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HERMITE–HADAMARD TYPE INTEGRAL INEQUALITIES INVOLVING FRACTIONAL INTEGRALS FOR THE s–CONVEX FUNCTIONS IN THE FOURTH SENSE

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Abstract. In this paper, the Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals for s-convex functions in the fourth sense are obtained. By means of certain integral identities for fractional integrals, some inequality relations among fractional integrals of the fourth sense s-convex functions and the functions whose derivatives are s-convex in the fourth sense are set forth. Moreover, based on the obtained results, some inequality relations among special functions including beta, incomplete beta functions, and means including logarithmic and arithmetic mean are given as applications.

1. Introduction

Inequalities associated with notions of calculus are indispensable tools in characterizing convex functions. Certain properties of convex functions are expressed via derivatives and integrals. For example, Hermite-Hadamard, Fejer, and Ostrowski inequalities characterize convex functions by associating them with integrals. The progress on generalizations and extensions in calculus requires the reinvestigation of these inequalities in light of these developments, which cover a vast area in the literature. Even a quick scan of the special issues of related journals on inequalities in quantum and fractional calculus can give an idea about the significance of the topic. On the other hand, similar advances in the generalization of convex functions have been ongoing. Novel abstract convexity types are introduced day by day, e.g., s-convexities such as s-convexity in the first sense, s-convexity in the second sense, s-convexity in the third sense and s-convexity in the fourth sense [3, 8, 14, 19, 24]. These types of convexity are more general, complementary classes obtained by adding a parameter to the definition of classical convexity, as will be explained below.

Suppose that $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}$. f is said to be a convex function on A if

$$f(\lambda x + \mu y) \leqslant \lambda f(x) + \mu f(y) \tag{1.1}$$

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for all $x, y \in A$ and all $\lambda, \mu \geqslant 0$ such that $\lambda + \mu = 1$.

In 1961, Orlicz in [19] generalizes this concept by adding a parameter s in $(0,1]$, replacing the condition $\lambda + \mu = 1$ with $\lambda^s + \mu^s = 1$, and replacing (1.1) with

$$f(\lambda x + \mu y) \leqslant \lambda^s f(x) + \mu^s f(y) \tag{1.2}$$

i.e., the concept of s-convex function in the first sense.

In 1978, Breckner in [3] set forth a similar concept by using the parameter s in $(0,1]$ and just replacing (1.1) with

$$f(\lambda x + \mu y) \leqslant \lambda^s f(x) + \mu^s f(y)$$

in the definition of a convex function, i.e., the concept of s-convex function in the second sense.

Recently, Kemali et al. in [14] defined the s-convex function in the third sense by replacing s in (1.2) with $\frac{1}{s}$ under the same conditions as the definition of s-convex function in the first sense. Similarly, Eken et al. in [8] defined the s-convex function in the fourth sense by replacing s in (1.2) with $\frac{1}{s}$ under the conditions of the definition of the s-convex function in the second sense.

As can be seen, s-convexity in the third sense is a complement to s-convexity in the first sense, in the following way: while $s \in (0,1]$, $\frac{1}{s}$ falls within $[1,\infty)$. Therefore, the parameter used as an exponent spans the interval $(0,\infty)$. The same situation applies to s-convexity in the second sense and s-convexity in the fourth sense. It is evident that in every case, $s = 1$ corresponds to classical convexity.

s-convexity is especially utilized in the generation of Mandelbrot and Julia sets to enhance the iterative methods applied in fractal construction. It provides a more generalized framework for fixed-point theory, allowing the extension of classical iteration schemes (such as Picard and Ishikawa) by incorporating s-convex combinations. This enables the development of new escape criteria and fractal structures, leading to a broader class of complex fractals. Studies have shown that applying s-convexity in iterative processes can lead to novel variations of Mandelbrot and Julia sets, offering deeper insights into their geometric and dynamic properties [16].

The properties of these functions, based on the aforementioned inequalities and expressed via the ordinary integral, especially the Hermite-Hadamard inequality, have been studied by many authors [1, 5–7, 13, 22, 23]. Studies on this inequality in terms of fractional integrals are of great interest to researchers [2, 4, 10, 11, 18, 21]. Although there are many types of fractional integrals, Riemann-Liouville fractional integrals are the oldest and most commonly practiced ones [12, 15]. A large number of researchers deal with the reexpression of the Hermite-Hadamard inequality in terms of fractional integrals. In this study, we established the Hermite-Hadamard inequality via Riemann-Liouville fractional integrals for the s-convex functions in the fourth sense. Additionally, using some integral identities expressed for Riemann-Liouville fractional integrals, we set relations between different orders of Riemann-Liouville fractional integrals of s-convex functions in the fourth sense. In the last part of the paper, to exemplify the applications of the results, we present various propositions that expose the inequality relations among special means, including the generalized logarithmic mean, and special functions, including the beta and incomplete beta functions.

2. Preliminaries

Let us express some notations, essential concepts, and auxiliary statements in this section.

DEFINITION 2.1. [8] Let U be a convex subset of real numbers $\mathbb{R}$ and $s \in (0,1]$. A function $f : U \to \mathbb{R}$ is called s-convex function in the fourth sense if

$$f(tx+(1-t)y) \leqslant t^{\frac{1}{s}} f(x) + (1-t)^{\frac{1}{s}} f(y)$$

for all $x, y \in U$ and $t \in [0,1]$.

In [8], it is shown that if f is fourth sense s-convex, then the image set of f is contained in $(-\infty, 0]$. Eken in [9] shows that $f(x) = -x^{\frac{1}{s}}$ on $[0,\infty)$ is a fourth sense s-convex function. For the sake of clarity, s-convex function in the fourth sense is referred to hereafter as s-convex function.

THEOREM 2.2. [13] *Let $f : \mathbb{R} \to \mathbb{R}_-$ be an s-convex function and integrable on $[a,b] \subseteq \mathbb{R}$. Then*

$$2^{\frac{1}{s}-1} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{s[f(a)+f(b)]}{s+1}.$$

Let us recall some special functions, some of which are used in the notions of fractional integrals and the others are encountered results and applications. Gamma function, beta function, incomplete beta function are given respectively, as follows

$$\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty e^{-x} x^{\alpha-1} dx \quad (\alpha > 0), \\
\mathrm{Beta}(\alpha,\beta) &= \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad (\alpha,\beta > 0) \\
B_\theta(\alpha,\beta) &= \int_0^\theta x^{\alpha-1} (1-x)^{\beta-1} dx \quad (0 < \theta < 1, \quad \alpha,\beta > 0).
\end{aligned}$$

For negative values of α, β, incomplete beta function can be calculated through analytic continuation. Also, hypergeometric function which is defined by

$${}_2F_1(\beta,\gamma;\eta;z) = 1 + \frac{\beta\gamma}{\eta} z + \frac{\beta(\beta+1)\gamma(\gamma+1)}{\eta(\eta+1)} \frac{z^2}{2!} + \cdots + \frac{(\beta)_n(\gamma)_n}{(\eta)_n} \frac{z^n}{n!} + \cdots$$

on $|z| \leqslant 1$ where $(\cdot)_n$ is the Pochhammer symbol, i.e.

$$(\mu)_n = \mu(\mu+1)(\mu+2)\cdots(\mu+n-1)$$

where $\beta, \gamma, \eta \in \mathbb{R}$ such that η is not zero or negative integer. It is defined on $[-1,1]$ provided that $\beta + \gamma - \eta < 0$.

DEFINITION 2.3. [15] Let $f:[a,b]\to\mathbb{R}$ where $a,b\in\mathbb{R}$ such that $a<b$ and $f\in L_1[a,b]$. The left-sided Riemann-Liouville integral $J_{a^+}^{\alpha}f$ and the right-sided Riemann-Liouville integral $J_{b^-}^{\alpha}f$ of order $\alpha>0$ with $a\geqslant 0$ are defined by

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt,\quad x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt,\quad x<b$$

respectively.

LEMMA 2.4. [20] *Let a,b,α be real numbers such that $a<b,\ \ \alpha>0$. Assume that $f:[a,b]\to\mathbb{R}$ and f' is integrable function. Then*

$$\begin{aligned}\frac{f(a)+f(b)}{2}&-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\\&=\frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)dt.\end{aligned}$$

LEMMA 2.5. [17] *Under the conditions of Lemma 2.4, the following equality holds*

$$\begin{aligned}\frac{f(a)+f(b)}{2}&-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\\&=\frac{b-a}{2^{\alpha+2}}\left[\int_0^1\left[(1-t)^{\alpha}-(1+t)^{\alpha}\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right.\\&\quad\left.+\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\right].\end{aligned}$$

3. Main results

Throughout this section, Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals for s-convex function are set forth. Then, by using two lemmas that links the derivative and the Riemann-Liouville fractional integrals of a functions, some results are obtained for s-convex functions.

THEOREM 3.1. *Let a,b,α be real number with $a<b,\ \ \alpha>0$. Assume that $f:[a,b]\to\mathbb{R}$ is integrable function. If f is an s-convex function, then*

$$\begin{aligned}2^{\frac{1}{s}}f\left(\frac{a+b}{2}\right)&\leqslant\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\\&\leqslant\frac{\alpha}{\alpha s+1}[f(a)+f(b)]\left(\mathrm{Beta}\left(\alpha,\frac{1}{s}\right)+s\right)\end{aligned}\tag{3.1}$$

holds.

Proof. Using s-convexity of f and taking $\lambda = \frac{1}{2}$ in Definition 2.1, we can write the following inequality for all $x, y \in [a,b]$:

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2^{\frac{1}{s}}}. \tag{3.2}$$

Choosing $x = ta+(1-t)b,\ y = (1-t)a+tb$ and using (3.2), one has

$$2^{\frac{1}{s}} f\left(\frac{a+b}{2}\right) \leqslant f(ta+(1-t)b) + f((1-t)a+tb). \tag{3.3}$$

By multiplying both sides of (3.3) by $t^{\alpha-1}$, then integrating with respect to t over $[0,1]$, it is obtained that

$$\begin{aligned}
\frac{2^{\frac{1}{s}}}{\alpha} f\left(\frac{a+b}{2}\right) &\leqslant \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt \\
&= \int_a^b \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{du}{b-a} + \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\frac{dv}{b-a} \\
&= \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]
\end{aligned}$$

thus,

$$2^{\frac{1}{s}} f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]. \tag{3.4}$$

For the second part of the inequality in (3.1), from the s-convexity of f, we have

$$f(ta+(1-t)b) \leqslant t^{\frac{1}{s}} f(a) + (1-t)^{\frac{1}{s}} f(b)$$

and

$$f((1-t)a+tb) \leqslant (1-t)^{\frac{1}{s}} f(a) + t^{\frac{1}{s}} f(b)$$

for $t \in [0,1]$. Adding these inequalities side by side, we have

$$f(ta+(1-t)b) + f((1-t)a+tb) \leqslant \left((1-t)^{\frac{1}{s}} + t^{\frac{1}{s}}\right)(f(a)+f(b)). \tag{3.5}$$

Then multiplication both sides of (3.5) by $t^{\alpha-1}$ and integration on $[0,1]$ with respect to t over $[0,1]$ yield

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt \\
&\qquad \leqslant [f(a)+f(b)] \int_0^1 t^{\alpha-1}\left((1-t)^{\frac{1}{s}} + t^{\frac{1}{s}}\right)dt \\
&\qquad = [f(a)+f(b)]\left(\frac{1}{\alpha s+1}\operatorname{Beta}\left(\alpha,\frac{1}{s}\right) + \frac{s}{\alpha s+1}\right) \\
&\qquad = \frac{1}{\alpha s+1}[f(a)+f(b)]\left(\operatorname{Beta}\left(\alpha,\frac{1}{s}\right) + s\right).
\end{aligned}$$

So,

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\leqslant\frac{\alpha}{\alpha s+1}[f(a)+f(b)]\left(\mathrm{Beta}\left(\alpha,\frac{1}{s}\right)+s\right). \tag{3.6}$$

By combining (3.4) and (3.6), the desired inequality is obtained. □

Note that the case $\alpha=1$ in Theorem 3.1 gives Theorem 2.2.

Using Lemma 2.4, we expose the relation between the Riemann–Liouville integrals of the functions and their derivatives, whose derivatives are s-convex.

THEOREM 3.2. *Let a,b,α be real number with $a<b$ and $\alpha>0$. Assume that $f:[a,b]\to\mathbb{R}$ and f' is integrable function. If f' is an s-convex function, then*

$$\begin{aligned}\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b^-}^{\alpha+1}f'(a)-H(b,a)&\leqslant\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\\&\leqslant H(a,b)-\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a^+}^{\alpha+1}f'(b)\end{aligned} \tag{3.7}$$

where

$$H(a,b)=\frac{b-a}{2}\frac{s}{\alpha s+s+1}\left(\frac{\alpha}{\alpha s+1}\mathrm{Beta}\left(\alpha,\frac{1}{s}\right)f'(a)+f'(b)\right).$$

Proof. From Lemma 2.4 and Definition 2.3

$$\begin{aligned}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]+\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a^+}^{\alpha+1}f'(b)\\=\frac{b-a}{2}\int_0^1(1-t)^{\alpha}f'(ta+(1-t)b)dt.\end{aligned}$$

By the s-convexity of f', it follows that

$$\frac{b-a}{2}\int_0^1(1-t)^{\alpha}f'(ta+(1-t)b)dt\leqslant\frac{b-a}{2}\int_0^1(1-t)^{\alpha}t^{\frac{1}{s}}f'(a)+(1-t)^{\frac{1}{s}+\alpha}f'(b)dt.$$

Using

$$\int_0^1(1-t)^{\alpha}t^{\frac{1}{s}}dt=\frac{\alpha s}{(as+s+1)(\alpha s+1)}\mathrm{Beta}\left(\alpha,\frac{1}{s}\right), \tag{3.8}$$

we have

$$\begin{aligned}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]+\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a^+}^{\alpha+1}f'(b)\\\leqslant\frac{b-a}{2}\frac{s}{as+s+1}\left(\frac{\alpha}{\alpha s+1}\mathrm{Beta}\left(\alpha,\frac{1}{s}\right)f'(a)+f'(b)\right).\end{aligned} \tag{3.9}$$

For the other side of the inequality, using Lemma 2.4 and the definition of Riemann-Liouville integrals, we have

$$-\frac{b-a}{2}\int_0^1 t^\alpha f'(ta+(1-t)b)dt = \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{b^-}^{\alpha+1} f'(a). \tag{3.10}$$

From s-convexity of f' and (3.8), we have

$$\begin{aligned}\frac{b-a}{2}\int_0^1 t^\alpha f'(ta+(1-t)b)dt &\leqslant \frac{b-a}{2}\int_0^1 t^{\alpha+\frac{1}{s}} f'(a) + (1-t)^{\frac{1}{s}} t^\alpha f'(b)dt\\ &= \frac{b-a}{2}\frac{s}{\alpha s+s+1}\left(\frac{\alpha}{\alpha s+1}\operatorname{Beta}\left(\alpha,\frac{1}{s}\right) f'(b) + f'(a)\right).\end{aligned} \tag{3.11}$$

Applying (3.11) in (3.10), we get

$$\begin{aligned}-\frac{b-a}{2}\frac{s}{\alpha s+s+1}\left(\frac{\alpha}{\alpha s+1}\operatorname{Beta}\left(\alpha,\frac{1}{s}\right) f'(b) + f'(a)\right) + \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{b^-}^{\alpha+1} f'(a)\\ \leqslant \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_a{+}^\alpha f(b) + J_b{-}^\alpha f(a)\right].\end{aligned} \tag{3.12}$$

Combining (3.9) and (3.12), we have the result. □

From Theorem 3.2, one can easily find the inequality relation between Riemann-Liouville integral sum of $f'(x)$ with order $\alpha+1$ and the ordinary derivatives of f or function values at the points a, b.

COROLLARY 3.3. *Under the conditions of Theorem 3.2, the following inequality holds*

$$\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J_{a^+}^{\alpha+1} f'(b) + J_{b^-}^{\alpha+1} f'(a)\right] \leqslant H(a,b) + H(b,a) \tag{3.13}$$

where $H(a,b)$ is given as in Theorem 3.2.

THEOREM 3.4. *Let a,b,α be real number with $a<b$ and $\alpha>0$. Assume $f:[a,b]\to\mathbb{R}$ and f' integrable function. If f is an s-convex function, then*

$$\begin{aligned}\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) + J_{\frac{a+b}{2}^-}^{\alpha+1} f'(a) - J_{a^+}^{\alpha+1} f'(b) + J_{\frac{a+b}{2}^+}^{\alpha+1} f'(b)\right)\\ \leqslant \frac{b-a}{2^{\alpha+2+\frac{1}{s}}}\left[H_1(a,b)f'(b) + H_2(a,b)f'(a)\right]\end{aligned} \tag{3.14}$$

where

$$H_1(a,b) = \frac{s\left(2^{\alpha+\frac{1}{s}+1}-1\right)}{\alpha s+s+1} - \frac{{}_2F_1(1,-\frac{1}{s};\alpha+2;-1)}{\alpha+1}$$

and

$$H_2(a,b) = \frac{s\,{}_2F_1(1,-\alpha;2+\frac{1}{s};-1)}{s+1} - \frac{s}{\alpha s+s+1}.$$

Proof. From Lemma 2.5, we can write

$$\begin{aligned}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]&\\ -\frac{b-a}{2^{\alpha+2}}\int_0^1\left[(1-t)^{\alpha}-(1+t)^{\alpha}\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt&\\ =\frac{b-a}{2^{\alpha+2}}\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt.&\qquad(3.15)\end{aligned}$$

Making substitution $u=\frac{1+t}{2}a+\frac{1-t}{2}b$ and algebraic manipulations, using Definition 2.3, we can express the integral on the left-hand side of the equality with respect to fractional integrals as follows

$$\begin{aligned}&\frac{b-a}{2^{\alpha+2}}\int_0^1\left[(1-t)^{\alpha}-(1+t)^{\alpha}\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\ &=\frac{b-a}{2^{\alpha+2}}\int_a^{\frac{a+b}{2}}\left[\left(\frac{2(u-a)}{b-a}\right)^{\alpha}-\left(\frac{2(b-u)}{b-a}\right)^{\alpha}\right]f'(u)\frac{2du}{b-a}\\ &=\frac{1}{2(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}(u-a)^{\alpha}f'(u)\,du-\frac{1}{2(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}(b-u)^{\alpha}f'(u)\,du\\ &=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}J_{\frac{a+b}{2}-}^{\alpha+1}f'(a)-\frac{1}{2(b-a)^{\alpha}}\left(\int_a^b(b-u)^{\alpha}f'(u)\,du-\int_{\frac{a+b}{2}}^b(b-u)^{\alpha}f'(u)\,du\right)\\ &=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}J_{\frac{a+b}{2}-}^{\alpha+1}f'(a)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha+1}f'(b)-J_{\frac{a+b}{2}+}^{\alpha+1}f'(b)\right)\\ &=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{\frac{a+b}{2}-}^{\alpha+1}f'(a)-J_{a^+}^{\alpha+1}f'(b)+J_{\frac{a+b}{2}+}^{\alpha+1}f'(b)\right).\end{aligned}$$

Now putting this in (3.15), we have

$$\begin{aligned}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)+J_{\frac{a+b}{2}-}^{\alpha+1}f'(a)-J_{a^+}^{\alpha+1}f'(b)+J_{\frac{a+b}{2}+}^{\alpha+1}f'(b)\right)&\\ =\frac{b-a}{2^{\alpha+2}}\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt&\qquad(3.16)\end{aligned}$$

From the s-convexity of f', it follows on the right-hand side of (3.16) that

$$\begin{aligned}&\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\\ &\leqslant\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]\left(\left(\frac{1+t}{2}\right)^{\frac{1}{s}}f'(b)+\left(\frac{1-t}{2}\right)^{\frac{1}{s}}f'(a)\right)dt\\ &=\frac{1}{2^{\frac{1}{s}}}\int_0^1\left((1+t)^{\alpha+\frac{1}{s}}-(1-t)^{\alpha}(1+t)^{\frac{1}{s}}\right)f'(b)dt\\ &\quad+\frac{1}{2^{\frac{1}{s}}}\int_0^1\left((1-t)^{\frac{1}{s}}(1+t)^{\alpha}-(1-t)^{\alpha+\frac{1}{s}}\right)f'(a)dt\end{aligned}$$

$$= \frac{1}{2^{\frac{1}{s}}}\left(\frac{s\left(2^{\alpha+\frac{1}{s}+1}-1\right)}{\alpha s+s+1} - \frac{{}_2F_1(1,-\frac{1}{s};\alpha+2;-1)}{\alpha+1}\right) f'(b)$$
$$+ \frac{1}{2^{\frac{1}{s}}}\left(\frac{s\,{}_2F_1(1,-\alpha;2+\frac{1}{s};-1)}{s+1} - \frac{s}{\alpha s+s+1}\right) f'(a).$$

Using this inequality in (3.16), we have the result. □

THEOREM 3.5. *Let a,b,α be real number with $a<b$ and $\alpha>0$. Assume $f:[a,b]\to\mathbb{R}$ and f' integrable function. If f is an s-convex function, then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)+J^{\alpha+1}_{b^-}f'(a)-J^{\alpha+1}_{\frac{a+b}{2}-}f'(a)-J^{\alpha+1}_{\frac{a+b}{2}+}f(b)\right)$$
$$-\frac{f(a)+f(b)}{2}\leqslant \frac{b-a}{2^{\alpha+2+\frac{1}{s}}}\left[H_1(a,b)f'(a)+H_2(a,b)f'(b)\right] \tag{3.17}$$

where $H_1(a,b)$ and $H_2(a,b)$ is defined as in Theorem 3.4.

Proof. By Lemma 2.5,

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]$$
$$-\frac{b-a}{2^{\alpha+2}}\int_0^1\left[(1+t)^\alpha-(1-t)^\alpha\right]f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt$$
$$=\frac{b-a}{2^{\alpha+2}}\int_0^1\left[(1-t)^\alpha-(1+t)^\alpha\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt.$$

Making the substitution $u=\frac{1+t}{2}b+\frac{1-t}{2}a$ and algebraic manipulations and using the definition of Riemann-Liouville integrals, we can write the integral on the left-hand side of the equality with respect to fractional integrals as follows

$$\frac{b-a}{2^{\alpha+2}}\int_0^1\left[(1+t)^\alpha-(1-t)^\alpha\right]f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt$$
$$=\frac{b-a}{2^{\alpha+2}}\int_{\frac{a+b}{2}}^{b}\left[\left(\frac{2(u-a)}{b-a}\right)^\alpha-\left(\frac{2(b-u)}{b-a}\right)^\alpha\right]f'(u)\frac{2du}{b-a}$$
$$=\frac{1}{2(b-a)^\alpha}\int_{\frac{a+b}{2}}^{b}(u-a)^\alpha f'(u)\,du-\frac{1}{2(b-a)^\alpha}\int_{\frac{a+b}{2}}^{b}(b-u)^\alpha f'(u)\,du$$
$$=\frac{1}{2(b-a)^\alpha}\left(\int_a^b(u-a)^\alpha f'(u)\,du-\int_a^{\frac{a+b}{2}}(u-a)^\alpha f'(u)\,du\right)$$
$$-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}J^{\alpha+1}_{\frac{a+b}{2}+}f'(b)$$

$$= \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{b^-}^{\alpha+1}f'(a) - J_{\frac{a+b}{2}^-}^{\alpha+1}f'(a)\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}J_{\frac{a+b}{2}^+}^{\alpha+1}f'(b)$$
$$= \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{b^-}^{\alpha+1}f'(a) - J_{\frac{a+b}{2}^-}^{\alpha+1}f'(a) - J_{\frac{a+b}{2}^+}^{\alpha+1}f'(b)\right).$$

Thus

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a) + J_{b^-}^{\alpha+1}f'(a) - J_{\frac{a+b}{2}^-}^{\alpha+1}f'(a) - J_{\frac{a+b}{2}^+}^{\alpha+1}f'(b)\right)$$
$$= \frac{b-a}{2^{\alpha+2}}\int_0^1 \left[(1-t)^{\alpha} - (1+t)^{\alpha}\right] f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt. \tag{3.18}$$

From s-convexity of f', we have

$$\int_0^1 \left[(1-t)^{\alpha} - (1+t)^{\alpha}\right] f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt$$
$$\geqslant -\int_0^1 \left[(1+t)^{\alpha} - (1-t)^{\alpha}\right]\left(\left(\frac{1+t}{2}\right)^{\frac{1}{s}} f'(a) + \left(\frac{1-t}{2}\right)^{\frac{1}{s}} f'(b)\right) dt$$
$$= -\frac{1}{2^{\frac{1}{s}}}\int_0^1 \left[(1+t)^{\alpha+\frac{1}{s}} - (1-t)^{\alpha}(1+t)^{\frac{1}{s}}\right] f'(a)dt$$
$$- \frac{1}{2^{\frac{1}{s}}}\int_0^1 \left[(1-t)^{\frac{1}{s}}(1+t)^{\alpha} - (1-t)^{\alpha+\frac{1}{s}}\right] f'(b)dt$$
$$= -\frac{1}{2^{\frac{1}{s}}}\left(\frac{{}_2F_1(1,-\frac{1}{s};\alpha+2;-1)}{\alpha+1} - \frac{s\left(2^{\alpha+\frac{1}{s}+1}-1\right)}{\alpha s+s+1}\right) f'(a)$$
$$- \frac{1}{2^{\frac{1}{s}}}\left[\frac{s}{\alpha s+s+1} - \frac{s\,{}_2F_1(1,-\alpha;2+\frac{1}{s};-1)}{s+1}\right] f'(b). \tag{3.19}$$

Applying (3.19) in (3.18), we get the result. □

4. Applications

This section exemplifies the usage of each of the results in previous section. In the following result, an upper bound for beta function is obtained.

PROPOSITION 4.1. *Let* $\alpha \geqslant 0$ *and* $y \geqslant 1$. *Then*

$$\mathrm{Beta}(\alpha, y) \leqslant \frac{1}{\alpha}.$$

Proof. Taking $f(x) = -x^{\frac{1}{s}}$ with $s \in (0,1]$ and $a = 0$ and $b = 1$ in Theorem 3.1 and using the right-hand side part of (3.1) yield to

$$\frac{\alpha}{\alpha s+1}\left(\mathrm{Beta}\left(\alpha, \frac{1}{s}\right) + s\right) \leqslant 1.$$

Then writing $y = \frac{1}{s}$ where $\frac{1}{s} > 1$ gives the required inequality. $\square$

Next proposition states a relation involving beta and incomplete beta functions.

PROPOSITION 4.2. *Let a, b, α, s positive real numbers with $s \in (0,1]$. Then*

$$\frac{2s(a^{\frac{1}{s}} + b^{\frac{1}{s}})}{\alpha(\alpha s + s + 1)} \left(\frac{\alpha}{\alpha s + 1} \mathrm{Beta}\left(\alpha, \frac{1}{s}\right) + 1 \right) \leqslant \frac{b^{\alpha + \frac{1}{s} + 1} W_1(a,b) + a^{\alpha + \frac{1}{s} + 1} W_2(a,b)}{(b-a)^{\alpha+1}} \tag{4.1}$$

where

$$W_1(a,b) = B_{\frac{a}{b}}\left(\frac{1}{s} + 1, \alpha + 1\right) - \mathrm{Beta}\left(\alpha + 1, \frac{1}{s} + 1\right)$$

and

$$W_2(a,b) = B_{\frac{a}{b}}\left(-\alpha - \frac{1}{s} - 1, \alpha + 1\right) - \mathrm{Beta}\left(\alpha + 1, -\alpha - \frac{1}{s} - 1\right).$$

Proof. Let us take $f(x) = -\frac{s}{s+1} x^{\frac{1}{s}+1}$ with $s \in (0,1]$ where f' is s-convex on $[a,b]$ in Corollary 3.3. We obtain the following inequality via Mathematica 13.1:

$$J_{b^-}^{\alpha+1} f'(a) = -\frac{1}{\Gamma(\alpha)} a^{\alpha + \frac{1}{s} + 1} \Bigg(\mathrm{Beta}\left(\alpha + 1, -\alpha - \frac{1}{s} - 1\right) - B_{\frac{a}{b}}\left(-\alpha - \frac{1}{s} - 1, \alpha + 1\right) \Bigg) \tag{4.2}$$

$$J_{a^+}^{\alpha+1} f'(b) = -\frac{1}{\Gamma(\alpha)} b^{\alpha + \frac{1}{s} + 1} \left(\mathrm{Beta}\left(\alpha + 1, \frac{1}{s} + 1\right) - B_{\frac{a}{b}}\left(\frac{1}{s} + 1, \alpha + 1\right) \right). \tag{4.3}$$

Putting (4.2) and (4.3) in (3.13), we get the result. $\square$

Using proposition above, one can conclude the following quick result.

PROPOSITION 4.3. *Let $t \in (0,1)$ and $2 < x,\ 1 < y$. Then*

$$B_t(x,y) + t^{x+y-1} B_t(-x-y+1, y) < \mathrm{Beta}(x,y) + t^{x+y-1} \mathrm{Beta}(-x-y+1, y).$$

Proof. It is clear that the left part of (4.1) is positive, so is the right part. So

$$0 < b^{\alpha + \frac{1}{s} + 1} W_1(a,b) + a^{\alpha + \frac{1}{s} + 1} W_2(a,b).$$

Dividing both sides with $b^{\alpha + \frac{1}{s} + 1}$, then making substitutions $t = \frac{a}{b},\ x = \frac{1}{s} + 1,\ y = \alpha + 1$, we get the result. $\square$

It is possible to derive some inequalities involving special means from the results as follows

PROPOSITION 4.4. *Let a,b,α positive real numbers. If $x>2$, then*

$$\frac{b-a}{2^{\alpha+x+1}}\frac{x-1}{(x+1)x}\left[\left(2^{x+1}+\frac{x+2}{x-1}+1\right)b^{x-1}+\frac{2}{x-1}a^{x-1}\right]$$
$$\leqslant \frac{1}{x}A(a^x,b^x)+\frac{1}{2}\frac{1}{\Gamma(\alpha)}\left(L_{x+1}^{x}\left(a,\frac{a+b}{2}\right)+\left(\frac{a+b}{2}\right)L_x^{x-1}\left(a,\frac{a+b}{2}\right)-\frac{2}{x}L_{x+1}^{x}(a,b)\right)$$

where $A(\cdot,\cdot)$ and $L\_(\cdot,\cdot)$ denote the arithmetic and generalized logarithmic mean, i.e.

$$A(a,b)=\frac{a+b}{2}\quad \textit{and}\quad L_p(a,b)=\begin{cases} a, & \textit{if } a=b\\ \left(\frac{a^{p+1}-b^{p+1}}{(p+1)(a-b)}\right)^{1/p}, & \textit{else}\end{cases}$$

for positive real numbers a,b,p.

Proof. Assume $\alpha=1$, $f(x)=-\frac{s}{s+1}x^{\frac{1}{s}+1}$ on $[a,b]$ in Theorem 3.4. We obtain fractional integrals as follows

$$J_{a+}^{1}f(b)+J_{b-}^{1}f(a)=-\frac{1}{\Gamma(\alpha)}\frac{2s^2}{(s+1)(2s+1)}\left(b^{\frac{1}{s}+2}-a^{\frac{1}{s}+2}\right),$$

$$J_{\frac{a+b}{2}-}^{2}f'(a)=-\left(\frac{\left(\frac{a+b}{2}\right)^{\frac{1}{s}+2}}{\frac{1}{s}+2}-\frac{a\left(\frac{a+b}{2}\right)^{\frac{1}{s}+1}}{\frac{1}{s}+1}-\frac{a^{\frac{1}{s}+2}}{\frac{1}{s}+2}+\frac{a^{\frac{1}{s}+2}}{\frac{1}{s}+1}\right),$$

$$J_{\frac{a+b}{2}+}^{2}f'(b)=-\left(\frac{b^{\frac{1}{s}+2}}{\frac{1}{s}+1}-\frac{b^{\frac{1}{s}+2}}{\frac{1}{s}+2}-\frac{b\left(\frac{a+b}{2}\right)^{\frac{1}{s}+1}}{\frac{1}{s}+1}+\frac{\left(\frac{a+b}{2}\right)^{\frac{1}{s}+2}}{\frac{1}{s}+2}\right),$$

$$J_{\frac{a+b}{2}+}^{2}f'(b)=-\left(\frac{b^{\frac{1}{s}+2}}{\frac{1}{s}+1}-\frac{b^{\frac{1}{s}+2}}{\frac{1}{s}+2}-\frac{b\left(\frac{a+b}{2}\right)^{\frac{1}{s}+1}}{\frac{1}{s}+1}+\frac{\left(\frac{a+b}{2}\right)^{\frac{1}{s}+2}}{\frac{1}{s}+2}\right)$$

and

$$J_{a+}^{2}f'(b)=-\left(\frac{b^{\frac{1}{s}+2}}{\frac{1}{s}+1}-\frac{b^{\frac{1}{s}+2}}{\frac{1}{s}+2}-\frac{ba^{\frac{1}{s}+1}}{\frac{1}{s}+1}+\frac{a^{\frac{1}{s}+2}}{\frac{1}{s}+2}\right).$$

Then, replacing $1+\frac{1}{s}=x$ and putting them in (3.14) and making some time-consuming adjustments and simplifications, we have the result. □

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