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REFINEMENTS OF ACZÉL–POPOVICIU AND BELLMAN INEQUALITIES FOR LINEAR ISOTONIC FUNCTIONALS

JINYAN MIAO* AND SILVESTRU SEVER DRAGOMIR

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Abstract. A refinement of reverse Jensen inequality for linear isotonic functionals is established. As consequences, refinements of the Aczél-Popoviciu and Bellman inequalities for linear isotonic functionals are obtained. Some particular inequalities are also deduced.

1. Introduction

In the following, we recall some notions and results mentioned in [3], [8] about *linear isotonic functionals.*

Let E be a nonempty set and L a class of real-valued functions $f: E \to \mathbb{R}$ satisfying the following condition:

(L_1) If $f, g \in L$, then $\alpha f + \beta g \in L$ for all $\alpha, \beta \in \mathbb{R}$.

We consider linear isotonic functionals defined on a class L, that is, a mapping $A: L \to \mathbb{R}$ satisfying the following conditions:

(A_1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.

(A_2) If $f \in L$ and $f(t) \geqslant 0$ for all $t \in E$, then $A(f) \geqslant 0$.

Now we recall a reverse of Jensen's inequality for linear isotonic functionals given in [8, p. 124] and [3].

THEOREM 1. *Let E, L and A be as above and $\varphi: I \to \mathbb{R}$ be a continuous convex function defined on an interval $I \subseteq \mathbb{R}$. Assume that $p \in L$ with $p(t) \geqslant 0$ for all $t \in E$ and $0 < A(p) < u$ for some $u \in \mathbb{R}$. Further, assume that $g: E \to I$ is such that $pg \in L$ and $p\varphi(g) \in L$. If $\alpha \in I$ and $(u\alpha - A(pg))/(u - A(p)) \in I$, then we have*

$$\frac{u\varphi(\alpha) - A(p\varphi(g))}{u - A(p)} \leqslant \varphi\left(\frac{u\alpha - A(pg)}{u - A(p)}\right). \tag{1}$$

The following result was also obtained in [3].

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* Corresponding author.

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THEOREM 2. *Suppose the assumptions of Theorem 1 are satisfied. Additionally, let* $q \in L$ *be such that* qg, $q\varphi(g) \in L$. *Also, assume that* $0 \leqslant q(t) \leqslant p(t)$ *for all* $t \in E$ *and* $0 < A(q) < A(p)$. *If* $\alpha \in I$ *and* $(u\alpha - A(pg))/(u - A(p)) \in I$, *then we have*

$$0 \leqslant A(q\varphi(g)) - A(q)\varphi\left(\frac{A(qg)}{A(q)}\right) \leqslant [u - A(p)]\varphi\left(\frac{u\alpha - A(pg)}{u - A(p)}\right) - [u\varphi(\alpha) - A(p\varphi(g))]. \tag{2}$$

The following result for Bellman inequality for linear isotonic functionals is also proved in Theorem 2.5 from [3].

THEOREM 3. *Let* E, L *and* A *be as above. Assume* $p, q : E \to \mathbb{R}$ *are such that* $0 \leqslant q(t) \leqslant p(t)$ *for all* $t \in E$. *Let* $f, g : E \to \mathbb{R}$ *be given functions with* $f(t), g(t) \geqslant 0$ *for all* $t \in E$ *and such that* pf^r, qf^r, pg^r, qg^r, $p(f+g)^r$, $q(f+g)^r \in L$ *for some* $r > 1$. *If*

$$0 < A(qf^r) \leqslant A(pf^r) < a^r, \quad 0 < A(qg^r) \leqslant A(pg^r) < b^r,$$

for some $a, b > 0$, *then we have*

$$0 \leqslant [A(qf^r)^{1/r} + A(qg^r)^{1/r}]^r - A(q(f+g)^r) \leqslant (a+b)^r - A(p(f+g)^r) - [(a^r - A(pf^r))^{1/r} + (b^r - A(pg^r))^{1/r}]^r.$$

Motivated by these results we obtain in this paper other similar results with applications to Aczél-Popoviciu inequality and Bellman inequality for linear isotonic functionals, as well as some other special cases.

2. Extension for linear isotonic functionals

First we prove an important result that provides a refinement of the inequality (1). It is similar but actually more accurate than Theorem 2 obtained in [3].

THEOREM 4. *Suppose the assumptions of Theorem 1 are satisfied. Additionally, let* $q \in L$ *be such that* $qg \in L$ *and* $q\varphi(g) \in L$. *Also, assume that* $0 \leqslant q(t) \leqslant p(t)$ *for all* $t \in E$ *and* $0 < A(q) < A(p)$. *If* $\alpha \in I$, $(u\alpha - A(pg))/(u - A(p)) \in I$ *and* $(u\alpha + A(qg))/(u + A(q)) \in I$, *then we have*

$$0 \leqslant u\varphi(\alpha) + A(q\varphi(g)) - [u + A(q)]\varphi\left(\frac{u\alpha + A(qg)}{u + A(q)}\right) \leqslant [u - A(p)]\varphi\left(\frac{u\alpha - A(pg)}{u - A(p)}\right) - [u\varphi(\alpha) - A(p\varphi(g))]. \tag{3}$$

Proof. The first inequality in (3) is a consequence of Jensen's inequality for linear isotonic functionals (see [8, pp. 112–113]) and Jensen's inequality.

Further, we have $A(p-q)=A(p)-A(q)>0$ and $2u-A(p-q)=2u-A(p)+A(q)>0$. Also

$$\begin{aligned}&\frac{2u\alpha-A((p-q)g)}{2u-A(p-q)}\\&=\frac{(u-A(p))\frac{u\alpha-A(pg)}{u-A(p)}+(u+A(q))\frac{u\alpha+A(qg)}{u+A(q)}}{u-A(p)+u+A(q)}\in I.\end{aligned}$$

Therefore, we can apply (1) with p replaced by $p-q$ and u replaced by $2u$ to obtain

$$\begin{aligned}&\frac{2u\varphi(\alpha)-A(p\varphi(g))+A(q\varphi(g))}{2u-A(p)+A(q)}\\&\leqslant\varphi\left(\frac{2u\alpha-A(pg)+A(qg)}{2u-A(p)+A(q)}\right)\\&\leqslant\frac{u-A(p)}{2u-A(p)+A(q)}\varphi\left(\frac{u\alpha-A(pg)}{u-A(p)}\right)\\&\quad+\frac{u+A(q)}{2u-A(p)+A(q)}\varphi\left(\frac{u\alpha+A(qg)}{u+A(q)}\right).\end{aligned}\tag{4}$$

The second inequality in (4) follows by Jensen inequality. Now it is easy to get the second inequality of (3) from (4). □

Before the main cases, first, as a corollary of Theorem 4, we get the refinement for Theorem 2.

COROLLARY 1. *Suppose the assumptions of Theorem 1 are satisfied. Additionally, let $q\in L$ be such that $qg\in L$ and $q\varphi(g)\in L$. Also, assume that $0\leqslant q(t)\leqslant p(t)$ for all $t\in E$ and $0<A(q)<A(p)$. If $\alpha\in I$, $(u\alpha-A(pg))/(u-A(p))\in I$ and $(u\alpha+A(qg))/(u+A(q))\in I$, then we have*

$$\begin{aligned}0&\leqslant A\left(q\varphi(g)\right)-A\left(q\right)\varphi\left(\frac{A(qg)}{A(q)}\right)\\&\leqslant u\varphi(\alpha)+A(q\varphi(g))-[u+A(q)]\varphi\left(\frac{u\alpha+A(qg)}{u+A(q)}\right)\\&\leqslant[u-A(p)]\varphi\left(\frac{u\alpha-A(pg)}{u-A(p)}\right)-[u\varphi(\alpha)-A(p\varphi(g))].\end{aligned}\tag{5}$$

Proof. The first and third inequalities in (5) have been proven in Theorem 2 and Theorem 4. For the second inequality in (5), due to Jensen inequality we have:

$$A\left(q\right)\varphi\left(\frac{A(qg)}{A(q)}\right)+u\varphi(\alpha)\geqslant[u+A(q)]\varphi\left(\frac{u\alpha+A(qg)}{u+A(q)}\right),$$

which is equivalent to the second inequality. □

From Theorem 4 we get the following refinement of Aczél-Popoviciu inequality.

THEOREM 5. *Let E,L and A be as above. Assume that $p,q : E \to \mathbb{R}$ are such that $0 \leqslant q(t) \leqslant p(t)$ for all $t \in E$. Also assume that $a > 0$ and $b > 0$ are given real numbers. Let $f,g : E \to \mathbb{R}$ be given functions such that $f(t) > 0$, $g(t) \geqslant 0$ for all $t \in E$ and pf^r, qf^r, pg^s, qg^s, pfg, $qfg \in L$, for some $r,s > 1$ with $1/r + 1/s = 1$. If*

$$0 < A(qf^r) \leqslant A(pf^r) < a^r, \quad 0 < A(qg^s) \leqslant A(pg^s) < b^s,$$

then we have

$$\begin{aligned} 0 &\leqslant [a^r + A(qf^r)]^{1/r}[b^s + A(qg^s)]^{1/s} - ab - A(qfg) \\ &\leqslant ab - A(pfg) - [a^r - A(pf^r)]^{1/r}[b^s - A(pg^s)]^{1/s}. \end{aligned} \tag{6}$$

Proof. Set $u = a^r$, $\alpha = a^{-r}b^s$ and replace p,q and g with $\widetilde{p} = pf^r$, $\widetilde{q} = qf^r$ and $\widetilde{g} = f^{-r}g^s$, respectively, and then apply Theorem 4 to the convex function $\varphi(x) = -x^{1/s}$ defined on $I = (0,\infty)$. In this case (3) reduces to (6). □

With Theorem 5 we can prove the following corollary.

COROLLARY 2. *Let all the assumptions be as above in Theorem 5, then we have*

$$\begin{aligned} &\frac{[a^r - A(pf^r)]^{1/r}[b^s - A(pg^s)]^{1/s}}{ab - A(pfg)} \\ &\leqslant \frac{ab + A(qfg)}{[a^r + A(qf^r)]^{1/r}[b^s + A(qg^s)]^{1/s}} \leqslant 1. \end{aligned} \tag{7}$$

Proof. Notice the fact that

$$\begin{aligned} &[a^r + A(qf^r)]^{1/r}[b^s + A(qg^s)]^{1/s} \\ &\geqslant ab + A(qfg) \geqslant ab - A(pfg) \\ &\geqslant [a^r - A(pf^r)]^{1/r}[b^s - A(pg^s)]^{1/s} > 0. \end{aligned} \tag{8}$$

Consider (6) and (8) together, from the elementary inequality

$$y_1 \geqslant y_2 \geqslant y_3 \geqslant y_4 > 0, \quad y_3 - y_4 \geqslant y_1 - y_2 \geqslant 0 \Rightarrow \frac{y_4}{y_3} \leqslant \frac{y_2}{y_1} \leqslant 1$$

we get (7). □

The following result is similar to Theorem 2.5 in [3] mentioned in the introduction.

THEOREM 6. *Let E,L and A be as above. Assume $p,q : E \to \mathbb{R}$ are such that $0 \leqslant q(t) \leqslant p(t)$ for all $t \in E$. Let $f,g : E \to \mathbb{R}$ be given functions with $f(t),g(t) \geqslant 0$ for all $t \in E$ and such that pf^r, qf^r, pg^r, qg^r, $p(f+g)^r$, $q(f+g)^r \in L$ for some $r > 1$. If*

$$0 < A(qf^r) \leqslant A(pf^r) < a^r, \quad 0 < A(qg^r) \leqslant A(pg^r) < b^r,$$

for some $a,b > 0$, then we have

$$\begin{aligned} 0 &\leqslant [a^r + A(qf^r)]^{1/r} + [b^r + A(qg^r)]^{1/r} - [(a+b)^r + A(q(f+g)^r)]^{1/r} \\ &\leqslant [(a+b)^r - A(p(f+g)^r)]^{1/r} - [a^r - A(pf^r)]^{1/r} - [b^r - A(pg^r)]^{1/r}. \end{aligned} \tag{9}$$

Proof. The first inequality in (9) is a consequence of Minkowski inequality for linear isotonic functionals [8, p. 114] and Minkowski inequality.

Next, the Bellman inequality for linear isotonic functionals [8, pp. 125–126] can be restated in the weighted form as follows.

If $p,f,g:E\to\mathbb{R}$ are such that $p(t),f(t),g(t)\geqslant 0$ for all $t\in E$ and pf^r, pg^r, $p(f+g)^r\in L$ for some $r>1$, then

$$[(a+b)^r-A(p(f+g)^r)]^{1/r}\geqslant[a^r-A(pf^r)]^{1/r}+[b^r-A(pg^r)]^{1/r}, \tag{10}$$

provided $a^r>A(pf^r)$, $b^r>A(pg^r)$.

Now use the proof by contradiction. Assume that

$$\begin{aligned}&[a^r+A(qf^r)]^{1/r}+[b^r+A(qg^r)]^{1/r}\\&\quad+[a^r-A(pf^r)]^{1/r}+[b^r-A(pg^r)]^{1/r}\\&>[(a+b)^r+A(q(f+g)^r)]^{1/r}+[(a+b)^r-A(p(f+g)^r)]^{1/r}\end{aligned} \tag{11}$$

holds for some functions. From the fact that

$$\begin{aligned}&[a^r+A(qf^r)]^{1/r}+[b^r+A(qg^r)]^{1/r}\\&\geqslant[(a+b)^r+A(q(f+g)^r)]^{1/r}\\&\geqslant[(a+b)^r-A(p(f+g)^r)]^{1/r}\\&\geqslant[a^r-A(pf^r)]^{1/r}+[b^r-A(pg^r)]^{1/r}>0,\end{aligned}$$

and (11), we have

$$\begin{aligned}&([a^r+A(qf^r)]^{1/r}+[b^r+A(qg^r)]^{1/r})^r\\&\quad+([a^r-A(pf^r)]^{1/r}+[b^r-A(pg^r)]^{1/r})^r\\&>(a+b)^r+A(q(f+g)^r)+(a+b)^r-A(p(f+g)^r).\end{aligned} \tag{12}$$

However, by reverse Minkowski inequality, we have

$$\begin{aligned}&([a^r+A(qf^r)]^{1/r}+[b^r+A(qg^r)])^r\\&\quad+([a^r-A(pf^r)]^{1/r}+[b^r-A(pg^r)]^{1/r})^r\\&\leqslant([2a^r-A((p-q)f^r)]^{1/r}+[2b^r-A((p-q)g^r)]^{1/r})^r.\end{aligned} \tag{13}$$

Further, apply (10) with p replaced by $p-q$ and a,b replaced by $2^{1/r}a$, $2^{1/r}b$, we have

$$\begin{aligned}&[2a^r-A((p-q)f^r)]^{1/r}+[2b^r-A((p-q)g^r)]^{1/r}\\&\leqslant[2(a+b)^r-A((p-q)(f+g)^r)]^{1/r},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&([2a^r-A((p-q)f^r)]^{1/r}+[2b^r-A((p-q)g^r]^{1/r})^r\\&\leqslant(a+b)^r+A(q(f+g)^r)+(a+b)^r-A(p(f+g)^r).\end{aligned} \tag{14}$$

Combine (13) and (14), (12) can not hold, so the assumption is wrong, hence the inequality (9) is proved. □

Set the convex function $\varphi(x) = -\ln x$ in Theorem 4, and choose the interval $I = (0,\infty)$, we have the proposition below.

PROPOSITION 1. *Let E, L and A be as above. Assume that $p,q \in L$ with $p(t) \geqslant q(t) \geqslant 0$ for all $t \in E$ and $0 < A(q) < A(p) < u$ for some $u \in \mathbb{R}$. Further, assume that $g : E \to (0,\infty)$ is such that pg, qg, $p\ln g$, $q\ln g \in L$. If $\alpha \in (0,\infty)$ and $(u\alpha - A(pg))/(u-A(p)) \in (0,\infty)$, then we have*

$$\left(\frac{u\alpha + A(qg)}{u+A(q)}\right)^{u+A(q)} \left(\frac{u\alpha - A(pg)}{u-A(p)}\right)^{u-A(p)} \alpha^{-2u} \leqslant \exp[A((q-p)\ln g)].$$

Set the convex function $\varphi(t)$ be the entropy function $t\ln t$ in Theorem 4, and choose the interval $I = (0,\infty)$, with some simplification we can also state the following result.

PROPOSITION 2. *Let E, L and A be as above. Assume that $p,q \in L$ with $p(t) \geqslant q(t) \geqslant 0$ for all $t \in E$ and $0 < A(q) < A(p) < u$ for some $u \in \mathbb{R}$. Further, assume that $g : E \to (0,\infty)$ is such that pg, qg, $pg\ln g$, $qg\ln g \in L$. If $\alpha \in (0,\infty)$ and $(u\alpha - A(pg))/(u-A(p)) \in (0,\infty)$, then we have*

$$\left(\frac{u\alpha + A(qg)}{u+A(q)}\right)^{u\alpha+A(qg)} \left(\frac{u\alpha - A(pg)}{u-A(p)}\right)^{u\alpha-A(pg)} \alpha^{-2u\alpha} \geqslant \exp[A((q-p)g\ln g)].$$

As a unified conclusion of Theorem 2 in [3] and Theorem 4, we explore some consequence of Corollary 1. Set the convex function $\varphi(x) = x^2$ in Corollary 1, we have the following refinement of Aczél inequality.

THEOREM 7. *Let E, L and A be as above. Assume that $p,q : E \to \mathbb{R}$ are such that $0 \leqslant q(t) \leqslant p(t)$ for all $t \in E$. Also assume that $a > 0$ and $b > 0$ are given real numbers. Let $f,g : E \to \mathbb{R}$ be given functions such that pf^2, qf^2, pg^2, qg^2, pfg, $qfg \in L$. If $f(t) \neq 0$ for all $t \in E$ and*

$$0 < A(qf^2) < A(pf^2) < a^2,$$

then we have

$$\begin{aligned} 0 &\leqslant A(qg^2) - \frac{A(qfg)^2}{A(qf^2)} \\ &\leqslant b^2 + A(qg^2) - \frac{[ab + A(qfg)]^2}{a^2 + A(qf^2)} \\ &\leqslant \frac{[ab - A(pfg)]^2}{a^2 - A(pf^2)} - [b^2 - A(pg^2)]. \end{aligned}$$

Proof. Note that $a \neq 0$ so that $u = a^2$ and $\alpha = b/a$ are well defined real numbers. Also, $\widetilde{p} = pf^2$, $\widetilde{q} = qf^2$, and $\widetilde{g} = g/f$ are well defined real-valued functions on E. Moreover, it is easy to see that we can replace p, q and g with $\widetilde{p}, \widetilde{q}$ and $\widetilde{g}$, respectively, and then apply Corollary 1 to the convex function $\varphi(x) = x^2$. $\square$

REMARK 1. In [3], the following part of Theorem 7

$$0 \leqslant A(qg^2) - \frac{A(qfg)^2}{A(qf^2)} \leqslant \frac{[ab - A(pfg)]^2}{a^2 - A(pf^2)} - [b^2 - A(pg^2)]$$

has been proven. Here we use a similar approach to deduce a more accurate refinement of this result in [3].

3. Examples for integral inequalities

Let $(\Omega, \mathscr{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathscr{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathscr{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geqslant 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)\,|f(x)|\,d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

THEOREM 8. *Let $\Omega, L_w(\Omega, \mu)$ be as above. Assume that $p, q : \Omega \to \mathbb{R}$ are such that $0 \leqslant q(t) \leqslant p(t)$ for all $t \in \Omega$. Also assume that $a > 0$ and $b > 0$ are given real numbers. Let $f, g : \Omega \to \mathbb{R}$ be given functions such that $f(t) > 0$, $g(t) \geqslant 0$ for all $t \in \Omega$ and pf^r, qf^r, pg^s, qg^s, pfg, $qfg \in L_w(\Omega, \mu)$, for some $r, s > 1$ with $1/r + 1/s = 1$. If*

$$0 < \int_\Omega wqf^r d\mu \leqslant \int_\Omega wpf^r d\mu < a^r, \quad 0 < \int_\Omega wqg^s d\mu \leqslant \int_\Omega wpg^s d\mu < b^s,$$

then we have

$$\begin{aligned} 0 &\leqslant [a^r + \int_\Omega wqf^r d\mu]^{1/r} [b^s + \int_\Omega wqg^s d\mu]^{1/s} - ab - \int_\Omega wqfg d\mu \\ &\leqslant ab - \int_\Omega wpfg d\mu - [a^r - \int_\Omega wpf^r d\mu]^{1/r} [b^s - \int_\Omega wpg^s d\mu]^{1/s}. \end{aligned} \tag{15}$$

Proof. In Theorem 4, we can choose $L = L_w(\Omega, \mu)$, because

$$\begin{aligned} &\int_\Omega w(x)\,|\alpha f_1(x) + \beta f_2(x)|\,d\mu(x) \\ &\leqslant |\alpha| \int_\Omega w(x)\,|f_1(x)|\,d\mu(x) + |\beta| \int_\Omega w(x)\,|f_2(x)|\,d\mu(x) < \infty. \end{aligned}$$

Further, let

$$A(f)=\int_{\Omega} w(x) f(x) d\mu(x),$$

in Theorem 4, it's easy to affirm that A is a linear isotonic functional.

Set $u=a^r$, $\alpha=a^{-r}b^s$ and replace p,q and g with $\widetilde{p}=pf^r$, $\widetilde{q}=qf^r$ and $\widetilde{g}=f^{-r}g^s$, respectively, and then apply Theorem 4 to the convex function $\varphi(x)=-x^{1/s}$ defined on $I=(0,\infty)$. In this case (3) reduces to (15). □

Set $\Omega=[a,b]\subset\mathbb{R}^1$, $w(x)\equiv 1$, $\mu(x)=x$ in Theorem 8, we get some integral inequalities in [6].

If μ is the discrete measure on a finite set I then for $A=\sum_{i\in I}$ we can get the discrete case of the integral inequalities stated below.

THEOREM 9. *Let $E=\{2,\dots,n\}$, $L=\mathbb{R}^{n-1}$. Assume $p=(p_i)$, $q=(q_i)$ such that $0\leqslant q_i\leqslant p_i$ for all $i\in E$. Also assume that $a_1>0$ and $b_1>0$ are given real numbers. Let $a=(a_i)$, $b=(b_i)$, $w=(w_i)$ be given sequences such that $a_i>0$, $b_i\geqslant 0$, $w_i\geqslant 0$ for all $i\in E$ and $r,s>1$ with $1/r+1/s=1$. If*

$$0<\sum_{i=2}^{n} w_i q_i a_i^r \leqslant \sum_{i=2}^{n} w_i p_i a_i^r < a_1^r,$$
$$0<\sum_{i=2}^{n} w_i q_i b_i^s \leqslant \sum_{i=2}^{n} w_i p_i b_i^s < b_1^s,$$

then we have

$$\begin{aligned}0&\leqslant [a_1^r+\sum_{i=2}^{n} w_i q_i a_i^r]^{1/r}[b_1^s+\sum_{i=2}^{n} w_i q_i b_i^s]^{1/s}-a_1b_1-\sum_{i=2}^{n} w_i q_i a_i b_i \\ &\leqslant a_1b_1-\sum_{i=2}^{n} w_i p_i a_i b_i-[a_1^r-\sum_{i=2}^{n} w_i p_i a_i^r]^{1/r}[b_1^s-\sum_{i=2}^{n} w_i p_i b_i^s]^{1/s}.\end{aligned}\tag{16}$$

Proof. In Theorem 4, let

$$A(f)=\sum_{i=2}^{n} w_i f_i,$$

it's easy to affirm that A is a linear isotonic functional on $\mathbb{R}^{n-1}$.

Set $u=a_1^r$, $\alpha=a_1^{-r}b_1^s$ and replace p,q and g with $\widetilde{p}=pa^r$, $\widetilde{q}=qa^r$ and $\widetilde{g}=a^{-r}b^s$, respectively, and then apply Theorem 4 to the convex function $\varphi(x)=-x^{1/s}$ defined on $I=(0,\infty)$. In this case (3) reduces to (16). □

Set $w_i=1$, $p_i=q_i=1$ in Theorem 9, we get some discrete inequalities in [6].

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JinYan Miao
ISILC
Victoria University
Melbourne, Australia
e-mail: 954599851@qq.com

Silvestru Sever Dragomir
ISILC
Victoria University
Melbourne, Australia
e-mail: sever.dragomir@vu.edu.au