} e^{-||y||_\beta^\alpha t}g(y)dy\Big)dt \\
&= \frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}\Big(t^{-m}\int_0^\infty e^{-xt}f^{(m)}(x)dx\Big)\Big(\int_{\{y\in\mathbf{R}_+^n\}} e^{-||y||_\beta^\alpha t}g(y)dy\Big)dt \\
&= \frac{1}{\Gamma(\lambda+m)}\int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty f^{(m)}(x)g(y)\left[\int_0^\infty t^{\lambda-1}e^{-(x+||y||_\beta^\alpha)t}dt\right]dxdy \\
&= \frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty \frac{f^{(m)}(x)g(y)}{(x+||y||_\beta^\alpha)^\lambda}dxdy.
\end{aligned}$$

Then by (14), we have (17). For $\lambda=\lambda_1+\lambda_2$ in (17), we have (18). The theorem is proved. □

THEOREM 2. *If $\lambda=\lambda_1+\lambda_2$, then the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

in (17) *is the best possible.*

Proof. For $\lambda=\lambda_1+\lambda_2$, by (17), we have (18). We need to prove that the constant factor in (18) is the best possible. For any $0<\varepsilon<p\lambda_1$, we set

$$\widetilde{f}^{(m)}(x) := \begin{cases} 0, & 0<x<1, \\ \prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})x^{\lambda_1-\frac{\varepsilon}{p}-1}, & x\geqslant 1, \end{cases}$$

$$\widetilde{g}(y) := \begin{cases} 0, & ||y||_\beta^\alpha<1 \\ ||y||_\beta^{\alpha(\lambda_2-\frac{\varepsilon}{q})-n}, & ||y||_\beta^\alpha\geqslant 1, \end{cases}$$

$$\widetilde{f}(x) := \begin{cases} 0, & 0<x<1, \\ \prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})\int_1^x(\int_1^{t_m}\cdots\int_1^{t_2} t_1^{\lambda_1-\frac{\varepsilon}{p}-1}dt_1\cdots dt_{m-1})dt_m, & x\geqslant 1, \end{cases}$$

namely,

$$\widetilde{f}(x)=\begin{cases}0, & 0<x<1,\\ x^{\lambda_1+m-\frac{\varepsilon}{p}-1}-p_{m-1}(x), & x\geqslant 1,\end{cases}$$

where, for $m\in\mathbf{N}$, $p_{m-1}(x)$ is indicated as a nonnegative polynomial of $(m-1)$-order satisfying $p_{m-1}(1)=1$. We denote $p_{-1}(1):=0$, $\prod_{i=0}^{-1}(a+i):=1$. Then the above expression satisfies for $m\in\mathbf{N}_0$.

If there exists a constant M, satisfying

$$0<M\leqslant\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2) \tag{20}$$

such that (18) is valid when we replace the constant factor in (18) by M, then in particular, by (18) and (8) $(c=\varepsilon)$, we have

$$\begin{aligned}\widetilde{I}&:=\int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty\frac{\widetilde{f}(x)\widetilde{g}(y)}{(x+||y||_\beta^\alpha)^{\lambda+m}}dxdy\\
&<M\left[\int_0^\infty x^{p(1-\lambda_1)-1}(\widetilde{f}^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\int_{\{y\in\mathbf{R}_+^n\}}||y||_\beta^{q(n-\alpha\lambda_2)-n}\widetilde{g}^q(y)dy\right]^{\frac{1}{q}}\\
&=M\prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\varepsilon}{p}\right)\left(\int_1^\infty x^{-1-\varepsilon}dx\right)^{\frac{1}{p}}\left(\int_{\{y\in\mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}}||y||_\beta^{-\alpha\varepsilon-n}dy\right)^{\frac{1}{q}}\\
&=\frac{M}{\varepsilon}\prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\varepsilon}{p}\right)\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{q}}.\end{aligned}$$

By (8), (9) and (11) $(s=\lambda+m>0,\ s_1=\lambda_1-\frac{\varepsilon}{p}+m\in(0,\lambda+m))$, we have

$$\begin{aligned}\widetilde{I}&=\int_{\{y\in\mathbf{R}_+^n\}}\int_0^\infty\frac{\widetilde{f}(x)\widetilde{g}(y)}{(x+||y||_\beta^\alpha)^{\lambda+m}}dxdy\\
&=\int_{\{y\in\mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}}\int_1^\infty\frac{(x^{\lambda_1+m-\frac{\varepsilon}{p}-1}-p_{m-1}(x))||y||_\beta^{\alpha(\lambda_2-\frac{\varepsilon}{q})-n}}{(x+||y||_\beta^\alpha)^{\lambda+m}}dxdy\\
&=\int_{\{y\in\mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}}||y||_\beta^{-\alpha\varepsilon-n}\left[||y||_\beta^{\alpha(\lambda_2+\frac{\varepsilon}{p})}\int_0^\infty\frac{x^{\lambda_1+m-\frac{\varepsilon}{p}-1}}{(x+||y||_\beta^\alpha)^{\lambda+m}}dx\right]dy\\
&\quad-\int_{\{y\in\mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}}||y||_\beta^{-\alpha\varepsilon-n}\left[||y||_\beta^{\alpha(\lambda_2+\frac{\varepsilon}{p})}\int_0^1\frac{x^{\lambda_1+m-\frac{\varepsilon}{p}-1}}{(x+||y||_\beta^\alpha)^{\lambda+m}}dx\right]dy\\
&\quad-\int_{\{y\in\mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}}\int_1^\infty\frac{p_{m-1}(x)||y||_\beta^{\alpha(\lambda_2-\frac{\varepsilon}{q})-n}}{(x+||y||_\beta^\alpha)^{\lambda+m}}dxdy\end{aligned}$$

$$
\begin{aligned}
&\geqslant \int_{\{y\in \mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}} ||y||_\beta^{-\alpha\varepsilon-n}\omega_{\lambda+m}\left(\lambda_1-\frac{\varepsilon}{p}+m,y\right)dy \\
&\quad -\int_{\{y\in \mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}} ||y||_\beta^{-\alpha\varepsilon-n}\left[||y||_\beta^{\alpha(\lambda_2+\frac{\varepsilon}{p})}\int_0^1 \frac{x^{\lambda_1+m-\frac{\varepsilon}{p}-1}}{(||y||_\beta^\alpha)^{\lambda+m}}dx\right]dy \\
&\quad -\int_{\{y\in \mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}} \frac{||y||_\beta^{\alpha(\lambda_2-\frac{\varepsilon}{q})-n}}{(||y||_\beta^\alpha)^{\lambda_2+(\lambda_1/2)}}dy\int_1^\infty \frac{p_{m-1}(x)}{x^{(\lambda_1/2)+m}}dx \\
&= B\left(\lambda_1-\frac{\varepsilon}{p}+m,\lambda_2+\frac{\varepsilon}{p}\right)\int_{\{y\in \mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}} |y||_\beta^{-\alpha\varepsilon-n}dy \\
&\quad -\left[\frac{1}{\lambda_1+m-\frac{\varepsilon}{p}}\int_{\{y\in \mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}} ||y||_\beta^{-\alpha(\lambda_1+\frac{\varepsilon}{q}+m)-n}dy\right. \\
&\quad \left.+\int_{\{y\in \mathbf{R}_+^n;||y||_\beta^\alpha\geqslant 1\}} ||y||_\beta^{-\alpha(+\frac{\varepsilon}{q})-n}dy\int_1^\infty O\left(\frac{1}{x^{(\lambda_1/2)+1}}\right)dx\right] \\
&= \frac{B(\lambda_1-\frac{\varepsilon}{p}+m,\lambda_2+\frac{\varepsilon}{p})\Gamma^n(\frac{1}{\beta})}{\varepsilon\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}-O_1(1).
\end{aligned}
$$

Hence, in view of the above results, we have the following inequality

$$
\begin{aligned}
&\frac{B(\lambda_1-\frac{\varepsilon}{p}+m,\lambda_2+\frac{\varepsilon}{p})\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}-\varepsilon O_1(1) \\
&\quad < \varepsilon\widetilde{I} < M\prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\varepsilon}{p}\right)\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{q}}.
\end{aligned}
$$

For $\varepsilon\to 0^+$, in view of the continuity of the beta function, we have

$$
\frac{B(\lambda_1+m,\lambda_2)\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\leqslant M\prod_{i=0}^{m-1}(\lambda_1+i)\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{q}},
$$

namely,

$$
\begin{aligned}
&\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2) \\
&= \frac{B(\lambda_1+m,\lambda_2)}{\prod_{i=0}^{m-1}(\lambda_1+i)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}\leqslant M.
\end{aligned}
$$

By (20), it follows that

$$
M=\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)
$$

is the best constant factor of (18) (namely, for $\lambda=\lambda_1+\lambda_2$ in (17)).

The theorem is proved. □

THEOREM 3. *If the same constant factor in* (17) *is the best possible, then we have* $\lambda=\lambda_1+\lambda_2$.

Proof. For $\widehat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, we find $\widehat{\lambda}_1+\widehat{\lambda}_2=\lambda$, $0<\widehat{\lambda}_1,\ \widehat{\lambda}_2<\lambda$.

By Hölder's inequality (cf. [19]), we obtain

$$
\begin{aligned}
B(\widehat{\lambda}_1,\widehat{\lambda}_2) &= \int_0^\infty \frac{u^{\widehat{\lambda}_1-1}}{(1+u)^\lambda}du = \int_0^\infty \frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}}{(1+u)^\lambda}du \\
&= \int_0^\infty \frac{1}{(1+u)^\lambda}(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du \\
&\leqslant \left[\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^\lambda}du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda}du\right]^{\frac{1}{q}} \\
&= B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1).
\end{aligned}
\tag{21}
$$

Since the constant factor

$$
\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)
$$

in (17) is the best possible. Compare with the constant factors in (17) and (18) (by substitution of $\lambda_1=\widehat{\lambda}_1$, $\lambda_2=\widehat{\lambda}_2$), we have

$$
\begin{aligned}
&\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\
\leqslant\ &\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B(\widehat{\lambda}_1,\widehat{\lambda}_2),
\end{aligned}
$$

namely, $B(\widehat{\lambda}_1,\widehat{\lambda}_2)\geqslant B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$. Hence, (21) keeps the form of equality. The necessary and sufficient condition for taking an equal sign is: there exist constants A and B, such that they are not both zero, and (cf. [19]) $Au^{\lambda-\lambda_2-1}=Bu^{\lambda_1-1}$ a.e. in $\mathbf{R}_+$. Assuming that $A\neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1}=\frac{B}{A}$ a.e. in $\mathbf{R}_+$. It follows that $\lambda-\lambda_2-\lambda_1=0$, and then $\lambda=\lambda_1+\lambda_2$.

The theorem is proved. □

COROLLARY 1. *For $n=1$ in (17), we have the following inequality:*

$$\int_{\mathbf{0}}^{\infty}\int_{0}^{\infty}\frac{f(x)g(y)}{(x+y^{\alpha})^{\lambda+m}}dxdy$$
$$<\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$
$$\times\left[\int_{0}^{\infty}x^{p(1-\widehat{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbf{0}}^{\infty}y^{q(1-\alpha\widehat{\lambda}_2)-1}g^q(y)dy\right]^{\frac{1}{q}}. \tag{22}$$

In particular, for $\lambda=\lambda_1+\lambda_2$, by (22), we have

$$\int_{\mathbf{0}}^{\infty}\int_{0}^{\infty}\frac{f(x)g(y)}{(x+y^{\alpha})^{\lambda+m}}dxdy$$
$$<\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)\left[\int_{0}^{\infty}x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_{\mathbf{0}}^{\infty}y^{q(1-\alpha\lambda_2)-1}g^q(y)dy\right]^{\frac{1}{q}}. \tag{23}$$

where the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)$ is the best possible.

COROLLARY 2. *If $\lambda=\lambda_1+\lambda_2$, then the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

in (23) is the best possible. On the other hand, if the same constant factor in (23) is the best possible, then we have $\lambda=\lambda_1+\lambda_2$.

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REFERENCES

[1] V. ADIYASUREN, T. BATBOLD AND E. A. AZAR, *A new discrete Hilbert-type inequality involving partial sums*, Journal of Inequalities and Applications, (2019) **2019**: 127.

[2] V. ADIYASUREN, T. BATBOLD AND M. KRNIĆ, *Hilbert-type inequalities involving differential operators, the best constants and applications*, Math. Inequal. Appl. **18** (2015), 111–124.

[3] L. E. AZAR, *The connection between Hilbert and Hardy inequalities*, Journal of Inequalities and Applications (2013) **2013**: 452.

[4] Q. CHEN, B. HE, Y. HONG AND Z. LI, *Equivalent parameter conditions for the validity of half-discrete Hilbert-type multiple integral inequality with generalized homogeneous kernel*, J. Funct. Spaces, (2020), **2020**, 7414861.

[5] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[6] B. HE, *A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor*, Journal of Mathematical Analysis and Applications **431** (2015), 902–990.

[7] B. HE, Y. HONG AND Z. LI, *Conditions for the validity of a class of optimal Hilbert type multiple integral inequalities with non-homogeneous*, Journal of Inequalities and Applications, (2021), **2021**: 64.

[8] Y. HONG, *On the structure character of Hilbert's type integral inequality with homogeneous kernel and applications*, J. Jilin. Univ. Sci. Ed. **55** (2), (2017), 189–194.

[9] Y. HONG, Q. L. HUANG, B. C. YANG AND J. Q. LIAO, *The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications*, Journal of Inequalities and Applications, (2017), **2017**: 316.

[10] Y. HONG AND Y. M. WEN, *A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor*, Ann. Math. **37A** (3), (2016), 329–336.

[11] Y. HONG, Y. R. ZHONG AND B. C. YANG, *A more accurate half-discrete multidimensional Hilbert-type inequality involving one multiple upper limit function*, Axioms, (2023), **12**, 211, https://doi.org/10.3390/axioms12020211.

[12] Y. HONG, Y. R. ZHONG AND B. C. YANG, *A more accurate half-discrete multidimensional Hilbert-type inequality involving one multiple upper limit function*, Journal of inequalities and Applications, (2023), **2023**: 74.

[13] Y. HONG, Y. R. ZHONG AND B. C. YANG, *On a more accurate half-discrete multidimensional Hilbert-type inequality involving one derivative function of m-order*, Journal of Inequalities and Applications, (2023), **2023**: 74.

[14] Q. L. HUANG, *A new extension of Hardy-Hilbert-type inequality*, Journal of Inequalities and Applications (2015), **2015**: 397.

[15] M. KRNIĆ AND J. PEČARIĆ, *Extension of Hilbert's inequality*, J. Math. Anal. Appl. **324** (1) (2006), 150–160.

[16] M. KRNIĆ AND J. PEČARIĆ, *General Hilbert's and Hardy's inequalities*, Mathematical Inequalities & Applications **8** (1) (2005), 29–51.

[17] J. C. KUANG, *Applied Inequalitie*, Shandong Science and Technology Press, Jinan, 2004.

[18] J. C. KUANG, *Real and Functional Analysis (Continuation)*, vol. 2, Higher Education Press, Beijing, 2015.

[19] J. Q. LIAO AND B. C. YANG, *A new reverse half-discrete Hilbert-type inequality with one partial sum involving one derivative function of higher order*, Open Mathematics, (2023), **21**: 20230139.

[20] R. C. LUO, B. C. YANG AND L. P. HE, *A Hardy-Hilbert-type integral inequality involving two multiple upper-limit functions*, Journal of Inequalities and Applications, (2023), **2023**: 19.

[21] H. M. MO AND B. C. YANG, *On a new Hilbert-type integral involving the upper limit functions*, Journal of Inequalities and Applications, (2020), **2020**: 5.

[22] I. PERIĆ AND P. VUKOVIĆ, *Multiple Hilbert's type inequalities with a homogeneous kernel*, Banach Journal of Mathematical Analysis **5** (2) (2011), 33–43.

[23] A. Z. WANG AND B. C. YANG, *A new half-discrete Hilbert-type inequality involving one multiple upper limit function and one partial sums*, Annals Mathematics **45A** (1) (2024), 25–38.

[24] Z. T. XIE, Z. ZENG AND Y. F. SUN, *A new Hilbert-type inequality with the homogeneous kernel of degree* -2, Advances and Applications in Mathematical Sciences **12** (7) (2013), 391–401.

[25] D. M. XIN, *A Hilbert-type integral inequality with the homogeneous kernel of zero degree*, Mathematical Theory and Applications **30** (2) (2010), 70–74.

[26] J. S. XU, *Hardy-Hilbert's inequalities with two parameters*, Adv. Math. **36** (2) (2007), 63–76.

[27] B. C. YANG, *The Norm of Operator and Hilbert-type Inequality*, Science Press, Beijing, 2009.

[28] Z. ZENG, K. RAJA RAMA GANDHI AND Z. T. XIE, *A new Hilbert-type inequality with the homogeneous kernel of degree* -2 *and with the integral*, Bulletin of Mathematical Sciences and Applications **3** (1) (2014), 11–20.

[29] Z. H. ZENG AND B. C. YANG, *A Hardy-Hilbert-type integral inequality involving the derivative functions of n-order*, Journal of South China Normal University (Natural Science Editon) **56** (4) (2024), 123–128.

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