Journal of
Mathematical
Inequalities
Volume 19, Number 3 (2025), 779–792 doi:10.7153/jmi-2025-19-50

DIAGONAL–SCHUR COMPLEMENTS OF DASHNIC–ZUSMANOVICH TYPE MATRICES

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(*Communicated by J. Liu*)

Abstract. The subclasses of nonsingular H-matrices are very important in the field of numerical algebra and related fields. As a subclass of H-matrices, the class of Dashnic-Zusmanovich type matrices was first mentioned in 2018. This paper investigates the closure properties of diagonal-Schur complements on DZT matrices. We prove that the diagonal-Schur complements of DZT matrices with respect to any index set are still in the same matrix class, which improves the corresponding result obtained by Li, Huang, and Zhao in 2022 (Chaoqian Li, Zhengyu Huang and Jianxing Zhao, Linear and Multilinear Algebra, 70 (2022): 4071–4096). Numerical examples are given to verify the correctness of the proposed results.

1. Introduction

H-matrices are widely used in many fields due to their excellent algebraic properties, like computational mathematics, control theory, economics, and dynamic systems, etc [1,3,8,15,16]. To our knowledge, H-matrices contain many well-known subclasses, such as SDD (strictly diagonally dominant) matrices, Σ-SDD matrices, DSDD (doubly strictly diagonally dominant) matrices, Nekrasov matrices, Σ-Nekrasov matrices, DZ (Dashnic-Zumanovich) matrices, and so on.

For any $A = (a_{ij}) \in C^{n\times n}$, denote $\langle n\rangle = \{1,2,\ldots,n\}$ and

$$r_i(A) = \sum_{j\neq i}^{n} |a_{ij}|, \tag{1.1}$$

$$r_i^S(A) = \sum_{j\neq i, j\in S}^{n} |a_{ij}|, \quad \text{(where } S \text{ is a given nonempty subset of } \langle n\rangle\text{)}. \tag{1.2}$$

The definition of DZ matrix can be traced back to [6].

DEFINITION 1.1. Let $A = (a_{ij}) \in C^{n\times n}$. We say that A is a Dashnic-Zusmanovich (DZ) matrix if there exists an index j such that for any $i(\neq j) \in \langle n\rangle$, it holds that

$$\left(|a_{ii}| - r_i^{\langle n\rangle\setminus\{j\}}(A)\right)|a_{jj}| > |a_{ij}| r_j(A). \tag{1.3}$$

Mathematics subject classification (2020): 15A45, 15A48.
Keywords and phrases: H-matrix, Dashnic-Zusmanovich type matrix, diagonal-Schur complement.
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Paper JMI-19-50

In 2018, Zhao, Liu, Li and Li [19] introduced a new subclass of H-matrices: DZT (Dashnic-Zusmanovich type) matrices.

DEFINITION 1.2. Let $A=(a_{ij})\in C^{n\times n}$. We say that A is a Dashnic-Zusmanovich type (DZT) matrix if for each $i\in\langle n\rangle$, there exists an index $j(\neq i)$ such that (1.3) holds.

Although these two classes of matrices are similar in form, neither of them contains the other one as a subclass. The SDD matrices are belong to DZT matrices.

Recently, many results about DZT matrices have been obtained, such as infinity norm bounds for the inverse and linear complementarity problems [10]. Kolotilina [9] gave DZT matrices a brief equivalent expression. This new expression brings a lot of convenience in proving that a given matrix belongs to DZT matrices.

LEMMA 1.1. [9, Theorem 2.1] *Given a matrix $A\in C^{n\times n}$ with $N^-(A)\neq\emptyset$. Then A is a DZT matrix if and only if $\Gamma_i(A)\neq\emptyset$ for all $i\in N^-(A)$, where*

$$N^+(A)=\{i\in\langle n\rangle : |a_{ii}|>r_i(A)\},\tag{1.4}$$

$$N^-(A)=\{i\in\langle n\rangle : |a_{ii}|\leqslant r_i(A)\},\tag{1.5}$$

and

$$\Gamma_i(A)=\{j\in\langle n\rangle\setminus\{i\} : \big(|a_{ii}|-r_i^{\langle n\rangle\setminus\{j\}}(A)\big)|a_{jj}|>|a_{ij}|r_j(A)\},\ \ i\in\langle n\rangle.\tag{1.6}$$

There is a very close connection between Schur complements and diagonal-Schur complements, and the diagonal-Schur complement is an important tool in numerical analysis, control theory, matrix theory and statistics [11, 13, 14, 17]. For instance, the following structural perturbation of stationary linear large-scale systems are usually considered in control theory [13, 17]:

$$\frac{dx}{dt}=Ax\tag{1.7}$$

where A is an $n\times n$ complex matrix and x is an n-dimensional vector. The matrix A is often written as: $A=\tilde{A}+\breve{A}$, where $\tilde{A}$ is a diagonal matrix. The matrix $\breve{A}=A-\tilde{A}$ is related to a diagonal-Schur complement of some matrix. Hence, an interesting and important problem is whether some important properties of the original matrix are inherited by its diagonal-Schur complements, more generally, whether the original matrix and its diagonal-Schur complements are in the same matrix class.

In 2004, it has been proved that diagonal-Schur complements of SDD matrices are SDD matrices [13]. Then in 2008, it has been proved that the diagonal-Schur complements of an H-matrix, DSDD matrix, and γ-SDD matrix remain an H-matrix, DSDD matrix, and γ-SDD matrix, respectively [14]. The closure property of diagonal-Schur complements of Dashnic-Zumanovich matrices are obtained in 2009 [5]. It was proved that the diagonal-Schur complement of a Nekrasov matrix is closed [5, 17]. For more results about diagonal-Schur complements, one can refer to [4, 12] and the references therein.

For DZT matrices, Li, Huang and Zhao [11] proved that the diagonal-Schur complements of Dashnic-Zumanovich type matrices are still Dashnic-Zumanovich type matrices under certain conditions.

THEOREM 1.1. [11, Theorem 3.5] *Given a DZT matrix A, let $N^+(A) \subseteq \alpha \subset \langle n \rangle$. For every $j_s \in \bar{\alpha}$, if*

$$|a_{i_t i_t}| > \sum_{v=1, v \neq t}^{k} |a_{i_t i_v}| + |a_{i_t j_s}|, \quad \text{for any} \quad i_t \in \alpha,$$

then $A/_\circ \alpha$ is an SDD matrix, where k is the cardinal number of α.

THEOREM 1.2. [11, Theorem 3.7] *Given a DZT matrix A, let $\emptyset \neq \alpha \subset N^+(A) \subset \langle n \rangle$. Then $A/_\circ \alpha$ is a DZT matrix.*

In order to completely investigate the closure properties of diagonal-Schur complements of DZT matrices, we follow a different approach from the previous works. We consider the relations of $N^+(A)$, $N^-(A)$, $N^+(A/_\circ \alpha)$ and $N^-(A/_\circ \alpha)$. Adopting the new expression of DZT matrices introduced in [9], we prove that the diagonal-Schur complements of DZT matrices with respect to any index set are also DZT matrices. An example is given to verify the correctness of the proposed results.

2. The preliminaries

In this paper, the set of all $n \times n$ complex matrices are denoted by $C^{n\times n}$ (we always assume $n \geqslant 2$ in this paper). Now we introduce some notations and symbols. For any $A = (a_{ij}) \in C^{n\times n}$, the determinant of A is denoted by $det(A)$. The comparison matrix of A is denoted by $\mu(A) = (u_{ij})_{n\times n}$ where $u_{ij} = |a_{ij}|$ for $i = j$ and $u_{ij} = -|a_{ij}|$ for $i \neq j$. For $A = (a_{ij}) \in C^{n\times n}$ and $B = (b_{ij}) \in C^{n\times n}$, $A \circ B$ is defined as $(a_{ij} b_{ij})$. Let α and β be given subsets of $\langle n \rangle$. $|\alpha|$ stands for the cardinal number of α. $A(\alpha, \beta)$ stands for the sub-matrix of A lying in the rows indexed by α and the columns indexed by β. $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. If $A(\alpha)$ is nonsingular, then the Schur complement of $A \in C^{n\times n}$ with respect to $A(\alpha)$ is denoted by A/α, i.e.,

$$A/\alpha = A(\bar{\alpha}) - A(\bar{\alpha}, \alpha)[A(\alpha)]^{-1} A(\alpha, \bar{\alpha}),$$

where $\bar{\alpha} = \langle n \rangle \setminus \alpha$. The diagonal-Schur complement of $A \in C^{n\times n}$ with respect to $A(\alpha)$ is denoted by $A/_\circ \alpha$, i.e.,

$$A/_\circ \alpha = A(\bar{\alpha}) - \{A(\bar{\alpha}, \alpha)[A(\alpha)]^{-1} A(\alpha, \bar{\alpha})\} \circ I.$$

DEFINITION 2.1. Let $A \in C^{n\times n}$. The matrix A is called an M-matrix if it can be written in the form of $A = sI - P$ where I is the identity matrix, P is a nonnegative matrix, $s > \rho(P)$ and $\rho(P)$ is the spectral radius of P.

DEFINITION 2.2. Let $A \in C^{n\times n}$. The matrix A is called an H-matrix if $\mu(A)$ is an M-matrix.

DEFINITION 2.3. Let $A = (a_{ij}) \in C^{n\times n}$. We say that A is a strictly diagonally dominant (SDD) matrix if for all $i \in \langle n \rangle$, it holds that $|a_{ii}| > r_i(A)$.

In this paper, the set of $n\times n$ nonsingular H-matrices, SDD matrices, DZ-matrices, and DZT matrices, are denoted by H_n, SDD_n, DZ_n and DZT_n, respectively.

LEMMA 2.1. [7, p. 131] *If A is an H-matrix, then $[\mu(A)]^{-1}\geqslant|A^{-1}|$.*

LEMMA 2.2. [7, p. 117] *If A is an M-matrix, then $det(A)>0$.*

LEMMA 2.3. [18, p. 5] *Let $A\in C^{n\times n}$ and α be a nonempty proper subset of $\langle n\rangle$. If $A(\alpha)$ is nonsingular, then $det(A)=det(A(\alpha))det(A/\alpha)$.*

LEMMA 2.4. [9, Lemma 2.2] *Given a matrix $A\in DZT_n$, then*
(i) $a_{ii}\neq 0$ for all $i\in\langle n\rangle$;
(ii) $\Gamma_i(A)\subseteq N^+(A)$ for all $i\in N^-(A)$;
(iii) $|a_{ii}|>r_i^{\langle n\rangle\backslash\{j\}}(A)$ if $j\in\Gamma_i(A)$;
(iv) $|a_{ii}|-r_i^{\langle n\rangle\backslash\{j\}}(A)>\frac{|a_{ij}|r_j(A)}{|a_{jj}|}$ if $j\in\Gamma_i(A)$.

By Lemma 1.1, if there exists an index $j\in N^+(A)$ such that $j\in\Gamma_i(A)$ for all $i\in N^-(A)$. Then $A\in DZT_n\cap DZ_n$.

REMARK 2.1. For $A\in DZT_n$ $(n\geqslant 2)$, it holds trivially that $N^+(A)\neq\emptyset$. We always assume $N^+(A)\subset\langle n\rangle$ because it holds that $A\in SDD_n$ if $N^+(A)=\langle n\rangle$ and the Schur and diagonal-Schur complements of SDD matrices are also SDD matrices [2,13].

REMARK 2.2. For any given $A\in DZT_n$, we always assume α is a nonempty proper subset of $\langle n\rangle$, i.e., $\emptyset\neq\alpha\subset\langle n\rangle$. The elements in both of α and $\bar{\alpha}$ are listed in increasing order, i.e.,

$$\alpha=\{i_1,i_2,\cdots,i_k\},\quad i_1<i_2<\cdots<i_k; \tag{2.1}$$
$$\bar{\alpha}=\{j_1,j_2,\cdots,j_l\},\quad j_1<j_2<\cdots<j_l. \tag{2.2}$$

Denote

$$x_u=(a_{j_ui_1},\ldots,a_{j_ui_k})^T,\quad j_u\in\bar{\alpha}; \tag{2.3}$$
$$y_u=(a_{i_1j_u},\ldots,a_{i_kj_u})^T,\quad j_u\in\bar{\alpha}. \tag{2.4}$$

Then

$$|x_u|=(|a_{j_ui_1}|,\ldots,|a_{j_ui_k}|)^T,\quad j_u\in\bar{\alpha};$$
$$|y_u|=(|a_{i_1j_u}|,\ldots,|a_{i_kj_u}|)^T,\quad j_u\in\bar{\alpha}.$$

The symbols x_u and y_u are frequently used in studying the diagonal-Schur complements.

3. Diagonal-Schur complements of DZT matrices

In this section, we will show that the diagonal-Schur complements of DZT matrices are also DZT matrices with respect to any index set. First, we show that $A/_{\circ}\alpha$ is a DZT matrix for any $\emptyset \neq \alpha \subset \langle n \rangle$.

LEMMA 3.1. *Let $A \in DZT_n$. For any nonempty proper subset α of $\langle n \rangle$, we have $A(\alpha) \in DZT_{|\alpha|}$.*

Proof. Denote $A = (a_{ij})$. Let α and $\bar{\alpha}$ be defined as in (2.1) and (2.2), respectively. Then

$$A(\alpha) = \begin{bmatrix} a_{i_1 i_1} & \dots & a_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k i_1} & \dots & a_{i_k i_k} \end{bmatrix}.$$

We consider the following conditions: (i) $N^+(A) \subseteq \alpha$, (ii) $\alpha \cap N^+(A) = \emptyset$, (iii) $\alpha \cap N^+(A) \neq \emptyset$ and $\alpha \cap N^-(A) \neq \emptyset$.

(i) $N^+(A) \subseteq \alpha$. For any $t \in \langle k \rangle$, if $i_t \in N^+(A)$, it is clear that $t \in N^+(A(\alpha))$, i.e.,

$$N^+(A(\alpha)) \supseteq \{t \in \langle k \rangle : i_t \in N^+(A)\},$$

and

$$N^-(A(\alpha)) \subseteq \{t \in \langle k \rangle : i_t \in N^-(A)\}.$$

By Lemma 1.1, we only need to show $\Gamma_t(N^-(A(\alpha)) \neq \emptyset$ for any $t \in \langle k \rangle$ with $i_t \in N^-(A)$. Since $N^+(A) \subseteq \alpha$, by Lemma 2.4, there exists i_u $(u \neq t) \in N^+(A) \subseteq \alpha$ such that $i_u \in \Gamma_{i_t}(A)$, i.e.,

$$\left(|a_{i_t i_t}| - r_{i_t}^{\langle n \rangle \setminus \{i_u\}}(A)\right)|a_{i_u i_u}| > |a_{i_t i_u}| r_{i_u}(A). \tag{3.1}$$

Then, it holds that

$$\begin{aligned} &\left(|a_{i_t i_t}| - r_t^{\langle k \rangle \setminus \{u\}}(A(\alpha))\right)|a_{i_u i_u}| \\ &= \left(|a_{i_t i_t}| - r_{i_t}^{\alpha \setminus \{i_u\}}(A)\right)|a_{i_u i_u}| \\ &\geqslant \left(|a_{i_t i_t}| - r_{i_t}^{\langle n \rangle \setminus \{i_u\}}(A)\right)|a_{i_u i_u}| \\ &> |a_{i_t i_u}| r_{i_u}(A) \\ &\geqslant |a_{i_t i_u}| r_u(A(\alpha)), \end{aligned} \tag{3.2}$$

which implies that $\Gamma_t(A(\alpha)) \neq \emptyset$. By Lemma 1.1, $A(\alpha) \in DZT_{|\alpha|}$.

(ii) $\alpha \cap N^+(A) = \emptyset$, i.e., $N^+(A) \subseteq \bar{\alpha}$. Then for any $t \in \langle k \rangle$, there exists $j_v \in N^+(A)$ such that $j_v \in \Gamma_{i_t}(A)$. Hence, by Lemma 2.4, we have

$$|a_{i_t i_t}| > r_{i_t}^{\langle n \rangle \setminus \{j_v\}}(A) \geqslant r_{i_t}^{\alpha}(A) = r_t(A(\alpha)), \tag{3.3}$$

which implies that $A(\alpha) \in SDD_{|\alpha|}$.

(iii) $\alpha \cap N^{+}(A) \neq \emptyset$ and $\alpha \cap N^{-}(A) \neq \emptyset$. It leads that $\bar{\alpha} \cap N^{+}(A) \neq \emptyset$. For any $t \in \langle k \rangle$, if $i_t \in N^{+}(A)$, it holds that $t \in N^{+}(A(\alpha))$. If $i_t \in N^{-}(A)$ and $\Gamma_{i_t}(A) \cap \alpha = \emptyset$, then $\Gamma_{i_t}(A) \cap \bar{\alpha} \neq \emptyset$. Suppose that there exists $j_v \in N^{+}(A)$ such that $j_v \in \Gamma_{i_t}(A)$. Then (3.3) holds. Thus we have

$$N^{+}(A(\alpha)) \supseteq \{t \in \langle k \rangle : i_t \in N^{+}(A), \text{ or } i_t \in N^{-}(A) \text{ with } \Gamma_{i_t}(A) \cap \alpha = \emptyset\},$$
$$N^{-}(A(\alpha)) \subseteq \{t \in \langle k \rangle : i_t \in N^{-}(A) \text{ with } \Gamma_{i_t}(A) \cap \alpha \neq \emptyset\}.$$

For any $i_t \in N^{-}(A)$ with $\Gamma_{i_t}(A) \cap \alpha \neq \emptyset$, there exists $i_u \in N^{+}(A) \cap \alpha$ $(u \neq t)$ such that $i_u \in \Gamma_{i_t}(A)$, then (3.1) and (3.2) hold. It follows that $\Gamma_t(A(\alpha)) \neq \emptyset$. By Lemma 1.1, $A(\alpha) \in DZT_{|\alpha|}$. $\square$

Given $A = (a_{ij}) \in DZT_n$, let $\alpha, \bar{\alpha}, x_u, y_u$ be defined as in (2.1)–(2.4), respectively. Denote $A/_\circ\alpha = (a'_{tu})$. Then

$$a'_{uu} = |a_{j_u j_u}| - |x_u^T [A(\alpha)]^{-1} y_u|,$$

and

$$a'_{tu} = |a_{j_t j_u}|, \quad t \neq u.$$

It is clear that

$$r_u(A/_\circ\alpha) = r_{j_u}^{\bar{\alpha}}(A). \tag{3.4}$$

For $u \in \langle l \rangle$, since

$$\begin{aligned} &|a'_{uu}| - r_u(A/_\circ\alpha) \\ \geqslant\ & \big(|a_{j_u j_u}| - |x_u^T [A(\alpha)]^{-1} y_u|\big) - r_{j_u}^{\bar{\alpha}}(A) \\ \geqslant\ & |a_{j_u j_u}| - |x_u^T| [\mu(A(\alpha))]^{-1} |y_u| - r_{j_u}^{\bar{\alpha}}(A), \end{aligned}$$

we can see that the value of $|x_u^T| [\mu(A(\alpha))]^{-1} |y_u|$ plays a key role in investigating whether $u \in N^{+}(A/_\circ\alpha)$ or not. Hence we first discuss the bound of $|x_u^T| [\mu(A(\alpha))]^{-1} |y_u|$. For this purpose, we construct several new nonsingular H-matrices (C_u and D_u, see the proof of Lemma 3.1), then apply Lemma 2.3 to determine the Schur complements with respect to $\mu(A(\alpha))$ for these new matrices. At last, by Lemma 2.2, the bounds of $|x_u^T| [\mu(A(\alpha))]^{-1} |y_u|$ can be obtained. The results are formulated in the following technical lemma.

LEMMA 3.2. *Let $A \in DZT_n$ and let $\alpha, \bar{\alpha}, x_u, y_u$ be defined as in* (2.1)–(2.4)*, respectively. Given $u \in \langle l \rangle$, then the following results hold.*

(i) If $j_u \in N^{+}(A)$, it holds that

$$|x_u^T| [\mu(A(\alpha))]^{-1} |y_u| \leqslant \frac{|a_{j_u j_u}|}{r_{j_u}(A)} r_{j_u}^{\alpha}(A),$$

with equality if and only if $r_{j_u}^{\alpha}(A) = 0$, i.e., $|x_u^T| [\mu(A(\alpha))]^{-1} |y_u| = 0$ if $r_{j_u}^{\alpha}(A) = 0$; otherwise,

$$|x_u^T| [\mu(A(\alpha))]^{-1} |y_u| < \frac{|a_{j_u j_u}|}{r_{j_u}(A)} r_{j_u}^{\alpha}(A). \tag{3.5}$$

(ii) If $j_u \in N^-(A)$ *and* $\Gamma_{j_u}(A) \cap \alpha \neq \emptyset$, *it holds that*

$$|x_u^T|[\mu(A(\alpha))]^{-1}|y_u| < |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A) \leqslant r_{j_u}^{\alpha}(A). \tag{3.6}$$

Proof. (i) If $r_{j_u}^{\alpha}(A) = 0$, then we have $x_u = 0$ and then $|x_u^T|[\mu(A(\alpha))]^{-1}|y_u| = 0$. Now we consider the case $r_{j_u}^{\alpha}(A) \neq 0$. It follows that

$$\frac{|a_{j_u j_u}|}{r_{j_u}(A)} r_{j_u}^{\alpha}(A) > r_{j_u}^{\alpha}(A). \tag{3.7}$$

Let

$$C_u := \begin{bmatrix} \mu[A(\alpha)] & -|y_u| \\ -|x_u^T| & \gamma_u^* \end{bmatrix} \quad \left(\text{where } \gamma_u^* = \frac{|a_{j_u j_u}|}{r_{j_u}(A)} r_{j_u}^{\alpha}(A)\right).$$

It is easy to see that $k+1 \in N^+(C_u)$ by (3.7). For any $t \in \langle k \rangle$, if $i_t \in N^+(A)$, then $t \in N^+(C_u)$. Hence,

$$\begin{aligned} N^+(C_u) &\supseteq \{k+1\} \cup \{t \in \langle k \rangle : i_t \in N^+(A)\}, \\ N^-(C_u) &\subseteq \{t \in \langle k \rangle : i_t \in N^-(A)\}. \end{aligned}$$

For any $t \in \langle k \rangle$ with $i_t \in N^-(A)$, if $j_u \in \Gamma_{i_t}(A)$, then

$$\left(|a_{i_t i_t}| - r_{i_t}^{\langle n \rangle \setminus \{j_u\}}(A)\right)|a_{j_u j_u}| > |a_{i_t j_u}| r_{j_u}(A).$$

Together with $r_{j_u}^{\alpha}(A) \neq 0$, we have

$$\begin{aligned} &\left(|a_{i_t i_t}| - r_t^{\langle k+1 \rangle \setminus \{k+1\}}(C_u)\right)\gamma_u^* \\ = &\left(|a_{i_t i_t}| - r_{i_t}^{\alpha}(A)\right)\gamma_u^* \\ \geqslant &\left(|a_{i_t i_t}| - r_{i_t}^{\langle n \rangle \setminus \{j_u\}}(A)\right)|a_{j_u j_u}| \frac{r_{j_u}^{\alpha}(A)}{r_{j_u}(A)} \\ > &|a_{i_t j_u}| r_{j_u}(A) \frac{r_{j_u}^{\alpha}(A)}{r_{j_u}(A)} \\ = &|a_{i_t j_u}| r_{j_u}^{\alpha}(A) = |a_{i_t j_u}| r_{k+1}(C_u). \end{aligned}$$

That is, $k+1 \in \Gamma_t(C_u)$. If $j_u \notin \Gamma_{i_t}(A)$, either there exists $s \in \langle k \rangle$ $(s \neq t)$ such that $i_s \in \Gamma_{i_t}(A)$, or there exists $v \in \langle l \rangle$ $(v \neq u)$ such that $j_v \in \Gamma_{i_t}(A)$. If the former situation holds, we have

$$\left(|a_{i_t i_t}| - r_{i_t}^{\langle n \rangle \setminus \{i_s\}}(A)\right)|a_{i_s i_s}| > |a_{i_t i_s}| r_{i_s}(A).$$

Then it holds that

$$\begin{aligned} &\left(|a_{i_t i_t}| - r_t^{\langle k+1 \rangle \setminus \{s\}}(C_u)\right)|a_{i_s i_s}| \\ = &\left(|a_{i_t i_t}| - r_{i_t}^{(\alpha \cup \{j_u\}) \setminus \{i_s\}}(A)\right)|a_{i_s i_s}| \\ \geqslant &\left(|a_{i_t i_t}| - r_{i_t}^{\langle n \rangle \setminus \{i_s\}}(A)\right)|a_{i_s i_s}| \\ > &|a_{i_t i_s}| r_{i_s}(A) \geqslant |a_{i_t i_s}| r_{i_s}^{\alpha \cup \{j_u\}}(A) = |a_{i_t i_s}| r_s(C_u). \end{aligned}$$

Hence, $s \in \Gamma_t(C_u)$. If the latter situation holds, by lemma 2.4, we have

$$|a_{i_t i_t}| > r_{i_t}^{\langle n\rangle\setminus\{j_v\}}(A) \geqslant r_{i_t}^{\alpha}(A) + |a_{i_t j_u}| = r_t(C_u),$$

which implies that $t \in N^+(C_u)$. Then $C_u \in DZT_{k+1}$ by Lemma 1.1.

By Lemma 2.2, we have $det(\mu[A(\alpha)]) > 0$ and $det(C_u) > 0$. By Lemma 2.3, it holds that

$$det(C_u) = det(\mu[A(\alpha)])\big(\gamma_u^* - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u|\big),$$

which implies that (3.5) holds.

(ii) Let

$$D_u := \begin{bmatrix} \mu[A(\alpha)] & -|y_u| \\ -|x_u^T| & |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A) \end{bmatrix}.$$

Since $j_u \in N^-(A)$, then $|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A) \leqslant r_{j_u}^{\alpha}(A)$. Hence we have $k+1 \in N^-(D_u)$. For any $t \in \langle k\rangle$, if $i_t \in N^+(A)$, then $t \in N^+(D_u)$. If $i_t \in N^-(A)$ with $\Gamma_{i_t}(A) \cap \alpha = \emptyset$, there exists $v \in \langle l\rangle$ such that $j_v \in \Gamma_{i_t}(A)$. Since $j_u \in N^-(A)$, we have $v \neq u$. Then

$$|a_{i_t i_t}| > r_{i_t}^{\langle n\rangle\setminus\{j_v\}}(A) \geqslant r_{i_t}^{\alpha\cup\{j_u\}}(A) = r_t(A(D_u)).$$

We have $t \in N^+(D_u)$. Hence, it holds that

$$\begin{aligned} N^+(D_u) &\supseteq \{t \in \langle k\rangle : i_t \in N^+(A), \text{ or } i_t \in N^-(A) \text{ with } \Gamma_{i_t}(A) \cap \alpha = \emptyset\}, \\ N^-(D_u) &\subseteq \{k+1\} \cup \{t \in \langle k\rangle : i_t \in N^-(A) \text{ with } \Gamma_{i_t}(A) \cap \alpha \neq \emptyset\}. \end{aligned}$$

Denote

$$\Lambda = \{k+1\} \cup \{t \in \langle k\rangle : i_t \in N^-(A) \text{ with } \Gamma_{i_t}(A) \cap \alpha \neq \emptyset\}.$$

To prove D_u is a DZT matrix, by Lemma 1.1, it is sufficient to show that $\Gamma_t(D_u) \neq \emptyset$ for any $t \in \Lambda$. For $t \in \langle k\rangle$ with $i_t \in N^-(A)$ and $\Gamma_{i_t}(A) \cap \alpha \neq \emptyset$, there exists $s \in \langle k\rangle$ $(s \neq t)$ such that $i_s \in \Gamma_{i_t}(A)$, using the similar deduction of (i), it holds that $\Gamma_t(D_u) \neq \emptyset$. Simultaneously, since $\Gamma_{j_u}(A) \cap \alpha \neq \emptyset$, there exists $i_s \in \alpha$ such that $i_s \in \Gamma_{j_u}(A)$. Hence, by Lemma 2.4, we have

$$|a_{j_u j_u}| > r_{j_u}^{\langle n\rangle - \{i_s\}}(A) \geqslant r_{j_u}^{\bar{\alpha}}(A),$$

and

$$\begin{aligned} &\big((|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A)) - r_{k+1}^{\langle k+1\rangle\setminus\{s\}}(D_u)\big)|a_{i_s i_s}| \\ &= \big(|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A) - r_{j_u}^{\alpha\setminus\{i_s\}}(A)\big)|a_{i_s i_s}| \\ &= \big(|a_{j_u j_u}| - r_{j_u}^{\langle n\rangle\setminus\{i_s\}}(A)\big)|a_{i_s i_s}| \\ &> |a_{j_u i_s}| r_{i_s}(A) \quad (\text{by } i_s \in \Gamma_{j_u}(A)) \\ &\geqslant |a_{j_u i_s}| r_{i_s}^{\alpha\cup\{j_u\}}(A) = |a_{j_u i_s}| r_s(D_u). \end{aligned}$$

It follows that $\Gamma_{k+1}(D_u) \neq \emptyset$. Then we have $D_u \in DZT_{k+1}$ by Lemma 1.1. By the similar deduction of (i) again, we get $|x_u^T|[\mu(A(\alpha))]^{-1}|y_u| < |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A)$. And $|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A) \leqslant r_{j_u}^{\alpha}(A)$ holds trivially since $j_u \in N^-(A)$. Hence, (3.6) holds. $\square$

LEMMA 3.3. *Let $A \in DZT_n$ and let α and $\bar{\alpha}$ be defined as in (2.1)–(2.2), respectively. Then*

$$N^+(A/_\circ\alpha) \supseteq \{u \in \langle l\rangle : j_u \in N^+(A), \quad or \quad j_u \in N^-(A) \quad with \quad \Gamma_{j_u}(A) \cap \alpha \neq \emptyset\}.$$

Proof. Denote $A/_\circ\alpha = (a'_{uv})$. We first consider the case $j_u \in N^+(A)$. If $r^{\alpha}_{j_u}(A) = 0$, then $x_u = 0$. It follows from (3.4) that

$$|a'_{uu}| = |a_{j_u j_u}| > r_{j_u}(A) = r^{\bar{\alpha}}_{j_u}(A) = r_u(A/_\circ\alpha),$$

which implies that $u \in N^+(A/_\circ\alpha)$. If $r^{\alpha}_{j_u}(A) \neq 0$, we have

$$\begin{aligned}
|a'_{uu}| &\geqslant |a_{j_u j_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| \\
&> |a_{j_u j_u}| - \frac{|a_{j_u j_u}|}{r_{j_u}(A)} r^{\alpha}_{j_u}(A) \quad \text{(by (3.5))} \\
&= \frac{|a_{j_u j_u}|}{r_{j_u}(A)}(r_{j_u}(A) - r^{\alpha}_{j_u}(A)) \\
&= \frac{|a_{j_u j_u}|}{r_{j_u}(A)} r_u(A/_\circ\alpha) \quad \text{(by (3.4))} \\
&\geqslant r_u(A/_\circ\alpha),
\end{aligned}$$

which implies that $u \in N^+(A/_\circ\alpha)$.

Now we consider the case $j_u \in N^-(A)$ with $\Gamma_{j_u}(A) \cap \alpha \neq \emptyset$. By (3.6), we have

$$|a'_{uu}| \geqslant |a_{j_u j_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| > |a_{j_u j_u}| - (|a_{j_u j_u}| - r^{\bar{\alpha}}_{j_u}(A)) = r_u(A/_\circ\alpha),$$

which implies that $u \in N^+(A/_\circ\alpha)$. $\square$

Given a DZT matrix A, denote

$$\Theta = \{u \in \langle l\rangle : j_u \in N^-(A) \text{ with } \Gamma_{j_u}(A) \cap \alpha = \emptyset\}. \tag{3.8}$$

THEOREM 3.1. *Let $A \in DZT_n$ and $\emptyset \neq \alpha \subset \langle n\rangle$. Let Θ be defined in (3.8). Then*
(i) $A/_\circ\alpha \in SDD_{|\bar{\alpha}|}$ if $\Theta = \emptyset$;
(ii) $A/_\circ\alpha \in DZT_{|\bar{\alpha}|}$ if $\Theta \neq \emptyset$. Particularly, $A/_\circ\alpha \in DZ_{|\bar{\alpha}|} \cap DZT_{|\bar{\alpha}|}$ if $\bigcap_{u\in\Theta} \Gamma_{j_u}(A) \neq \emptyset$.

Proof. (i) If $\Theta = \emptyset$, then $A/_\circ\alpha \in SDD_{|\bar{\alpha}|}$ holds trivially by Lemma 3.3.

(ii) If $\Theta \neq \emptyset$, by Lemma 3.3, we have $N^-(A/_\circ\alpha) \subseteq \Theta$. For any $u \in \Theta$, it holds that $\Gamma_{j_u}(A) \subseteq \bar{\alpha}$ since $\Gamma_{j_u}(A) \cap \alpha = \emptyset$. Suppose that there is an index $v \in \langle l\rangle$ $(v \neq u)$ such that $j_v \in \Gamma_{j_u}(A)$. Then $j_v \in N^+(A)$ by Lemma 2.4. Denote $A/_\circ\alpha = (a'_{uv})$. It holds that

$$\begin{aligned}
&\left(|a'_{uu}| - r_u^{\langle l\rangle\backslash\{v\}}(A/_\circ\alpha)\right)|a'_{vv}| \qquad (3.9)\\
&= \left(|a'_{uu}| - r_{j_u}^{\bar{\alpha}\backslash\{j_v\}}(A)\right)|a'_{vv}|\\
&\geqslant \left(|a_{j_uj_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| - r_{j_u}^{\langle n\rangle\backslash\{j_v\}}(A) + r_{j_u}^{\alpha}(A)\right)\left(|a_{j_vj_v}| - |x_v^T|[\mu(A(\alpha))]^{-1}|y_v|\right)\\
&\geqslant \left(|a_{j_uj_u}| - r_{j_u}^{\langle n\rangle\backslash\{j_v\}}(A)\right)\left(|a_{j_vj_v}| - |x_v^T|[\mu(A(\alpha))]^{-1}|y_v|\right) \quad \text{(by (3.6))}\\
&> \left(|a_{j_uj_u}| - r_{j_u}^{\langle n\rangle\backslash\{j_v\}}(A)\right)\left(|a_{j_vj_v}| - \frac{|a_{j_vj_v}|}{r_{j_v}(A)} r_{j_v}^{\alpha}(A)\right) \quad \text{(by (3.5))}\\
&= \left(|a_{j_uj_u}| - r_{j_u}^{\langle n\rangle\backslash\{j_v\}}(A)\right)|a_{j_vj_v}|\frac{r_{j_v}^{\bar{\alpha}}(A)}{r_{j_v}(A)}\\
&\geqslant |a_{j_uj_v}| r_{j_v}(A)\frac{r_{j_v}^{\bar{\alpha}}(A)}{r_{j_v}(A)} \quad \text{(by } j_v \in \Gamma_{j_u}(A))\\
&= |a'_{uv}| r_v(A/_\circ\alpha).
\end{aligned}$$

That is, $\Gamma_u(A/_\circ\alpha) \neq \emptyset$ for any $u \in \Theta$. Hence, we have $A/_\circ\alpha \in DZT_{|\bar{\alpha}|}$ by Lemma 1.1.

If $\bigcap_{u\in\Theta} \Gamma_{j_u}(A) \neq \emptyset$, then $\bigcap_{u\in\Theta} \Gamma_{j_u}(A) \subseteq \bar{\alpha}$ since $\Gamma_{j_u}(A) \cap \alpha = \emptyset$ for each $u \in \Theta$. Thus there is an index $j_v \in N^+(A)$ such that $j_v \in \bigcap_{u\in\Theta} \Gamma_{j_u}(A)$. Then, $v \in \Gamma_u(A/_\circ\alpha)$ for all $u \in \Theta$ by using the similar deduction with (3.9). Hence, we have $A/_\circ\alpha \in DZ_{|\bar{\alpha}|}$ and $A/_\circ\alpha \in DZT_{|\bar{\alpha}|}$. The proof is completed. □

By Theorem 3.1, we can obtain the following results directly.

COROLLARY 3.1. *Let $A \in DZT_n$. For any $\emptyset \neq \alpha \subset \langle n\rangle$, we have $A/_\circ\alpha \in DZT_{|\bar{\alpha}|}$.*

COROLLARY 3.2. *Let $A \in DZT_n$ and $\emptyset \neq \alpha \subset \langle n\rangle$. If for each $i \in N^-(A)$, it holds that $\Gamma_i(A) \cap \alpha \neq \emptyset$, then $A/_\circ\alpha \in SDD_{|\bar{\alpha}|}$.*

Proof. It only need to note that $\Theta = \emptyset$. □

COROLLARY 3.3. *Let $A \in DZT_n$ and $\emptyset \neq \alpha \subset \langle n\rangle$. If $N^+(A) \subseteq \alpha$, then $A/_\circ\alpha \in SDD_{|\bar{\alpha}|}$.*

Proof. Since $N^+(A) \subseteq \alpha$, then $\Theta = \emptyset$. Hence $A/_\circ\alpha \in SDD_{|\bar{\alpha}|}$ by Theorem 3.1 (i). □

COROLLARY 3.4. *Let $A \in DZT_n$ and $\emptyset \neq \alpha \subset \langle n\rangle$. If $N^-(A) \subseteq \alpha$, then $A/_\circ\alpha \in SDD_{|\bar{\alpha}|}$.*

Proof. Since $N^-(A) \subseteq \alpha$, then $\bar{\alpha} \subseteq N^+(A)$, which implies that $\Theta = \emptyset$. Hence $A/_\circ\alpha \in SDD_{|\bar{\alpha}|}$ by Theorem 3.1 (i). □

REMARK 3.1. For a DZT matrix A, if α satisfies the conditions of Theorem 3.1 (ii) with $\bigcap_{u\in\Theta}\Gamma_{j_u}(A)=\emptyset$, $A/_\circ\alpha$ is not necessarily a DZ matrix. If α satisfies the conditions of Theorem 3.1 (ii) with $\bigcap_{u\in\Theta}\Gamma_{j_u}(A)\neq\emptyset$, $A/_\circ\alpha$ is not necessarily an SDD matrix. Here, an example is used to illustrate it and to show the correctness of Theorem 3.1.

EXAMPLE 3.1. Given a matrix,

$$A=\begin{bmatrix} 52 & 0 & 3 & 3 & 3 & 6\\ -5 & -18 & 1 & -4 & -5 & -5\\ -6 & -4 & -14 & 0 & 4 & 4\\ -5 & -5 & -1 & 15 & 1 & -4\\ 1 & 5 & -5 & 0 & -14 & 6\\ -3 & 2 & -2 & -2 & -5 & 34 \end{bmatrix}.$$

By computation, we have

$$N^+(A)=\{1,6\},\quad N^-(A)=\{2,3,4,5\},\quad \Gamma_1(A)=\{2,3,4,5,6\},$$

$$\Gamma_2(A)=\Gamma_4(A)=\{1,6\},\quad \Gamma_3(A)=\{1\};\quad \Gamma_5(A)=\{6\},\quad \Gamma_6(A)=\{1,2,3,4,5\}.$$

It is easy to see that A is a DZT but not a DZ matrix. There are $2^6-2=62$ nonempty proper index subsets of $\langle 6\rangle$, we calculate 62 diagonal-Schur complements of the matrix A (see Table 1). Among them, there are 32 nonempty proper index subsets satisfying $\Theta=\emptyset$ (the condition of Theorem 3.1 (i)) and the corresponding diagonal-Schur complements are all SDD matrices. There are 3 nonempty proper index subsets satisfying $\Theta\neq\emptyset$ and $\bigcap_{u\in\Theta}\Gamma_{j_u}(A)=\emptyset$ (the conditions of Theorem 3.1 (ii)), and the corresponding diagonal-Schur complements are all DZT matrices.

The rest 27 nonempty proper index subsets satisfying $\bigcap_{u\in\Theta}\Gamma_{j_u}(A)\neq\emptyset$ (the conditions of Theorem 3.1 (ii)), and the corresponding diagonal-Schur complements are all DZT and DZ matrices.

Remark that if α satisfies $\Theta\neq\emptyset$ and $\bigcap_{u\in\Theta}\Gamma_{j_u}(A)=\emptyset$ (the conditions of Theorem 3.1 (ii)), $A/_\circ\alpha$ is not necessarily a DZ matrix. Taking $\alpha=\{4\}$, then

$$A/_\circ\{4\}=\begin{bmatrix} 53 & 0 & 3 & 3 & 6\\ -5 & -19.3333 & 1 & -5 & -5\\ -6 & -4 & -14 & 4 & 4\\ 1 & 5 & -5 & -14 & 6\\ -3 & 2 & -2 & -5 & 33.4667 \end{bmatrix}:=B.$$

We have

$$N^+(B)=\{1,2,5\},\quad N^-(B)=\{3,4\},\quad \Gamma_1(B)=\{2,3,4,5\},$$

$$\Gamma_2(B)=\{1,3,4,5\},\quad \Gamma_3(B)=\{1\},\quad \Gamma_4(B)=\{5\},\quad \Gamma_5(B)=\{1,2,3,4\},$$

Table 1: *The closure properties of diagonal-Schur complements of the matrix A.*

α	Theorem	$A/_\circ\alpha$
$\{1,5\},\{1,6\},\{3,6\},\{1,2,5\},\{1,2,6\},\{1,3,5\}$, $\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,6\},\{3,4,6\}$, $\{1,2,3,5\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,4,6\}$, $\{1,3,4,5\},\{1,3,4,6\},\{1,3,5,6\},\{1,4,5,6\}$, $\{2,3,4,6\},\{2,3,5,6\},\{3,4,5,6\}$, $\{1,3,4,5,6\},\{1,2,3,5,6\},\{1,2,4,5,6\}$, $\{1,2,3,4,6\},\{1,3,6\},\{1,2,5,6\},\{2,3,4,5\}$, $\{3,5,6\},\{2,3,4,5,6\},\{1,2,3,4,5\}$	Theorem 3.1 (i)	SDD
$\{2\},\{4\},\{2,4\}$	Theorem 3.1 (ii), $\Theta \neq \emptyset$, $\bigcap_{u\in\Theta} \Gamma_{j_u}(A) = \emptyset$	DZT (not necessarily DZ)
$\{3\},\{5\},\{6\},\{1,2\},\{1,3\},\{1,4\},\{2,3\}$, $\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{4,5\},\{4,6\}$, $\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{2,3,5\}$, $\{1\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{4,5,6\}$, $\{2,4,5\},\{2,4,5,6\},\{5,6\},\{1,2,3,4\}$,	Theorem 3.1, (ii) $\bigcap_{u\in\Theta} \Gamma_{j_u}(A) \neq \emptyset$	DZT and DZ (not necessarily SDD)

and then $B = A/_\circ\{4\}$ is a DZT but not a DZ matrix. Simultaneously, if α satisfies $\bigcap_{u\in\Theta} \Gamma_{j_u}(A) \neq \emptyset$ (the conditions of Theorem 3.1 (ii)), $A/_\circ\alpha$ is not necessarily an *SDD* matrix. Taking $\alpha = \{1\}$, then

$$A/_\circ\{1\} = \begin{bmatrix} -18 & 1 & -4 & -5 & -5 \\ -4 & -13.6538 & 0 & 4 & 4 \\ -5 & -1 & 15.2885 & 1 & -4 \\ 5 & -5 & 0 & -14.0577 & 6 \\ 2 & -2 & -2 & -5 & 34.3462 \end{bmatrix}.$$

It is easy to check that $A/_\circ\{1\}$ is a DZ and DZT matrix but not an SDD matrix.

4. Conclusions

In this paper, we research diagonal Schur complements of DZT matrices by discussing the indices in $\bar{\alpha}$. Specifically, for a given DZT matrix $A = (a_{ij}) \in C^{n\times n}$ and a proper subset α of $\langle n\rangle$, we divide $\bar{\alpha}$ into three disjoint subsets, i.e., $\bar{\alpha} = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \bar{\alpha} \cap N^+(A), \\ S_2 &= \{j_u \in \bar{\alpha} \cap N^-(A) : \Gamma_{j_u}(A) \cap \alpha \neq \emptyset\}, \\ S_3 &= \{j_u \in \bar{\alpha} \cap N^-(A) : \Gamma_{j_u}(A) \cap \alpha = \emptyset\}. \end{aligned}$$

We prove that $u \in N^+(A/_\circ\alpha)$ if $j_u \in S_1 \cup S_2$, and $\Gamma_u(A/_\circ\alpha) \neq \emptyset$ for $j_u \in S_3$ (i.e., $u \in \Theta$). The main results are summarized as follows.

• $A(\alpha)$ is a DZT matrix.

• $A/_\circ\alpha$ is also a DZT matrix.

• $A/_\circ\alpha$ can be an SDD matrix under certain conditions. Specifically, $A/_\circ\alpha$ is an SDD matrix if $\Theta = \emptyset$. Particularly, $A/_\circ\alpha$ is an SDD matrix if $N^+(A) \subseteq \alpha$ or $N^-(A) \subseteq \alpha$.

• $A/_\circ\alpha$ can be also a DZ matrix under certain conditions. Specifically, $A/_\circ\alpha$ is also a DZ matrix if $\bigcap\limits_{u\in\Theta} \Gamma_{j_u}(A) \neq \emptyset$.

We believe that the method of discussing the indices in $\bar{\alpha}$ can be applied to the research on (diagonal) Schur complements for other matrix classes.

Acknowledgements. The work was supported by the National Natural Science Foundation of China (No. 12171323), Liaoning Provincial Department of Education Program (JYTMS20230281), and the Fundamental Research Funds for the Universities of Liaoning Province (20240218).

Declarations

Competing interests. The authors have no competing interests to declare that are relevant to the content of this article.

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(Received August 21, 2024)

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