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ON REVERSE HARDY INEQUALITY IN VARIABLE LEBESGUE SPACES

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Abstract. In this paper, we prove the reverse Hardy inequality for Hardy operator in weighted variable Lebesgue spaces with exponent less than one. In particular, we establish necessary and sufficient conditions on weight functions for the validity of the reverse Hardy inequality for Hardy operator in weighted variable Lebesgue spaces with negative exponents. It should be noted that in the case of variable Lebesgue space $L_{p(x)}(0,\infty)$ for $0<p(x)<1$, the obtained necessary and sufficient conditions on the weight functions are different and coincide for some classes of variable exponents. Also, we prove similar results for the dual Hardy operator. The results are illustrated by an example.

1. Introduction

In the literature, many authors, including G. H. Hardy, J. E. Littlewood and G. Pólya [24] consider the following standard form of discrete Hardy's inequality in discrete Lebesgue space with constant exponent. Let $p>1,\ p'=\frac{p}{p-1}$ and let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence of non-negative real numbers. Then

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)^p\right)^{\frac{1}{p}}\leqslant p'\left(\sum_{n=1}^{\infty}x_n\right)^{\frac{1}{p}}.$$

An integral version of Hardy's inequality states that if f is a Lebesgue measurable non-negative function, then

$$\left(\int_0^{\infty}\left(\frac{1}{x}\int_0^{x}f(t)\,dt\right)^p\right)^{\frac{1}{p}}\leqslant p'\left(\int_0^{\infty}[f(t)]^p\,dt\right)^{\frac{1}{p}}.$$

The development of the famous Hardy inequality in both discrete and continuous forms during the period 1906 to 1928 has its own history. Contributions of mathematicians other than G. H. Hardy, such as E. Landau, G. Polya, E. Schur, and M. Riesz, are important here. The prehistory was described in detail in [29], [30] and [31].

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A systematic investigation of the generalized Hardy inequality with weights that started in [12]. Namely, in [12] two-weight Hardy inequality in its equivalent differential form

$$\left(\int_0^\infty f^p(x)\,\omega(x)\,dx\right)^{\frac{1}{p}} \leqslant C\left(\int_0^\infty \left(f'(x)\right)^p\,v(x)\,dx\right)^{\frac{1}{p}},\ \ f(0)=f(+0)=0 \qquad (1)$$

was connected with the Euler-Lagrange differential equation. It should be mentioned that in [13] and [14] Hardy inequality was studied not only with the case $p>1$, but also with indices $p<1$. Beesack's approach was extended to a class of inequalities containing the Hardy inequality (1) as a special case (see, e.g., [13]). In particular, a necessary and sufficient condition on weight functions for validity (1) was obtained in [47] and [48]. The study of the case with different parameters p and q started in [15] and developed in [25], [31], [42], etc. In the case $p\neq q$ the other type criterion on weight functions for validity (1) was obtained in [23] and [46]. Namely, in [23] and [46] the inequality (1) was connected with a nonlinear ordinary differential equation in weighted Lebesgue spaces. Moreover, the Hardy inequality has numerous applications in the spectral theory of operators, in the theory of integral equations, in the theory of function spaces, etc (see e.g. [20], [21], [31], [41], [42] and others). Characterization of the mapping properties such as boundedness and compactness of the Hardy operator was considered in [2], [14], [15], [21]–[23], [25], [41], [42], [47]–[49] etc. The study of reverse Hardy's inequality in weighted Lebesgue spaces with a constant negative exponent was considered in the works [14], [28], [43], [44] and others. It should be noted that the integral inequality for harmonic means is a special case of the reverse Hardy inequality.

In the rapidly developing field of variable exponent spaces there has already been made an essential progress in studying classical integral operators, such as maximal and singular operators, Riesz potentials, Hardy operators and others. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions. For more details about the theory of variable Lebesgue spaces, we recommend an interested reader to see the books [16], [18] and [26]. In the variable Lebesgue spaces, the boundedness of the Hardy operator is studied in [3]–[5], [7], [8], [10], [11], [17], [19], [22], [27], [32]–[40], [45] etc. Moreover, in [35] the equivalent conditions on the exponent function for the boundedness of the Hardy operator in the variable Lebesgue space were obtained.

The remainder of the paper is structured as follows. Section 2 contained some preliminaries along with the standard ingredients used in the proofs. Our principal assertions, concerning the Hardy operator in the mentioned spaces are formulated and proved in Section 3. We establish necessary and sufficient conditions on weight functions for the validity of the reverse Hardy inequality for the Hardy operator in weighted variable Lebesgue spaces with negative exponents. But in the case of $L_{p(x),\omega}(0,\infty)$ space for $0<p(x)<1$, the obtained necessary conditions on weight functions differ from the sufficient conditions and coincide in some particular cases.

2. Preliminaries

We recall the definition of the weighted variable Lebesgue spaces with negative and positive exponents.

Let $x \in (0,\infty)$ and let $p(x)$ be a Lebesgue measurable function with values in $(-\infty,0)$ We suppose that $-\infty < \underline{p} \leqslant p(x) \leqslant \overline{p} < 0$, where $\underline{p} := \operatorname*{ess\,inf}_{x>0} p(x)$ and $\overline{p} := \operatorname*{ess\,sup}_{x>0} p(x)$. Let ω be a weight function defined on $(0,\infty)$. Given $p(\cdot)$, we define the conjugate exponent function $p'(\cdot)$ by the formula $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

DEFINITION 1. We denote by $L_{p(x),\omega}(0,\infty)$ the space of all Lebesgue measurable functions on $(0,\infty)$ such that for some $\lambda_0 > 0$

$$\int_0^\infty \left(\frac{|f(x)|\,\omega(x)}{\lambda_0}\right)^{p(x)} dx < \infty.$$

The weighted variable Lebesgue spaces $L_{p(x),\omega}(0,\infty)$ is defined by the following functional (see, [9])

$$A_{p(\cdot),\omega}(f) = \sup\left\{\lambda > 0 : \int_0^\infty \left(\frac{|f(x)|\,\omega(x)}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

It is obvious that $A_{p(\cdot),\omega}(f) = \|f^{-1}\|^{-1}_{L_{-p(\cdot),\omega^{-1}}(0,\infty)}$, where

$$\|g\|_{L_{-p(\cdot),\omega}(0,\infty)} = \inf\left\{\lambda > 0 : \int_0^\infty \left(\frac{|g(x)|\,\omega(x)}{\lambda}\right)^{-p(x)} dx \leqslant 1\right\}.$$

If $p(x) \equiv p = const$, then $L_{p(x),\omega}(0,\infty) = L_{p,\omega}(0,\infty)$ is the classical weighted Lebesgue space with negative exponent. Also,

$$A_{p,\omega}(f) = \left(\int_0^\infty [|f(x)|\,\omega(x)]^p \, dx\right)^{\frac{1}{p}}.$$

Let us give the following useful Lemma.

LEMMA 1. *Let* $x>0$ *and let* $-\infty<\underline{p}\leqslant p(x)\leqslant\overline{p}<0$. *Then the following inequality holds:*

$$\min\left\{\left(\int_0^\infty [|f(x)|\,\omega(x)]^{p(x)}\,dx\right)^{\frac{1}{\underline{p}}},\ \left(\int_0^\infty [|f(x)|\,\omega(x)]^{p(x)}\,dx\right)^{\frac{1}{\overline{p}}}\right\}\leqslant A_{p(\cdot),\omega}(f)$$
$$\leqslant\max\left\{\left(\int_0^\infty [|f(x)|\,\omega(x)]^{p(x)}\,dx\right)^{\frac{1}{\underline{p}}},\ \left(\int_0^\infty [|f(x)|\,\omega(x)]^{p(x)}\,dx\right)^{\frac{1}{\overline{p}}}\right\}$$

Lemma 1 is proved similarly in the case of positive exponents (see also [9]).

LEMMA 2. [6] *Let* $0<\underline{p}\leqslant p(x)\leqslant q(x)\leqslant\overline{q}<\infty$ *and let* $\frac{1}{r(x)}=\frac{1}{p(x)}-\frac{1}{q(x)}$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,\infty)$ *and satisfy the following condition:*

$$\left\|\frac{\omega_1}{\omega_2}\right\|_{L_{r(\cdot)}(0,\infty)}<\infty.$$

Then the following inequality holds

$$\|f\|_{L_{p(\cdot),\omega_1}(0,\infty)}\leqslant\left(A+B+\left\|\chi_{\Omega_2}\right\|_{L_\infty(0,\infty)}\right)^{\frac{1}{\underline{p}}}\left\|\frac{\omega_1}{\omega_2}\right\|_{L_{r(\cdot)}(0,\infty)}\|f\|_{L_{q(\cdot),\omega_2}(0,\infty)}.$$

Here $\Omega_1=\{x\in(0,\infty):\ p(x)<q(x)\}$, $\Omega_2=\{x\in(0,\infty):\ p(x)=q(x)\}$,

$$A=\operatorname*{ess\,sup}_{x\in\Omega_1}\frac{p(x)}{q(x)}\quad\textit{and}\quad B=\operatorname*{ess\,sup}_{x\in\Omega_1}\frac{q(x)-p(x)}{q(x)}.$$

Taking $p(x)=\underline{q}$ and $\omega_1=\omega_2=\omega$ in Lemma 2, we have the following corollary.

COROLLARY 1. *Let* $0<\underline{q}\leqslant q(x)\leqslant\overline{q}<\infty$ *and let* $\frac{1}{r(x)}=\frac{1}{\underline{q}}-\frac{1}{q(x)}$. *Suppose that* ω *is a weight function defined on* $(0,\infty)$. *Let*

$$\|1\|_{L_{r(\cdot)}(0,\infty)}<\infty.$$

Then the following inequality holds

$$\|f\|_{L_{\underline{q},\omega}(0,\infty)}\leqslant C_q\,\|f\|_{L_{q(\cdot),\omega}(0,\infty)}.$$

Here $\Omega=\{x\in(0,\infty):\ q(x)=\underline{q}\}$ *and* $C_q=\left(2-\frac{\underline{q}}{\overline{q}}+\|\chi_\Omega\|_{L_\infty(0,\infty)}\right)^{\frac{1}{\underline{q}}}\|1\|_{L_{r(\cdot)}(0,\infty)}$.

We need the generalized Minkowski inequality in variable Lebesgue space.

THEOREM 1. [1] *Let* $1 \leqslant \underline{p} \leqslant p(x) \leqslant \overline{p} < \infty$ *and* $x \in \mathbb{R}^m$. *Suppose* $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ *is a Lebesgue measurable function. Then the following inequality holds*

$$\left\| \int_{\mathbb{R}^m} f(x,\cdot)\, dx \right\|_{L_{p(\cdot)}(\mathbb{R}^n)} \leqslant \int_{\mathbb{R}^m} \|f(x,\cdot)\|_{L_{p(\cdot)}(\mathbb{R}^n)}\, dx.$$

Similarly to the proof of Theorem 1, we can prove the following reverse generalized Minkowski inequality for variable exponents $0 < \underline{p} \leqslant p(x) \leqslant \overline{p} < 1$.

THEOREM 2. *Let* $0 < \underline{p} \leqslant p(x) \leqslant \overline{p} < 1$ *and* $x \in \mathbb{R}^m$. *Suppose* $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ *is a non-negative Lebesgue measurable function. Then the following inequality holds*

$$\left\| \int_{\mathbb{R}^m} f(x,\cdot)\, dx \right\|_{L_{p(\cdot)}(\mathbb{R}^n)} \geqslant \int_{\mathbb{R}^m} \|f(x,\cdot)\|_{L_{p(\cdot)}(\mathbb{R}^n)}\, dx.$$

3. Main results

Let $f \in L_1^{loc}(0,\infty)$. In the sequel, we consider the classical Hardy operator $Hf(x) = \int_0^x f(t)\, dt$ and its dual defined by $H^\star f(x) = \int_x^\infty f(t)\, dt$.

THEOREM 3. *Let* $-\infty < \underline{q} \leqslant q(x) \leqslant p < 0$ *and let* $\alpha < 0$. *Suppose that* ω_1 *and* ω_2 *be weight functions defined on* $(0,\infty)$. *Then the inequality*

$$A_{q(\cdot),\omega_2}(Hf) \geqslant C \left(\int_0^\infty [f(t)\,\omega_1(t)]^p \, dt \right)^{\frac{1}{p}} \tag{2}$$

holds for all $f > 0$ *if and only if*

$$B(\alpha,p,q(\cdot)) = \inf_{t>0} h(t) A_{q(x)} \left([h(x)]^{\frac{1-\alpha}{\alpha}}\, \omega_2(x)\, \chi_{(t,\infty)}(x) \right) > 0,$$

where $h(t) = \left(\int_0^t [\omega_1(s)]^{-p'}\, ds \right)^{\frac{\alpha}{p'}}$.

Moreover, if C *is the best possible constant in* (2), *then*

$$\sup_{\alpha<0} \frac{B(\alpha,p,q(\cdot))}{(1-\alpha)^{\frac{1}{p'}}} \leqslant C \leqslant \inf_{\alpha<0} \left[\frac{(p')^p\,(p-1)\,\alpha}{(p')^p\,(p-1)\,\alpha + (1-\alpha)^p} \right]^{\frac{1}{p}} B(\alpha,p,q(\cdot)).$$

Proof. Sufficiency. By Hölder inequality for negative exponents, we have

$$Hf(x) = \int_0^x f(t)\,dt = \int_0^x f(t)\,\omega_1(t)\,h(t)\,[\omega_1(t)\,h(t)]^{-1}\,dt$$
$$\geqslant \left(\int_0^x [f(t)\,\omega_1(t)\,h(t)]^p\,dt\right)^{\frac{1}{p}} \left(\int_0^x [\omega_1(t)\,h(t)]^{-p'}\,dt\right)^{\frac{1}{p'}}.$$

We have

$$\left(\int_0^x [\omega_1(t)\,h(t)]^{-p'}\,dt\right)^{\frac{1}{p'}} = \left(\int_0^x [\omega_1(t)]^{-p'} \left(\int_0^t [\omega_1(s)]^{-p'}\,ds\right)^{-\alpha} dt\right)^{\frac{1}{p'}}$$
$$= \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \left(\int_0^x \frac{d}{dt}\left[\left(\int_0^t [\omega_1(s)]^{-p'}\,ds\right)^{1-\alpha}\right] dt\right)^{\frac{1}{p'}}$$
$$= \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \left(\int_0^x [\omega_1(s)]^{-p'}\,ds\right)^{\frac{1-\alpha}{p'}}$$
$$= \frac{1}{(1-\alpha)^{\frac{1}{p'}}} [h(x)]^{\frac{1-\alpha}{\alpha}}. \tag{3}$$

Next, by (3), one has

$$Hf(x) \geqslant \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \left(\int_0^x [f(t)\,\omega_1(t)\,h(t)]^p\,dt\right)^{\frac{1}{p}} [h(x)]^{\frac{1-\alpha}{\alpha}}$$
$$= \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \left(\int_0^\infty [f(t)\,\omega_1(t)\,h(t)]^p\,[h(x)]^{\frac{(1-\alpha)p}{\alpha}}\,\chi_{(0,x)}(t)dt\right)^{\frac{1}{p}}.$$

By Theorem 1 for variable exponent $\frac{q(x)}{p} \geqslant 1$, we get

$$A_{q(x),\,\omega_2}(Hf)$$
$$\geqslant \frac{1}{(1-\alpha)^{\frac{1}{p'}}} A_{q(x)}\left[\left(\int_0^\infty [f(t)\,\omega_1(t)\,h(t)]^p\,[h(x)]^{\frac{(1-\alpha)p}{\alpha}}\,[\omega_2(x)]^p\,\chi_{(0,x)}(t)dt\right)^{\frac{1}{p}}\right]$$
$$= \frac{1}{(1-\alpha)^{\frac{1}{p'}}} \left\|\int_0^\infty [f(t)\,\omega_1(t)\,h(t)]^p\,[h(\cdot)]^{\frac{(1-\alpha)p}{\alpha}}\,[\omega_2(\cdot)]^p\,\chi_{(0,\cdot)}(t)dt\right\|^{\frac{1}{p}}_{L_{\frac{q(\cdot)}{p}}(0,\infty)}$$

$$\geqslant \frac{1}{(1-\alpha)^{\frac{1}{p'}}}\left(\int_0^\infty [f(t)\,\omega_1(t)\,h(t)]^p \left\|[h(\cdot)]^{\frac{(1-\alpha)p}{\alpha}}\,[\omega_2(\cdot)]^p\,\chi_{(0,\cdot)}(t)\right\|_{L_{\frac{q(\cdot)}{p}}(0,\infty)} dt\right)^{\frac{1}{p}}$$

$$=\frac{1}{(1-\alpha)^{\frac{1}{p'}}}\left(\int_0^\infty [f(t)\omega_1(t)\,h(t)]^p\left[A_{q(x)}\Big([h(x)]^{\frac{1-\alpha}{\alpha}}\,\omega_2(x)\chi_{(t,\infty)}(x)\Big)\right]^p dt\right)^{\frac{1}{p}}. \tag{4}$$

Thus, by inequality (4) and by the definition of $B(\alpha,p,q(\cdot))$, we have

$$A_{q(x),\omega_2}(Hf)$$

$$\geqslant \frac{1}{(1-\alpha)^{\frac{1}{p'}}}\left(\int_0^\infty [f(t)\,\omega_1(t)\,h(t)]^p\left[A_{q(x)}\left([h(x)]^{\frac{1-\alpha}{\alpha}}\,\omega_2(x)\,\chi_{(t,\infty)}(x)\right)\right]^p dt\right)^{\frac{1}{p}}$$

$$\geqslant \frac{1}{(1-\alpha)^{\frac{1}{p'}}}\left(\sup_{t>0}[h(t)]^p\left[A_{q(x)}\left([h(x)]^{\frac{1-\alpha}{\alpha}}\,\omega_2(x)\,\chi_{(t,\infty)}(x)\right)\right]^p\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^\infty [f(t)\,\omega_1(t)]^p\,dt\right)^{\frac{1}{p}}$$

$$=\frac{1}{(1-\alpha)^{\frac{1}{p'}}}\left[\inf_{t>0} h(t)\,A_{q(x)}\left([h(x)]^{\frac{1-\alpha}{\alpha}}\,\omega_2(x)\,\chi_{(t,\infty)}(x)\right)\right]\left(\int_0^\infty [f(t)\,\omega_1(t)]^p\,dt\right)^{\frac{1}{p}}$$

$$=\frac{B(\alpha,p,q(\cdot))}{(1-\alpha)^{\frac{1}{p'}}}\left(\int_0^\infty [f(t)\,\omega_1(t)]^p\,dt\right)^{\frac{1}{p}}.$$

Necessity. We assume that (2) holds. So, one has

$$\left[A_{q(x),\omega_2}(Hf)\right]^{-1}\leqslant C^{-1}\left(\int_0^\infty [f(t)\,\omega_1(t)]^p\,dt\right)^{-\frac{1}{p}}. \tag{5}$$

But by Definition 1 $\left[A_{q(x),\omega_2}(Hf)\right]^{-1}=\left\|(Hf)^{-1}\right\|_{L_{-q(\cdot),\omega_2^{-1}}(0,\infty)}$.

Let $t>0$ be a fixed number and let

$$f_t(x)=\frac{p'}{1-\alpha}\,(h(t))^{-1-\frac{1}{\alpha(p-1)}}\,[\omega_1(x)]^{-p'}\,\chi_{(0,t)}(x)+(h(x))^{-1-\frac{1}{\alpha(p-1)}}\,[\omega_1(x)]^{-p'}\,\chi_{(t,\infty)}(x).$$

It is obvious that

$$[f_t(x)\,\omega_1(x)]^p$$

$$=\left(\frac{p'}{1-\alpha}\right)^p (h(t))^{-p-\frac{p'}{\alpha}}\,[\omega_1(x)]^{-p'}\,\chi_{(0,t)(x)}+(h(x))^{-p-\frac{p'}{\alpha}}\,[\omega_1(x)]^{-p'}\,\chi_{(t,\infty)(x)}.$$

So, one has

$$\left(\int_0^\infty [f_t(x)\,\omega_1(x)]^p\,dx\right)^{-\frac{1}{p}}$$

$$=\left[\left(\frac{p'}{1-\alpha}\right)^p (h(t))^{-p-\frac{p'}{\alpha}}\left(\int_0^t [\omega_1(s)]^{-p'}\,ds\right)+\int_t^\infty (h(x))^{-p-\frac{p'}{\alpha}}[\omega_1(x)]^{-p'}\,dx\right]^{-\frac{1}{p}}$$

$$=\left[\left(\frac{p'}{1-\alpha}\right)^p (h(t))^{-p}+\int_t^\infty\left(\int_0^x [\omega_1(s)]^{-p'}\,ds\right)^{-\alpha(p-1)-1}[\omega_1(x)]^{-p'}\,dx\right]^{-\frac{1}{p}}$$

$$=\left[\left(\frac{p'}{1-\alpha}\right)^p (h(t))^{-p}-\frac{1}{\alpha(p-1)}\left(\left(\int_0^\infty [\omega_1(s)]^{-p'}\,ds\right)^{-\alpha(p-1)}-(h(t))^{-p}\right)\right]^{-\frac{1}{p}}$$

$$=\left[\left(\left(\frac{p'}{1-\alpha}\right)^p+\frac{1}{\alpha(p-1)}\right)(h(t))^{-p}-\frac{1}{\alpha(p-1)}\left(\int_0^\infty [\omega_1(s)]^{-p'}\,ds\right)^{-\alpha(p-1)}\right]^{-\frac{1}{p}}$$

$$\leqslant\left(\left(\frac{p'}{1-\alpha}\right)^p+\frac{1}{\alpha(p-1)}\right)^{-\frac{1}{p}} h(t).$$

Next, we get

$$\left[A_{q(x),\omega_2}(Hf_t)\right]^{-1}=\left\|(Hf_t)^{-1}\right\|_{L_{-q(\cdot),\omega_2^{-1}}(0,\infty)}\geqslant\left\|(Hf_t)^{-1}\right\|_{L_{-q(\cdot),\omega_2^{-1}}(t,\infty)}.$$

Therefore, one has

$$\begin{aligned}
Hf_t(x)&=\int_0^x f_t(y)\,dy=\int_0^t f_t(y)\,dy+\int_t^x f_t(y)\,dy\\
&=\frac{p'}{1-\alpha}(h(t))^{-1-\frac{1}{\alpha(p-1)}}\int_0^t [\omega_1(y)]^{-p'}\,dy+\int_t^x (h(y))^{-1-\frac{1}{\alpha(p-1)}}[\omega_1(y)]^{-p'}\,dy\\
&=\frac{p'}{1-\alpha}(h(t))^{\frac{1-\alpha}{\alpha}}+\int_t^x\left(\int_0^y [\omega_1(s)]^{-p'}\,ds\right)^{-\frac{\alpha}{p'}-\frac{1}{p}}[\omega_1(y)]^{-p'}\,dy\\
&=\frac{p'}{1-\alpha}(h(t))^{\frac{1-\alpha}{\alpha}}+\frac{p'}{1-\alpha}\left[(h(x))^{\frac{1-\alpha}{\alpha}}-(h(t))^{\frac{1-\alpha}{\alpha}}\right]=\frac{p'}{1-\alpha}(h(x))^{\frac{1-\alpha}{\alpha}}.
\end{aligned}$$

So, by (5) for any $\alpha<0$, we get

$$\left(\frac{p'}{1-\alpha}\right)^{-1}\left[A_{q(x)}\left([h(x)]^{\frac{1-\alpha}{\alpha}}\,\omega_2(x)\,\chi_{(t,\infty)}(x)\right)\right]^{-1}\leqslant\left[A_{q(x),\omega_2}(Hf_t)\right]^{-1}$$

$$\leqslant C^{-1}\left(\left(\frac{p'}{1-\alpha}\right)^{p}+\frac{1}{\alpha(p-1)}\right)^{-\frac{1}{p}}h(t).$$

Finally, we have

$$C\leqslant\inf_{\alpha<0}\left[\frac{(p')^{p}\,(p-1)\,\alpha}{(p')^{p}\,(p-1)\,\alpha+(1-\alpha)^{p}}\right]^{\frac{1}{p}}B(\alpha,p,q(\cdot)).$$

The proof of Theorem 3 is complete. □

A similar Theorem holds for dual of Hardy operator.

THEOREM 4. *Let* $-\infty<\underline{q}\leqslant q(x)\leqslant p<0$ *and let* $\beta<0$*. Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,\infty)$*. Then the inequality*

$$A_{q(\cdot),\omega_2}(H^{\star}f)\geqslant C\left(\int_0^{\infty}[f(t)\,\omega_1(t)]^{p}\,dt\right)^{\frac{1}{p}}\tag{6}$$

holds for all $f>0$ *if and only if*

$$B^{\star}(\beta,p,q(\cdot))=\inf_{t>0}g(t)\,A_{q(x)}\left([g(x)]^{\frac{1-\beta}{\beta}}\,\omega_2(x)\,\chi_{(0,t)}(x)\right)>0,$$

where $g(t)=\left(\int_t^{\infty}[\omega_1(s)]^{-p'}\,ds\right)^{\frac{\beta}{p'}}$.

Moreover, if C *is the best possible constant in (6), then*

$$\sup_{\beta<0}\frac{B^{\star}(\beta,p,q(\cdot))}{(1-\beta)^{\frac{1}{p'}}}\leqslant C\leqslant\inf_{\beta<0}\left[\frac{(p')^{p}\,(p-1)\,\beta}{(p')^{p}\,(p-1)\,\beta+(1-\beta)^{p}}\right]^{\frac{1}{p}}B^{\star}(\beta,p,q(\cdot)).$$

REMARK 1. In the case $q(x)=q=const$ under other type condition on weight functions, Theorem 3 was proved in [14]. In the case $1<p\leqslant q<\infty$ the boundedness of the Hardy operator(reverse of inequality (2)) in weighted Lebesgue spaces was proved in [49] etc. In the case of weighted normed Lebesgue spaces with variable exponent, the boundedness of the Hardy type operator was proved in [3]–[5], [7], [8], [10], [11], [17], [19], [22], [27], [32]–[40], [45] etc.

EXAMPLE. Suppose $q(x)=q$ is a constant. Let $-\infty<q\leqslant p<0$ and let $\alpha<0$. We suppose that $\omega_1(x)=x^\gamma$ and $\omega_2(x)=x^{\gamma-\frac{1}{p'}-\frac{1}{q}}$. Then the inequality (1) holds if and only if $\gamma<\dfrac{1}{p'}$.

Moreover, if C is the best possible constant in (2), then

$$\left(\frac{|p'+q|}{(1-\gamma p')\,|q|}\right)^{\frac{1}{p'}+\frac{1}{q}}\leqslant C\leqslant\frac{(p')^{1+\frac{1}{q}}\,(1-p)^{\frac{1}{p}}}{(1-\gamma p')^{\frac{1}{p'}+\frac{1}{q}}\,|q|^{\frac{1}{q}}}\inf_{\alpha<0}\frac{|\alpha|^{\frac{1}{q}-\frac{1}{p}}}{\left[(p')^p\,(p-1)\alpha+(1-\alpha)^p\right]^{\frac{1}{p}}}.$$

It is obvious that, if $\displaystyle\inf_{\alpha<0}\frac{|\alpha|^{\frac{1}{q}-\frac{1}{p}}}{\left[(p')^p\,(p-1)\alpha+(1-\alpha)^p\right]^{\frac{1}{p}}}=\frac{|p'+q|^{\frac{1}{p'}+\frac{1}{q}}}{(p')^{1+\frac{1}{q}}\,(1-p)^{\frac{1}{p}}\,|q|^{\frac{1}{p'}}}$, then $C=\left(\dfrac{|p'+q|}{(1-\gamma p')\,|q|}\right)^{\frac{1}{p'}+\frac{1}{q}}$. In particular, if $p=q$, then $C=\dfrac{p'}{1-\gamma p'}$.

In the next two theorems we consider the exponents $0<p(x)<1$. Let

$$J(t)=\left(\int_t^\infty[\omega_1(s)]^{-p'}\,ds\right)^{\frac{1}{p'}}\left(\int_t^\infty[\omega_2(s)]^{\underline{q}}\,ds\right)^{\frac{1}{\underline{q}}}.$$

We introduce the following quantities

$$K=\inf_{t>0}(J(t))^{\frac{1}{p}}\inf_{t\leqslant s<\infty}(J(s))^{\frac{1}{p'}},$$

$$R=\inf_{t>0}\left(\int_t^\infty[\omega_1(s)]^{-p'}\,ds\right)^{\frac{1}{p'}}\|\omega_2\|_{L_{q(\cdot)}(t,\infty)}.$$

THEOREM 5. *Let* $0<\underline{q}\leqslant q(x)\leqslant p<1$ *and let* $\dfrac{1}{r(x)}=\dfrac{1}{\underline{q}}-\dfrac{1}{q(x)}$. *Let* $\|1\|_{L_{r(\cdot)(0,\infty)}}<\infty$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,\infty)$.

(i) If $K>0$, *then the inequality*

$$\|Hf\|_{L_{q(\cdot),\omega_2}(0,\infty)}\geqslant C\left(\int_0^\infty[f(t)\,\omega_1(t)]^p\,dt\right)^{\frac{1}{p}} \tag{7}$$

holds.

(ii) If (7) holds, then $R>0$.

Moreover, if C *is the best possible constant in (7), then*

$$\left|p'\right|^{\frac{1}{p'}}p^{\frac{1}{q}}C_q^{-1}K\leqslant C\leqslant R.$$

Proof. (*i*) Let $P(t)=\left(\int_t^\infty[\omega_1(s)]^{-p'}\,ds\right)^{\frac{1}{pp'}}$. By Hölder inequality for exponents $0<p<1$, we have

$$Hf(x)=\int_0^x f(t)\,dt=\int_0^x f(t)\,\omega_1(t)\,P(t)\,[\omega_1(t)\,P(t)]^{-1}\,dt$$
$$\geqslant\left(\int_0^x[f(t)\,\omega_1(t)\,P(t)]^p\,dt\right)^{\frac{1}{p}}\left(\int_0^x[\omega_1(t)\,P(t)]^{-p'}\,dt\right)^{\frac{1}{p'}}.$$

We have

$$\left(\int_0^x[\omega_1(t)\,P(t)]^{-p'}\,dt\right)^{\frac{1}{p'}}=\left(\int_0^x[\omega_1(t)]^{-p'}\left(\int_t^\infty[\omega_1(s)]^{-p'}\,ds\right)^{-\frac{1}{p}}\,dt\right)^{\frac{1}{p'}}$$
$$=\left(-p'\int_0^x\frac{d}{dt}\left[\left(\int_t^\infty[\omega_1(s)]^{-p'}\,ds\right)^{1-\frac{1}{p}}\right]dt\right)^{\frac{1}{p'}}$$
$$=\left|p'\right|^{\frac{1}{p'}}\left(\left[\int_x^\infty[\omega_1(s)]^{-p'}\,ds\right]^{\frac{1}{p'}}-\left[\int_0^\infty[\omega_1(s)]^{-p'}\,ds\right]^{\frac{1}{p'}}\right)^{\frac{1}{p'}}$$
$$\geqslant\left|p'\right|^{\frac{1}{p'}}\left(\int_x^\infty[\omega_1(s)]^{-p'}\,ds\right)^{\frac{1}{(p')^2}}. \tag{8}$$

Next, by (8), one has

$$Hf(x)\geqslant\left|p'\right|^{\frac{1}{p'}}\left(\int_x^\infty[f(t)\,\omega_1(t)\,P(t)]^p\,dt\right)^{\frac{1}{p}}\left(\int_x^\infty[\omega_1(s)]^{-p'}\,ds\right)^{\frac{1}{(p')^2}}$$
$$=\left|p'\right|^{\frac{1}{p'}}\left(\int_0^\infty[f(t)\,\omega_1(t)\,P(t)]^p\left(\int_x^\infty[\omega_1(s)]^{-p'}\,ds\right)^{\frac{p}{(p')^2}}\chi_{(0,x)}(t)dt\right)^{\frac{1}{p}}$$
$$=\left|p'\right|^{\frac{1}{p'}}\left(\int_0^\infty[f(t)\,\omega_1(t)\,P(t)]^p\,(J(x))^{\frac{p}{p'}}\left(\int_x^\infty[\omega_2(s)]^{\underline{q}}\,ds\right)^{-\frac{p}{\underline{q}p'}}\chi_{(0,x)}(t)dt\right)^{\frac{1}{p}}.$$

By Theorem 2 for variable exponent $\frac{q(x)}{p} \leqslant 1$ and by Corollary 1, we get

$$\|Hf\|_{L_{q(\cdot),\omega_2}(0,\infty)} = \|Hf\|_{L_{q(x),\omega_2}(0,\infty)}$$

$$\geqslant |p'|^{\frac{1}{p'}} \left\| \left(\int_0^\infty [f(t)\omega_1(t)P(t)]^p (J(x))^{\frac{p}{p'}} \left(\int_x^\infty [\omega_2(s)]^{\underline{q}} ds \right)^{-\frac{p}{\underline{q}p'}} \chi_{(0,x)}(t) dt \right)^{\frac{1}{p}} \right\|_{L_{q(x),\omega_2}(0,\infty)}$$

$$= |p'|^{\frac{1}{p'}} \left\| \int_0^\infty [f(t)\omega_1(t)P(t)]^p (J(x))^{\frac{p}{p'}} \left(\int_x^\infty [\omega_2(s)]^{\underline{q}} ds \right)^{-\frac{p}{\underline{q}p'}} \chi_{(0,x)}(t)\, dt \right\|^{\frac{1}{p}}_{L_{\frac{q(x)}{p},\omega_2}(0,\infty)}$$

$$\geqslant |p'|^{\frac{1}{p'}} \left(\int_0^\infty [f(t)\,\omega_1(t)\,P(t)]^p \left\| (J)^{\frac{p}{p'}} \left(\int_x^\infty [\omega_2(s)]^{\underline{q}} ds \right)^{-\frac{p}{\underline{q}p'}} \right\|_{L_{\frac{q(x)}{p},\omega_2}(t,\infty)} dt \right)^{\frac{1}{p}}$$

$$= |p'|^{\frac{1}{p'}} \left(\int_0^\infty [f(t)\,\omega_1(t)\,P(t)]^p \left\| (J)^{\frac{1}{p'}} \left(\int_x^\infty [\omega_2(s)]^{\underline{q}} ds \right)^{-\frac{1}{\underline{q}p'}} \right\|^p_{L_{q(x),\omega_2}(t,\infty)} dt \right)^{\frac{1}{p}}$$

$$\geqslant |p'|^{\frac{1}{p'}} \left(\int_0^\infty [f(t)\omega_1(t)P(t)]^p \inf_{t\leqslant x<\infty} (J(x))^{\frac{p}{p'}} \left\| \omega_2 \left(\int_x^\infty [\omega_2(s)]^{\underline{q}} ds \right)^{-\frac{1}{\underline{q}p'}} \right\|^p_{L_{q(x)}(t,\infty)} dt \right)^{\frac{1}{p}}$$

$$\geqslant |p'|^{\frac{1}{p'}} C_q^{-1} \left(\int_0^\infty [f(t)\,\omega_1(t)\,P(t)]^p \inf_{t\leqslant x<\infty} (J(x))^{\frac{p}{p'}} \left\| \omega_2 \left(\int_x^\infty [\omega_2(s)]^{\underline{q}} ds \right)^{-\frac{1}{\underline{q}p'}} \right\|^p_{L_{\underline{q}}(t,\infty)} dt \right)^{\frac{1}{p}}$$

$$= |p'|^{\frac{1}{p'}} p^{\frac{1}{\underline{q}}} C_q^{-1} \left(\int_0^\infty [f(t)\,\omega_1(t)\,P(t)]^p \inf_{t\leqslant x<\infty} (J(x))^{\frac{p}{p'}} \left(\int_t^\infty [\omega_2(x)]^{\underline{q}} dx \right)^{\frac{1}{\underline{q}}} dt \right)^{\frac{1}{p}}$$

$$= |p'|^{\frac{1}{p'}} p^{\frac{1}{\underline{q}}} C_q^{-1} \left(\int_0^\infty [f(t)\,\omega_1(t)\,P(t)]^p \inf_{t\leqslant x<\infty} (J(x))^{\frac{p}{p'}} J(t)\, (P(t))^{-p}\, dt \right)^{\frac{1}{p}}$$

$$\geqslant |p'|^{\frac{1}{p'}} p^{\frac{1}{\underline{q}}} C_q^{-1} K \left(\int_0^\infty [f(t)\,\omega_1(t)]^p\, dt \right)^{\frac{1}{p}}.$$

(ii) We suppose that (7) holds. Let $t>0$ be fixed and define

$$f_t(x)=(\omega_1(x))^{-p'}\,\chi_{[t,\infty)}(x).$$

So, one has

$$\|f_t\|_{L_{p,\omega_1}(0,\infty)}=\left(\int_t^\infty(\omega_1(x))^{-p'}\,dx\right)^{\frac{1}{p}}.$$

Next, we have

$$\|Hf_t\|_{L_{q(\cdot),\omega_2}(0,\infty)}=\left\|\omega_2\int_t^x(\omega_1(s))^{-p'}\,ds\right\|_{L_{q(x)}(t,\infty)}\leqslant\left(\int_t^\infty(\omega_1(s))^{-p'}\,ds\right)\|\omega_2\|_{L_{q(x)}(t,\infty)}.$$

Thus, from (7) we have

$$\left(\int_t^\infty(\omega_1(s))^{-p'}\,ds\right)\|\omega_2\|_{L_{q(x)}(t,\infty)}\geqslant C\left(\int_t^\infty(\omega_1(s))^{-p'}\,ds\right)^{\frac{1}{p}}.$$

So, we have that $C\leqslant\left(\int_t^\infty(\omega_1(s))^{-p'}\,ds\right)^{\frac{1}{p'}}\|\omega_2\|_{L_{q(x)}(t,\infty)}$.

The proof of Theorem 5 is complete. □

Obviously, $\inf\limits_{t\leqslant x<\infty}(J(x))^{\frac{1}{p'}}\leqslant(J(t))^{\frac{1}{p'}}$ for all $t>0$. Suppose that there exists a constant $0<M\leqslant 1$ such that

$$M\,(J(t))^{\frac{1}{p'}}\leqslant\inf_{t\leqslant x<\infty}(J(x))^{\frac{1}{p'}}\leqslant(J(t))^{\frac{1}{p'}}\quad\text{for all } t>0. \tag{9}$$

Thus, under condition (9) we have that $\inf\limits_{t\leqslant x<\infty}(J(x))^{\frac{1}{p'}}\approx(J(t))^{\frac{1}{p'}}$.

We have the following corollaries.

COROLLARY 2. *Let $q(x)=q=const$ and let $0<q\leqslant p<1$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$ and let J satisfy condition (9). Then the inequality (7) holds if and only if $K>0$.*

Moreover, if C is the best possible constant in (7), then

$$M\left|p'\right|^{\frac{1}{p'}}\,p^{\frac{1}{q}}\,K\leqslant C\leqslant K.$$

COROLLARY 3. [14] *Let $q(x)=q=const$ and let $0<q\leqslant p<1$. Suppose that ω_1 and ω_2 are weight functions defined on $(0,\infty)$ and let J be a nonincreasing function. Then the inequality (7) holds if and only if $K>0$.*

Moreover, if C is the best possible constant in (7), then

$$\left|p'\right|^{\frac{1}{p'}}\,p^{\frac{1}{q}}K\leqslant C\leqslant K.$$

The proof of Corollary 3 follows directly from Corollary 2 for $M = 1$.

Let

$$J^{\star}(t) = \left(\int_0^t [\omega_1(s)]^{-p'} \, ds \right)^{\frac{1}{p'}} \left(\int_0^t [\omega_2(s)]^{\underline{q}} \, ds \right)^{\frac{1}{\underline{q}}}.$$

We consider the following quantities

$$K^{\star} = \inf_{t>0} (J(t))^{\frac{1}{p}} \inf_{t \leqslant s < \infty} (J^{\star}(s))^{\frac{1}{p'}},$$

$$R^{\star} = \inf_{t>0} \left(\int_0^t [\omega_1(s)]^{-p'} \, ds \right)^{\frac{1}{p'}} \|\omega_2\|_{L_{q(\cdot)}(0,t)}.$$

A similar Theorem holds for dual of Hardy operator.

THEOREM 6. *Let* $0 < \underline{q} \leqslant q(x) \leqslant p < 1$ *and let* $\dfrac{1}{r(x)} = \dfrac{1}{\underline{q}} - \dfrac{1}{q(x)}$. *Let* $\|1\|_{L_{r(\cdot)(0,\infty)}} < \infty$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,\infty)$.

(i_1) *If* $K^{\star} > 0$, *then the inequality*

$$\|H^{\star} f\|_{L_{q(\cdot),\omega_2}(0,\infty)} \geqslant C \left(\int_0^{\infty} [f(t)\,\omega_1(t)]^p \, dt \right)^{\frac{1}{p}} \tag{10}$$

holds.

(ii_1) *If* (10) *holds, then* $R^{\star} > 0$.

Moreover, if C *is the best possible constant in (10), then*

$$|p'|^{\frac{1}{p'}} p^{\frac{1}{q}} C_q^{-1} K^{\star} \leqslant C \leqslant R^{\star}.$$

Clearly, $\inf\limits_{0<x\leqslant t} (J^{\star}(x))^{\frac{1}{p'}} \leqslant (J^{\star}(t))^{\frac{1}{p'}}$ for all $t > 0$. Suppose that there exists a constant $0 < M^{\star} \leqslant 1$ such that

$$M^{\star} (J^{\star}(t))^{\frac{1}{p'}} \leqslant \inf_{0<x\leqslant t} (J^{\star}(x))^{\frac{1}{p'}} \leqslant (J^{\star}(t))^{\frac{1}{p'}}. \tag{11}$$

Thus, under condition (11) we have that $\inf\limits_{0<x\leqslant t} (J^{\star}(x))^{\frac{1}{p'}} \approx (J^{\star}(t))^{\frac{1}{p'}}$.

We have the following corollaries.

COROLLARY 4. *Let* $q(x) = q = const$ *and let* $0 < q \leqslant p < 1$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,\infty)$ *and let* $J^{\star}$ *satisfies condition* (11). *Then the inequality* (10) *holds if and only if* $K^{\star} > 0$.

Moreover, if C *is the best possible constant in* (10), *then*

$$M^{\star} |p'|^{\frac{1}{p'}} p^{\frac{1}{q}} K^{\star} \leqslant C \leqslant K^{\star}.$$

COROLLARY 5. [14] *Let* $q(x) = q = const$ *and let* $0 < q \leqslant p < 1$. *Suppose that* ω_1 *and* ω_2 *are weight functions defined on* $(0,\infty)$ *and let* J *be a nondecreasing function. Then the inequality* (10) *holds if and only if* $K^\star > 0$.

Moreover, if C *is the best possible constant in* (10), *then*

$$\left|p'\right|^{\frac{1}{p'}}\, p^{\frac{1}{q}}\, K^\star \leqslant C \leqslant K^\star.$$

The proof of Corollary 5 follows directly from Corollary 4 for $M^\star = 1$.

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