Journal of Mathematical Inequalities
Volume 19, Number 3 (2025), 793–810 doi:10.7153/jmi-2025-19-51

SOME INEQUALITIES RELATED TO HARMONICALLY CONVEXITY WITH APPLICATIONS IN TSALLIS ENTROPY AND MEANS

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(*Communicated by M. Sababheh*)

Abstract. In this paper, we give some new inequalities related to harmonically convex functions. In fact, we introduce a certain function F including a harmonically convex function and using Jensen and Jensen-Mercer inequalities to obtain some new inequalities. Finally, we give some application for these inequalities in information theory, Tsallis entropy, mean theory and approximation theory.

1. Introduction and preliminaries

Convexity is one of the fundamental concepts in convex analysis. In fact, due to its simple understanding in terms of geometry, it has been of interest to many researchers. The most basic simple function that is introduced with the help of the concept of convexity is the convex function whose use goes back to the ancient Greeks [12]. The convex function has many applications in various sciences such as information theory, dynamic systems, physics, and probability theory (see [11, 17, 27, 28, 29, 30, 31, 32]). In addition, many inequalities have their roots in the concept of inequality. Nowadays, many authors have used this concept to construct and improve various inequalities (see [1, 3, 18, 24, 33]). One of the most important inequalities is Hermite–Hadamard which states that if $\phi : [a,b] \to \mathbb{R}$ is a convex function, then the following inequality holds:

$$\phi\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \phi(x)dx \leqslant \frac{\phi(a)+\phi(b)}{2} \tag{1.1}$$

which was first obtained Jaques Hadamard and Charles Hermite (see [14]). This inequality was used to approximate the average of convex functions and in fact, it says that the average value of a convex function is a value between the value of the function at the arithmetic mean of two points and the arithmetic mean of the value of the function from two points. In general, the researchers study

$$\phi(T(a,b)) \leqslant L(\phi(a),\phi(b))$$

Mathematics subject classification (2020): 37B40, 26B25, 94A17, 26D15, 26D20.
Keywords and phrases: Entropy, Tsallis entropy, Hermite–Hadamard inequality, harmonically convex, Jensen inequality and special means.
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Paper JMI-19-51

for a suitable pair of means T and L also function ϕ. Note that, a mean of two positive real numbers is defined as a function $M:(0,+\infty)\times(0,+\infty)\to(0,+\infty)$ with the property

$$\min\{a,b\}\leqslant M(a,b)\leqslant \max\{a,b\}$$

for each $a,b\in(0,+\infty)$. Hermite-Hadamard inequality in this case using convex functions ϕ when T and L are arithmetic mean is obtained. We mention here the arithmetic, geometric, harmonic and identric means of two numbers $a,b>0$ as

$$A(a,b)=\frac{a+b}{2},\quad G(a,b)=\sqrt{ab},\quad H(a,b)=\frac{2ab}{a+b}\quad\text{and}\quad I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

with $I(a,a):=a$ (see [21, 23]). Also, we use the symbol $S(a,b)$ for the adojoint of $I(a,b)$:

$$S(a,b):=I(a^{-1},b^{-1})^{-1}=e\left(\frac{a^b}{b^a}\right)^{\frac{1}{b-a}}.$$

Among other inequalities, Jensen's inequality was introduced in 1906 by Danish mathematician Johan Jensen, relates the value of a convex function which states that: If ϕ is convex on I, then

$$\phi\left(\sum_{i=1}^n p_i x_i\right)\leqslant\sum_{i=1}^n p_i\phi(x_i),$$

where $x_i\in I$ and $\sum_{i=1}^n p_i=1$ (see [20]).

Let ϕ be a harmonically convex function on I, $0\notin I$; $x_i\in I$; $i=1,\ldots,n$. Then the following Jensen type inequality holds (see [9]):

$$\phi\left(\left(\sum_{i=1}^n t_i x_i^{-1}\right)^{-1}\right)\leqslant\sum_{i=1}^n t_i\phi(x_i) \tag{1.2}$$

where $\sum_{i=1}^n t_i=1$ and $t_i\geqslant 0$.

Also, it has been used in the approximation of the expected value as well as the approximation of entropy. Various modifications have also been presented in [2, 5, 34]. Jensen-Mercer's inequality was introduced in 2003 by A. A. Mercer (see [22]) as follows. If h is a convex function on an interval $I:=[a,b]$, $x_i\in I$, $1\leqslant i\leqslant n$ and $\sum_{i=1}^n p_i=1$, $p_i\geqslant 0$, then

$$h\left(a+b-\sum_{i=1}^n p_i x_i\right)+\sum_{i=1}^n p_i h(x_i)\leqslant h(a)+h(b). \tag{1.3}$$

Generalizations for convex functions in operator form are also seen in the references [10, 13]. Therefore, due to the importance of convexity, the authors are interested in introducing various categories of functions that are related to the concept of convexity, for example, m-convex functions, harmonically convex, uniformly convex, p-convex, s-convex, logarithmic convex and etc. (see [3, 4, 15, 16, 19, 24, 35]).

In [26], Rooin proved some aspects of convex functions and some practical inequalities using the following function:

$$G:[0,1]\to\mathbb{R}$$
$$G(t)=\frac{\sum_{i=1}^{n} g\left(\sum_{j=1}^{n} a_{ij}(t)x_j\right)}{n}$$

where $a_{ij}:[0,1]\to\mathbb{R}$ are affine mappings for every i,j $(1\leqslant i,j\leqslant n)$, $x_1,\cdots,x_n$ are elements in a vector space X and $g:X\to\mathbb{R}$ is a convex function.

In this article, the goal is to obtain new inequalities with the help of harmonically convex functions, and finally, with the help of these results, we approximated some values such as the Napier number, as well as the approximation of arithmetic averages and harmonic averages, and the relationship between them, along with other applications of them.

The following definition is well known in the literature.

DEFINITION 1.1. [19] Let $f:[a,b]\longrightarrow\mathbb{R}$ be a function and $0\notin[a,b]$. Then f is harmonically convex, if

$$f\left(\frac{xy}{(1-t)x+ty}\right)\leqslant(1-t)f(y)+tf(x) \tag{1.4}$$

for every $t\in[0,1]$ and $x,y\in[a,b]$. If the inequality is reversed, f is harmonically concave.

THEOREM 1.2. [7] *Assume that* $f:[a,b]\to\mathbb{R}$ *is a harmonically convex function then*

$$f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\sum_{i=1}^{n}\frac{t_i}{a_i}}\right)\leqslant f(a)+f(b)-\sum_{i=1}^{n}t_if(a_i)$$

for all $a_i\in[a,b]$; $0\notin[a,b]$; $t_i\in[0,1]$; $\sum_{i=1}^{n}t_i=1$.

2. Main results

In the following we begin the definition of F related to harmonically convex function.

DEFINITION 2.1. Assume that $[a,b]$ is an interval such that $0\notin[a,b]$, f is a harmonically convex and $a\leqslant x_i\leqslant b$; $i=1,\cdots,n$. We define the function $F:[0,1]\to\mathbb{R}$ by

$$F(t):=\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{1}{\frac{1-t}{x_i}+\frac{t}{x_{n+1-i}}}\right).$$

THEOREM 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a harmonically function, and $F(t)$ be as in Definition 2.1. Then*

$$f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right)\leqslant F(t)\leqslant\sum_{i=1}^{n}\frac{f(x_i)}{n} \tag{2.1}$$

for every $a\leqslant x_i\leqslant b;\ i=1,\cdots,n$.

Proof. Since f is harmonically convex,

$$f\left(\frac{1}{\frac{1-t}{x_i}+\frac{t}{x_{n+1-i}}}\right)\leqslant tf(x_{n+1-i})+(1-t)f(x_i).$$

Also,

$$\sum_{i=1}^{n}[f(x_{n+1-i})-f(x_i)]=0.$$

Using definition $F(t)$ and (1.2), we have

$$\begin{aligned}F(t)&=\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{1}{\frac{1-t}{x_i}+\frac{t}{x_{n+1-i}}}\right)\\&\geqslant f\left(\frac{n}{\sum_{i=1}^{n}\left(\frac{1-t}{x_i}+\frac{t}{x_{n+1-i}}\right)}\right)=f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right),\end{aligned}$$

because $\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{x_{n+1-i}}\right)=0$, which implies the right-hand side of inequality (2.1).

On the other hand, in view of harmonically convex f, we have

$$\begin{aligned}F(t)&=\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{1}{\frac{1-t}{x_i}+\frac{t}{x_{n+1-i}}}\right)\\&\leqslant\frac{1}{n}\sum_{i=1}^{n}[(1-t)f(x_i)+tf(x_{n+1-i})]\\&=\frac{1}{n}\sum_{i=1}^{n}f(x_i),\end{aligned}$$

since $\sum_{i=1}^{n}f(x_i)=\sum_{i=1}^{n}f(x_{n+1-i})$. □

Another bound related to the function F as follows.

THEOREM 2.3. *If $f:[a,b]\to\mathbb{R}$ is a harmonically convex function, then*

$$F(t)\leqslant f(a)+f(b)-\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x_i}}\right)$$

Proof. Using Theorem 1.2, we get

$$f\left(\frac{1}{\frac{1-t}{x_i}+\frac{t}{x_{n+1-i}}}\right)=f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-(1-t)(\frac{1}{a}+\frac{1}{b}-\frac{1}{x_i})-t(\frac{1}{a}+\frac{1}{b}-\frac{1}{x_{n+1-i}})}\right)$$
$$\leqslant f(a)+f(b)-(1-t)f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x_i}}\right)-tf\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x_{n+1-i}}}\right) \tag{2.2}$$

for each $1\leqslant i\leqslant n$, we use this fact that

$$\sum_{i=1}^{n} f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x_i}}\right)=\sum_{i=1}^{n} f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x_{n+1-i}}}\right).$$

With summing on inequalities (2.2) from $i=1$ to $i=n$ and by division n we get

$$F(t)\leqslant f(a)+f(b)-\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x_i}}\right),$$

which completes the proof. □

COROLLARY 2.4. *In view of Theorems 2.2, 2.3, we arrive at*

(1) $f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}\right)\leqslant \int_0^1 F(t)dt\leqslant \sum_{i=1}^{n}\frac{f(x_i)}{n}$.

(2) $\int_0^1 F(t)dt\leqslant f(a)+f(b)-\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x_i}}\right)$.

In the next theorem we obtain some lower bounds for $F(t)$ and $\int_0^1 F(t)dt$.

THEOREM 2.5. *Assume that f is a harmonically convex function then*

(1) $F(t)\geqslant \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2}{\frac{1}{x_i}+\frac{1}{x_{n+1-i}}}\right)$

(2) $\int_0^1 F(t)dt\geqslant \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2}{\frac{1}{x_i}+\frac{1}{x_{n+1-i}}}\right)$

Proof. (1): For each $0\leqslant t\leqslant 1$ we have

$$f\left(\frac{2}{\frac{1}{x_i}+\frac{1}{x_{n+1-i}}}\right)=f\left(\frac{1}{\frac{1}{2}\left(\frac{t}{x_i}+\frac{1-t}{x_{n+1-i}}\right)+\frac{1}{2}(\frac{t}{x_{n+1-i}}+\frac{1-t}{x_i})}\right)$$
$$\leqslant \frac{1}{2}\left[f\left(\frac{1}{\frac{t}{x_i}+\frac{1-t}{x_{n+1-i}}}\right)+f\left(\frac{1}{\frac{t}{x_{n+1-i}}+\frac{1-t}{x_i}}\right)\right].$$

Now, with summing above inequalities from $i=1$ to $i=n$, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2}{\frac{1}{x_i}+\frac{1}{x_{n+1-i}}}\right) \geqslant F(t).$$

Note that, $\sum_{i=1}^{n} f\left(\frac{1}{\frac{t}{x_i}+\frac{1-t}{x_{n+1-i}}}\right) = \sum_{i=1}^{n} f\left(\frac{1}{\frac{1-t}{x_i}+\frac{t}{x_{n+1-i}}}\right)$.

(2): It is clear from (1). □

In the next, we give some properties of the function F.

THEOREM 2.6. *Assume that f is a harmonically convex function then*

(1) $F(t)$ *is a convex function on* $[0,1]$.

(2) $\max\limits_{t\in[0,1]} F(t) = F(0) = F(1)$.

(3) $\min\limits_{t\in[0,\frac{1}{2}]} F(t) = F(\frac{1}{2}) = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2}{\frac{1}{x_i}+\frac{1}{x_{n+1-i}}}\right)$.

Proof. (1) Let $x,y\in[0,1]$, $t\in[0,1]$. Then

$$\begin{aligned}
F(tx+(1-t)y) &= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{\frac{tx+(1-t)y}{x_i}+\frac{1-tx-(1-t)y}{x_{n+1-i}}}\right)\\
&= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{t\left(\frac{x}{x_i}+\frac{1-x}{x_{n+1-i}}\right)+(1-t)\left(\frac{y}{x_i}+\frac{1-y}{x_{n+1-i}}\right)}\right)\\
&\leqslant \frac{1}{n}\sum_{i=1}^{n}\left[tf\left(\frac{1}{\frac{x}{x_i}+\frac{1-x}{x_{n+1-i}}}\right)+(1-t)f\left(\frac{1}{\frac{y}{x_i}+\frac{1-y}{x_{n+1-i}}}\right)\right]\\
&= tF(x)+(1-t)F(y)
\end{aligned}$$

which implies that F is convex.

(2) In view of Theorem 2.2, we have

$$F(t) \leqslant \sum_{i=1}^{n}\frac{f(x_i)}{n} = F(1) = F(0).$$

(3) If $t \in [0, \frac{1}{2}]$ then

$$\begin{aligned}
F\left(\frac{1}{2}-t\right) &= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{\frac{\frac{1}{2}-t}{x_i}+\frac{\frac{1}{2}+t}{x_{n+1-i}}}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{\frac{\frac{1}{2}+t}{x_i}+\frac{\frac{1}{2}-t}{x_{n+1-i}}}\right) \\
&= F\left(\frac{1}{2}+t\right).
\end{aligned}$$

Let us proceed by contradiction and assume that $\frac{1}{2}$ is not minimum of $F(t)$ on $[0, \frac{1}{2}]$. So, there exists a $t \in (0, \frac{1}{2}]$ such that

$$F\left(\frac{1}{2}-t\right) = F\left(\frac{1}{2}+t\right) < F\left(\frac{1}{2}\right).$$

Also, F is convex hence we have

$$\begin{aligned}
F\left(\frac{1}{2}\right) &= F\left(\frac{1}{2}\left(\frac{1}{2}-t\right)+\frac{1}{2}\left(\frac{1}{2}+t\right)\right) \\
&\leqslant \frac{1}{2}F\left(\frac{1}{2}-t\right)+\frac{1}{2}F\left(\frac{1}{2}+t\right) \\
&< F\left(\frac{1}{2}\right)
\end{aligned}$$

which is a contradiction. So, $\min\limits_{t\in[0,\frac{1}{2}]} F(t) = F\left(\frac{1}{2}\right)$. □

Using Hermite-Hadamard inequality we obtain the following result.

COROLLARY 2.7. *From Theorem 2.6 and Hermite-Hadamard inequality, we get*

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2}{\frac{1}{x_i}+\frac{1}{x_{n+1-i}}}\right) \leqslant \int_0^1 F(t)dt \leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

Using Mercer inequality we obtain the following result.

COROLLARY 2.8. *From convexity $F(t)$ on $[0,1]$ and Mercer inequality we have*

$$\sum_{i=1}^{n} f\left(\frac{1}{\frac{\sum_{j=1}^{m} p_j t_j}{x_i}+\frac{\sum_{j=1}^{m} p_j(1-t_j)}{x_{n+1-i}}}\right) \leqslant 2\sum_{i=1}^{n} f(x_i) - n\sum_{i=1}^{n}\sum_{j=1}^{m} p_j f\left(\frac{1}{\frac{1-t_j}{x_i}+\frac{t_j}{x_{n+1-i}}}\right).$$

for every $0 \leqslant t_j \leqslant 1,\ 1 \leqslant j \leqslant m,\ 0 \leqslant p_j \leqslant 1,\ 1 \leqslant j \leqslant m,\ \sum_{j=1}^{m} p_j = 1.$

COROLLARY 2.9. *By the use of convexity F on $[0,1]$ and*

$$\min_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right), \quad \max_{t\in[0,1]} F(t) = F(0) = F(1)$$

we conclude that F is decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$.

3. Applications

In this section, we give some applications in information theory, entropy bounds, mean theory and approximation theory. In this section, we treat the function log as the logarithmic function of base e.

3.1. Applications in information theory

We use r-logarithmic function defined by $\log_r x := \dfrac{x^r - 1}{r}$, $(x > 0,\ r \neq 0)$ in this subsection. The function $\log_r x$ uniformly converges to the usual logarithmic function $\log x$ in the limit $r \to 0$.

DEFINITION 3.1. For a probability distribution $\mathbf{P} = \{p_i\}_{i=1}^n$ the Tsallis entropy $T_r(\mathbf{P})$ is defined by (see [36])

$$T_r(\mathbf{P}) := -\sum_{i=1}^n p_i^{1-r} \log_r p_i, \ \ (r \neq 0)$$

and the Shannon entropy of $\mathbf{P}$ is defined by

$$H(\mathbf{P}) := -\sum_{i=1}^n p_i \log p_i.$$

It is standard to use the convention $0 \log 0 \equiv 0$ in information theory [8]. However, to avoid such a case, we consider the positive probability distribution $\mathbf{P} := \{p_1, \cdots, p_n\}$ such that $p_i > 0$ for all $i = 1, \cdots, n$ and $\sum\limits_{i=1}^n p_i = 1$ in the sequel.

DEFINITION 3.2. For a positive probability distribution $\mathbf{P} = \{p_i\}_{i=1}^n$, we define another positive probability distribution $\mathbf{Q}_t(\mathbf{P}) = \{q_i\}_{i=1}^n$ by

$$q_i := (1-t)p_i + t p_{n+1-i}$$

for every $t \in [0,1]$. In the special case we have

$$\mathbf{Q}_{\frac{1}{2}}(\mathbf{P}) = \left\{ \frac{p_i + p_{n+1-i}}{2} \right\}.$$

LEMMA 3.3. *Let* $a,b>0$ *and* $f_r(x) := \frac{1}{x^{1-r}}\log_r \frac{1}{x}$. *If* $0 \neq r \leqslant 1$, *then the function* $f_r(x)$ *is harmonically convex. If* $r \geqslant 1$, *then* $f_r(x)$ *is harmonically concave.*

Proof. Let $g_r(t,a,b) := (1-t)f_r(b)+tf_r(a)-f_r\left(\frac{1}{(1-t)b^{-1}+ta^{-1}}\right)$ for $a,b>0$, $r \leqslant 1$ and $0 \leqslant t \leqslant 1$ where $f_r(x) := \frac{1}{x^{1-r}}\log_r \frac{1}{x}$. Since $g_r(0,a,b) = g_r(1,a,b) = 0$, it is sufficient to show $\frac{d^2 g_r(t,a,b)}{dt^2} \leqslant 0$. Indeed it is calculated in the following:

$$\frac{d^2 g_r(t,a,b)}{dt^2} = \frac{(r-1)(a-b)^2\left(\frac{ab}{(1-t)a+tb}\right)^r}{ab\{(1-t)a+tb\}} \leqslant 0$$

which implies $g_r(t,a,b) \geqslant 0$. If $r \geqslant 1$, then $\frac{d^2 g_r(t,a,b)}{dt^2} \geqslant 0$ which implies $g_r(t,a,b) \leqslant 0$. □

COROLLARY 3.4. *Let* $a,b>0$. *Then the function* $f(x) = -\frac{1}{x}\log x$ *is harmonically convex on* $[a,b]$.

Proof. Since $\log_r x$ uniformely convergences to $\log x$ when $r \to 0$, we see $\lim_{r\to 0} f_r(x) = -\frac{1}{x}\log x$. Thus, by Lemma 3.3, we conclude that $f(x) = -\frac{1}{x}\log x$ is harmonically convex. □

THEOREM 3.5. *Let* $0 \neq r \leqslant 1$ *and* $\mathbf{P} = \{p_i\}_{i=1}^n$ *be a positive probability distribution. Then*

$$T_r(\mathbf{P}) \leqslant T_r(\mathbf{Q}_t(\mathbf{P})) \leqslant \log_r n$$

for every $t \in [0,1]$.

Proof. If $f(x) := f_r(x) = \frac{1}{x^{1-r}}\log_r \frac{1}{x}$ and $x_i := \frac{1}{p_i}$; $i = 1,\ldots,n$, then

$$\begin{aligned} F(t) &= \frac{1}{n}\sum_{i=1}^{n} \frac{[p_i(1-t)+p_{n+1-i}t]^r - 1}{r[p_i(1-t)+p_{n+1-i}t]^{r-1}} \\ &= \frac{-1}{n} T_r(\mathbf{Q}_t(\mathbf{P})). \end{aligned} \tag{3.1}$$

Set $f(x) = f_r(x)$ and $x_i := \frac{1}{p_i}$; $i = 1,\ldots,n$ in Theorem 2.2, we get

$$n^{r-1}\log_r \frac{1}{n} \leqslant \frac{-T_r(\mathbf{Q}_t(\mathbf{P}))}{n} \leqslant \frac{-T_r(\mathbf{P})}{n}$$

for every $t \in [0,1]$ and $0 \neq r \leqslant 1$. This completes the proof. □

COROLLARY 3.6. *Let* $\mathbf{P} = \{p_i\}_{i=1}^n$ *be a positive probability distribution. Then*

$$H(\mathbf{P}) \leqslant H(\mathbf{Q}_t(\mathbf{P})) \leqslant \log n$$

for every $t \in [0,1]$.

Proof. An immediate consequence of Theorem 3.5, since

$$\lim_{r \to 0} T_r(\mathbf{P}) = H(\mathbf{P})$$

for every positive probability distribution $\mathbf{P}$. □

This corollary has the following useful proposition:

PROPOSITION 3.7. *Let* $\mathbf{P} = \{p_i\}_{i=1}^n$ *be a positive probability distribution. Then*

$$H(\mathbf{P}) \leqslant \sum_{i=1}^n p_i \log\left(\frac{2}{p_i + p_{n+1-i}}\right) \leqslant \log n$$

for every $t \in [0,1]$.

Proof. A consequence of Corollary 3.6 with $t = \frac{1}{2}$, because

$$\sum_{i=1}^n p_i \log\left(\frac{p_i + p_{n+1-i}}{2}\right) = \sum_{i=1}^n p_{n+1-i} \log\left(\frac{p_i + p_{n+1-i}}{2}\right).$$

This completes the proof of the proposition. □

EXAMPLE 3.8. Let $\mathbf{P} = \left\{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right\}$ and $H(\mathbf{P}) = \frac{2}{3}\log 2 + \frac{1}{2}\log 3$. On the other hand, we have

$$\mathbf{Q}_t(\mathbf{P}) = \left\{\frac{1}{6} + \frac{1}{3}t, \frac{1}{3}, \frac{1}{2} - \frac{1}{3}t\right\}.$$

Thus

$$H(\mathbf{Q}_t(\mathbf{P})) = \left(\frac{2t+1}{6}\right)\log\left(\frac{6}{2t+1}\right) + \left(\frac{3-2t}{6}\right)\log\left(\frac{6}{3-2t}\right) + \frac{1}{3}\log 3.$$

Therefore, by Proposition 3.7

$$\frac{2}{3}\log 2 + \frac{1}{6}\log 3 \leqslant \frac{2t+1}{6}\log\left(\frac{6}{2t+1}\right) + \frac{3-2t}{6}\log\left(\frac{6}{3-2t}\right) \leqslant \frac{2}{3}\log 3$$

for every $t \in [0,1]$.

Figure 1 illustrates the relations among $H(\mathbf{P})$, $H(\mathbf{Q_t}(\mathbf{P}))$ and $\log n$ for the above example. The dashed line, the solid curve and the dotted line represent $H(\mathbf{P})$, $H(\mathbf{Q_t}(\mathbf{P}))$ and $\log n$, respectively.

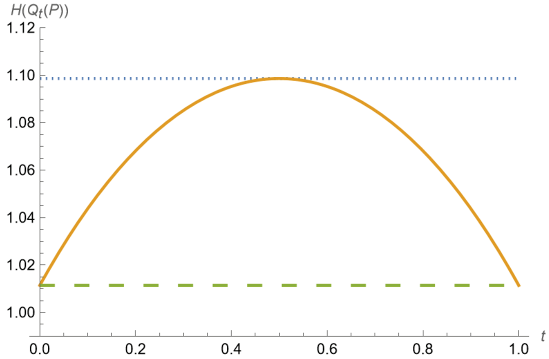

Figure 1: $H(\mathbf{P})$, $H(\mathbf{Q}_t(\mathbf{P}))$ *and* $\log n$ *in Example* 3.8.

THEOREM 3.9. *Let* $\mathbf{P}=\{p_i\}_{i=1}^n$ *be a positive probability distribution,* $t\in[0,1]$ *and* $r\leqslant 1$*;* $r\neq 0$*. Then*

$$T_r(\mathbf{Q}_t(\mathbf{P}))\leqslant T_r\left(\mathbf{Q}_{\frac{1}{2}}(\mathbf{P})\right).$$

Proof. Set $f(x)=f_r(x)=\dfrac{1}{x^{1-r}}\log_r\dfrac{1}{x}$ and $x_i:=\frac{1}{p_i}$; $i=1,\dots,n$ in Theorem 2.5 by (3.1), we get

$$\frac{-T_r(\mathbf{Q}_t(\mathbf{P}))}{n}\geqslant\frac{1}{n}\sum_{i=1}^n\frac{\left(\frac{p_i+p_{n+1-i}}{2}\right)^r-1}{r\left(\frac{p_i+p_{n+1-i}}{2}\right)^{r-1}}=-\frac{1}{n}T_r\left(\frac{p_i+p_{n+1-i}}{2}\right)$$

for every $t\in[0,1]$. This completes the proof. □

COROLLARY 3.10. *Let* $\mathbf{P}=\{p_i\}_{i=1}^n$ *be a positive probability distribution. Then*

$$H(\mathbf{Q}_t(\mathbf{P}))\leqslant H\left(\mathbf{Q}_{\frac{1}{2}}(\mathbf{P})\right)$$

for every $t\in[0,1]$.

Proof. It follows from Theorem 3.9 as $r\to 0$. □

Theorems 3.5 and 3.9, yield the following result.

COROLLARY 3.11. *Let* $r\leqslant 1$, $r\neq 0$ *and* $\mathbf{P}=\{p_i\}_{i=1}^n$ *be a positive probability distribution. Then*

$$\max_{0\leqslant t\leqslant 1}\{T_r(\mathbf{Q}_t(\mathbf{P}))\}=T_r\left(\mathbf{Q}_{\frac{1}{2}}(\mathbf{P})\right)\quad \textit{and}\quad \min_{0\leqslant t\leqslant 1}\{T_r(\mathbf{Q}_t(\mathbf{P}))\}=T_r(\mathbf{P}).$$

Corollaries 3.6 and 3.10, yield the following result.

COROLLARY 3.12. *If* $\mathbf{P} = \{p_i\}_{i=1}^n$ *is a positive probability distribution, then*

$$\max_{0\leqslant t\leqslant 1}\{H(\mathbf{Q}_t(\mathbf{P}))\} = H\left(\mathbf{Q}_{\frac{1}{2}}(\mathbf{P})\right) \quad \textit{and} \quad \min_{0\leqslant t\leqslant 1}\{H(\mathbf{Q}_t(\mathbf{P}))\} = H(\mathbf{P}).$$

3.2. Applications in analysis

In this subsection, we apply the harmonically convexity of $\log x$ to obtain some inequalities for some means.

PROPOSITION 3.13. *Let* $0 < a \leqslant x_i \leqslant b$.

(1) The following extension of GH-inequality holds:

$$\frac{n}{\sum_{i=1}^n \frac{1}{x_i}} \leqslant \sqrt[n]{\prod_{i=1}^n \frac{x_i x_{n+1-i}}{(1-t)x_{n+1-i}+tx_i}} \leqslant \sqrt[n]{\prod_{i=1}^n x_i}. \tag{3.2}$$

(2) For every positive integer n, *we have*

$$\frac{n}{\sum_{i=1}^n \frac{1}{x_i}} \leqslant \prod_{i=1}^n \left[\frac{x_i x_{n+1-i}\left(x_i^{\frac{x_i}{x_{n+1-i}-x_i}}\right)e}{\left(x_{n+1-i}^{\frac{x_{n+1-i}}{x_{n+1-i}-x_i}}\right)}\right]^{\frac{1}{n}} \leqslant \sqrt[n]{\prod_{i=1}^n x_i}. \tag{3.3}$$

Proof. (1) In Theorem 2.2 put $f(x) = \log x$, we obtain

$$F(t) = \frac{1}{n}\sum_{i=1}^n \log\left(\frac{1}{\frac{t}{x_{n+1-i}}+\frac{1-t}{x_i}}\right) = \log\sqrt[n]{\prod_{i=1}^n \frac{x_i x_{n+1-i}}{(1-t)x_{n+1-i}+tx_i}}.$$

By the use of Theorem 2.2, we obtain the desired inequality.

(2) In view of Corollary 2.4 (part (1)) with $f(x) = \log x$, we have

$$\begin{aligned}\int_0^1 F(t)dt &= \frac{1}{n}\sum_{i=1}^n \int_0^1 \log\left(\frac{x_i x_{n+1-i}}{tx_i+(1-t)x_{n+1-i}}\right)dt \\ &= \frac{1}{n}\sum_{i=1}^n \Bigg[\log(x_i x_{n+1-i}) \\ &\quad - \frac{1}{x_{n+1-i}-x_i}\left(x_{n+1-i}\log x_{n+1-i} - x_{n+1-i} - x_i\log x_i + x_i\right)\Bigg] \\ &= \frac{1}{n}\sum_{i=1}^n \log\frac{x_i x_{n+1-i}\left(x_i^{\frac{x_i}{x_{n+1-i}-x_i}}\times e\right)}{\left(x_{n+1-i}^{\frac{x_{n+1-i}}{x_{n+1-i}-x_i}}\right)},\end{aligned} \tag{3.4}$$

which completes the proof. □

The following extensions of AM–GM–HM inequality hold.

PROPOSITION 3.14. *Let $x_i > 0$; $i = 1, \dots, n$. Then we have the following*

$$\frac{n}{\sum\limits_{i=1}^{n} x_i^{-1}} \leqslant \sqrt[n]{\prod_{i=1}^{n} \frac{x_i x_{n+1-i}}{(1-t)x_{n+1-i} + t x_i}} \leqslant \sqrt[n]{\prod_{i=1}^{n} x_i}$$

$$\leqslant \sqrt[n]{\prod_{i=1}^{n} (1-t)x_i + t x_{n+1-i}} \leqslant \frac{1}{n}\sum_{i=1}^{n} x_i \tag{3.5}$$

for every $t \in [0,1]$.

Proof. By replacing x_i by x_i^{-1} in (3.2), we obtain

$$\sqrt[n]{\prod_{i=1}^{n} x_i} \leqslant \sqrt[n]{\prod_{i=1}^{n} (1-t)x_i + t x_{n+1-i}} \leqslant \frac{1}{n}\sum_{i=1}^{n} x_i. \tag{3.6}$$

Combine (3.2) and (3.6), we get (3.5). □

PROPOSITION 3.15. *Let $x_i > 0$; $i = 1, \dots, n$. Then we have the following*

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \leqslant \prod_{i=1}^{n} \left[\frac{x_i x_{n+1-i} \left(x_i^{\frac{x_i}{x_{n+1-i} - x_i}} \right) e}{\left(x_{n+1-i}^{\frac{x_{n+1-i}}{x_{n+1-i} - x_i}} \right)} \right]^{\frac{1}{n}} \leqslant \sqrt[n]{\prod_{i=1}^{n} x_i}$$

$$\leqslant \prod_{i=1}^{n} \left[\frac{x_i x_{n+1-i} \left(x_i^{\frac{x_{n+1-i}}{x_i - x_{n+1-i}}} \right)}{\left(x_{n+1-i}^{\frac{x_i}{x_i - x_{n+1-i}}} \right) e} \right]^{\frac{1}{n}} \leqslant \frac{\sum_{i=1}^{n} x_i}{n}. \tag{3.7}$$

Proof. Replace x_i by x_i^{-1} in (3.3), we arrive at

$$\sqrt[n]{\prod_{i=1}^{n} x_i} \leqslant \prod_{i=1}^{n} \left[\frac{x_i x_{n+1-i} \left(x_i^{\frac{x_{n+1-i}}{x_i - x_{n+1-i}}} \right)}{\left(x_{n+1-i}^{\frac{x_i}{x_i - x_{n+1-i}}} \right) e} \right]^{\frac{1}{n}} \leqslant \frac{\sum_{i=1}^{n} x_i}{n}. \tag{3.8}$$

Combine (3.3) and (3.8), we conclude (3.7). □

COROLLARY 3.16. *If $a,b>0$, then*

$$H(a,b) \leqslant \frac{ab}{\sqrt{(ta+(1-t)b)(tb+(1-t)a)}} \leqslant G(a,b)$$
$$\leqslant \sqrt{(ta+(1-t)b)(tb+(1-t)a)} \leqslant A(a,b)$$

for every $t \in [0,1]$.

Proof. Put $n=2$, $x_1 := a$ and $x_2 := b$ in Proposition 3.14. □

COROLLARY 3.17. *If $b \neq a > 0$, then*

$$H(a,b) \leqslant \frac{G^2(a,b)}{I(a,b)} \leqslant G(a,b) \leqslant \frac{G^2(a,b)}{S(a,b)} \leqslant A(a,b).$$

where

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \quad \textit{and} \quad S(a,b) := e\left(\frac{a^b}{b^a}\right)^{\frac{1}{b-a}}.$$

Proof. Put $n=2$, $x_1 := a$ and $x_2 := b$ in Proposition 3.15, we obtain

$$\frac{2ab}{a+b} \leqslant abe\left(\frac{a^{\frac{2a}{b-a}}}{b^{\frac{2b}{b-a}}}\right)^{\frac{1}{2}} \leqslant \sqrt{ab} \leqslant \frac{ab}{e}\left(\frac{b^{\frac{2a}{b-a}}}{a^{\frac{2b}{b-a}}}\right)^{\frac{1}{2}} \leqslant \frac{a+b}{2}.$$

This completes the proof. □

Note that the second and third inequalities in Corollary 3.17 revocer the known relation [6, Corollary 13 in Section 5 of Chapter II]:

$$S(a,b) \leqslant G(a,b) \leqslant I(a,b).$$

PROPOSITION 3.18. *Assume that $0 < a \leqslant x_i \leqslant b$. Then*

$$\prod_{i=1}^{n} \frac{x_i x_{n+1-i}\left(x_i^{\frac{x_i}{x_{n+1-i}-x_i}} \times e\right)}{x_{n+1-i}^{\frac{x_{n+1-i}}{x_{n+1-i}-x_i}}} \leqslant a^n b^n \prod_{i=1}^{n}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{x_i}\right).$$

Proof. In view of Corollary 2.4 (part (2)) with $f(x) = \log x$ and using (3.4), we have

$$\prod_{i=1}^{n}\left[\frac{x_i x_{n+1-i}\left(x_i^{\frac{x_i}{x_{n+1-i}-x_i}} \times e\right)}{x_{n+1-i}^{\frac{x_{n+1-i}}{x_{n+1-i}-x_i}}}\right]^{\frac{1}{n}} \leqslant ab\prod_{i=1}^{n}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{x_i}\right)^{\frac{1}{n}},$$

which completes the proof. □

PROPOSITION 3.19. *Assume that* $0 < a \leqslant x_i \leqslant b$. *Then*

$$\prod_{i=1}^{n} \frac{x_{n+1-i}x_i}{(1-t)x_{n+1-i}+tx_i} \leqslant a^n b^n \prod_{i=1}^{n} \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{x_i} \right).$$

Proof. Put $f(x) = \log x$ in Theorem 2.3. □

If we take $n := 2$, $x_1 := a$ and $x_2 := b$ in Proposition 3.19, then we have

$$G(a,b)^2 \leqslant A_t(b,a)A_t(a,b),$$

where $A_t(a,b) := (1-t)a + tb$, $(0 \leqslant t \leqslant 1)$ is the weighted arithmetic mean of two positive numbers a and b.

COROLLARY 3.20. *Let* $0 < a \leqslant x_1, x_2 \leqslant b$. *Then*

$$(S(x_1,x_2))^2 \leqslant \left(a+b-\frac{ab}{x_1} \right) \left(a+b-\frac{ab}{x_2} \right).$$

Proof. Set $n = 2$ in Proposition 3.18. □

We give a result related to e.

COROLLARY 3.21. *Let* $n \in \mathbb{N}$. *Then,*

$$\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n+1}\right) < e < \left(1+\frac{1}{n}\right)^{n+1/2}.$$

Proof. Setting $a := 1$ and $b := 1+\frac{1}{n}$ in $G(a,b) \leqslant I(a,b) \leqslant A(a,b)$, we obtain the inequalities:

$$\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n+1}\right) \leqslant e \leqslant \left(1+\frac{1}{n}\right)^{n+1/2}.$$

The equalities' conditions of the inequalities $G(a,b) \leqslant I(a,b) \leqslant A(a,b)$ are $a = b$. However it never occurs such a condition since $a := 1$ and $b := 1+\frac{1}{n}$. □

It is known that [25] and [37, Eq. (2)]:

$$\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n+1}\right) < e < \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n}\right).$$

Since it is not difficult to show $\left(1+\frac{1}{n}\right)^{1/2} < \left(1+\frac{1}{2n}\right)$ for $n \in \mathbb{N}$, Corollary 3.21 gives same lower bound and a tighter upper bound of e, compared to the above.

In the following proposition, we further approximate the number e.

PROPOSITION 3.22. *Let $n,m \in \mathbb{N}$. Then,*

$$\left(1+\frac{1}{mn}\right)^{mn}\left(1+\frac{1}{2mn+1}\right) < e < \left(1+\frac{1}{mn}\right)^{mn+1/2}.$$

Proof. The desired inequality follows from Corollary 3.17 by putting $a := m$, $b := m+\frac{1}{n}$, because we have

$$H(a,b) = \frac{2m(mn+1)}{2mn+1}, \quad \frac{G^2(a,b)}{I(a,b)} = \frac{em^{mn+1}n^{mn}}{(mn+1)^{mn}}$$
$$G(a,b) = \sqrt{m(m+1/n)}.$$

In this case, the equalities never occur. □

REMARK 3.23. If we take $m := n$ in Proposition 3.22, then we have

$$\frac{2\left(n^2+1\right)^{n^2+1}}{n^{2n^2}\left(2n^2+1\right)} < e < \frac{\left(n^2+1\right)^{n^2+\frac{1}{2}}}{n^{2n^2+1}}, \quad (n \in \mathbb{N}).$$

Declarations

Availability of data and materials. Not applicable.

Competing interests. The authors declare that they have no competing interests.

Funding. This research is supported by a grant (JSPS KAKENHI, Grant Number: 21K03341) awarded to the author, S. Furuichi.

Authors' contributions. Authors declare that they have contributed equally to this paper. All authors have read and approved this version.

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(Received August 23, 2024)

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