Journal of Mathematical Inequalities
Volume 19, Number 3 (2025), 839–864 doi:10.7153/jmi-2025-19-54

WEIGHTED BILINEAR DISCRETE HARDY–TYPE INEQUALITY FOR CLASS OF MATRIX OPERATOR

NAZERKE ZHANGABERGENOVA* AND AINUR TEMIRKHANOVA

(*Communicated by R. Oinarov*)

Abstract. The paper contains a characterization a weighted Hardy-type inequality involving a discrete bilinear operator with a matrix a weighted Lebesgue sequence spaces.

1. Introduction

Let $0<q,p,s<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Let $u=\{u_i\}_{i=1}^{\infty}$ be non-negative, $v=\{v_i\}_{i=1}^{\infty}$ and $w=\{w_i\}_{i=1}^{\infty}$ be positive sequences of real numbers. In the paper, we discuss the following bilinear discrete Hardy-type inequality

$$\left(\sum_{n=1}^{\infty}u_n^q\left|\sum_{i=1}^{n}g_i\right|^q\left|\sum_{j=1}^{n}a_{n,j}f_j\right|^q\right)^{\frac{1}{q}}\leqslant C\left(\sum_{i=1}^{\infty}|v_if_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}|w_ig_i|^s\right)^{\frac{1}{s}},\tag{1}$$

where C is a positive constant independent of f and g sequences of real numbers for which the right side of the inequality (1) is finite, and $(a_{n,j})$, $n\geqslant j\geqslant 1$, is a matrix, whose non-negative entries satisfy the discrete Oinarov condition: there exist $d\geqslant 1$, entries $a_{n,j}$ are almost non-decreasing in n and almost non-increasing in j, such that the inequalities

$$\frac{1}{d}(a_{n,k}+a_{k,j})\leqslant a_{n,j}\leqslant d(a_{n,k}+a_{k,j})\tag{2}$$

or, equivalently, the relation $a_{n,j}\approx a_{n,k}+a_{k,j}$ hold for all $n\geqslant k\geqslant j\geqslant 1$.

Since our work is related to discrete inequalities, we review the history of the discrete Hardy-type inequalities. In 1985, K. F. Andersen and H. P. Heinig presented sufficient conditions for the following two weighted discrete Hardy-type inequalities in the cases $1<p\leqslant q<\infty$ and $1<q<p<\infty$ for the first time in their works [2] and [7]:

$$\left(\sum_{k=1}^{\infty}u_k^q\left|\sum_{i=1}^{k}f_i\right|^q\right)^{\frac{1}{q}}\leqslant C\left(\sum_{i=1}^{\infty}|v_if_i|^p\right)^{\frac{1}{p}}.\tag{3}$$

Mathematics subject classification (2020): 26D15, 26D20.

Keywords and phrases: Hardy inequality, Hardy-type operator, bilinear operator, weight sequence, weighted sequence space.

* Corresponding author.

© ELEMENT, Zagreb
Paper JMI-19-54

M. Sh. Braverman and V. D. Stepanov [6] obtained the fulfillment criterion of the inequality (3) for the case $0 < q < 1 < p < \infty$. Moreover, G. Bennett described in detail the inequality (3) for almost all ratios of parameters p and q in the works [3], [4] and [5]. A detailed description and comprehensive overview of the development of continuous and discrete Hardy inequalities can be found in [12].

In later years, many authors have shown significant interest in studying the generalized inequality (3) for the class of matrix operators given by

$$\left(\sum_{k=1}^{\infty} u_k^q \left|\sum_{i=1}^{k} a_{k,i} f_i\right|^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}, \tag{4}$$

under certain assumptions on matrix $(a_{k,i})$ (see [13], [14], [15], [18], [21] and [22]). In particular, the discrete Hardy-type operators with a matrix satisfying the discrete Oinarov condition (2) were first investigated in [13] and [15]. In later years, such conditions on the matrix have been significantly generalized. In article [16], the expanding classes of matrices O_n, $n \geqslant 0$, were introduced and criteria of boundedness and compactness for a wide class of matrix operators were obtained in the weighted space of sequences. Currently, the inequality (3) involving various discrete quasilinear operators has been presented and explored by the authors of this paper (see [9], [17], [23] and [24]). The findings related to the aforementioned linear and quasilinear operators can be applied to characterize bilinear operators. In 2012, the following weighted integral Hardy-type inequalities involving bilinear operators were first considered in the paper [1]:

$$\left(\int_0^{\infty} \hat{u}^q(t)|(H_1\hat{g}\hat{f})(t)|^q dt\right)^{\frac{1}{q}} \leqslant C\left(\int_0^{\infty} |\hat{w}(t)\hat{g}(t)|^s dt\right)^{\frac{1}{s}}\left(\int_0^{\infty} |\hat{v}(t)\hat{f}(t)|^p dt\right)^{\frac{1}{p}}, \tag{5}$$

where $\hat{u},\hat{v},\hat{w}$ are locally integrable and positive measurable functions on $(0,\infty)$ and H_1 is a bilinear operator defined as follows

$$(H_1\hat{g}\hat{f})(t) = \int_0^t \hat{g}(x)dx \int_0^t \hat{f}(x)dx, \quad t \in (0,\infty).$$

To prove the results, the authors used the discretization technique, which is complex for the bilinear inequality. M. Křepela provided a much simpler proof for the characterization of the validity of these inequalities, using an iteration technique that reduces the problem to the corresponding linear operator case. Additionally, the author estimated the bilinear inequality for non-increasing functions in [11]. Furthermore, two-dimensional and multidimensional cases of the bilinear inequality have been extensively studied (see [19], [20]). Bilinear discrete Hardy-type inequality (1) when $a_{k,i} = 1$ for all $k \geqslant i \geqslant 1$ has already been explored for all possible cases of parameters $0 < p,q,s < \infty$ in the recent paper [8]. The weighted estimates of discrete bilinear operators, including Hardy operator and the matrix operator, have not been obtained yet, therefore, this paper is dedicated to addressing this problem. The aim of the paper is to estimate the inequality (1) for the following cases of parameters p,q and s

1) $\min\{p,s\} \leqslant \max\{p,s\} \leqslant q < \infty,\ p,s \in (0,1]$;
2) $0 < \min\{p,s\} \leqslant 1 < \max\{p,s\} \leqslant q < \infty$;
3) $1 < \min\{p,s\} \leqslant q < \max\{p,s\} < \infty$;
4) $\min\{p,s\} \leqslant q < \max\{p,s\} < \infty,\ 0 < \min\{p,s\} \leqslant 1,\ \max\{p,s\} > 1$;
5) $0 < q < s \leqslant 1 < p < \infty$.

The structure of the paper is as follows: Section 2 includes all the necessary auxiliary statements and definitions required to characterize inequality (1), while Section 3 contains the main results.

2. Auxiliary statements

In the section, we indicated some previous results that we need for the proof.

THEOREM A. (see [12]). *(i) If $0 < p \leqslant 1,\ p \leqslant q < \infty$, then inequality (3) holds if and only if $A_1 < \infty$, where*

$$A_1 = \sup_{j \geqslant 1} \left(\sum_{k=j}^{\infty} u_k^q \right)^{\frac{1}{q}} v_j^{-1}.$$

(ii) If $1 < p \leqslant q < \infty$, then inequality (3) holds if and only if $A_2 < \infty$, where

$$A_2 = \sup_{j \geqslant 1} \left(\sum_{i=1}^{j} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} u_k^q \right)^{\frac{1}{q}}.$$

(iii) If $0 < q < p < \infty$ and $p > 1$, then inequality (3) holds if and only if $A_3 < \infty$, where

$$A_3 = \left[\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} u_k^q \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{n} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_n^{-p'} \right]^{\frac{p-q}{pq}},$$

$$A_3 \approx \widetilde{A}_3 = \left[\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} u_k^q \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{n} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_n^q \right]^{\frac{p-q}{pq}}.$$

(iv) If $0 < q < p \leqslant 1$, then inequality (3) holds if and only if $A_4 < \infty$, where

$$A_4 = \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=n}^{\infty} u_k^q \right)^{\frac{q}{p-q}} \max_{1 \leqslant i \leqslant n} v_i^{-\frac{pq}{p-q}} \right)^{\frac{p-q}{pq}}.$$

Moreover, $C \approx A_i,\ i = 1,2,3,4$, where C is the best constant in (3).

THEOREM F. *(i)* (see [18]). *If* $0 < p \leqslant 1$, $p \leqslant q < \infty$ *and the entries of the matrix* $(a_{k,i})$ *be non-increasing in* i, *then inequality* (4) *holds if and only if* $F_1 < \infty$, *where*

$$F_1 = \sup_{j\geqslant 1}\left(\sum_{k=j}^{\infty} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} v_j^{-1}.$$

(ii) (see [15]). *If* $1 < p \leqslant q < \infty$ *and the entries of the matrix* $(a_{k,i})$ *satisfy condition* (2), *then inequality* (4) *holds if and only if* $F_2 = \max\{F_{21}, F_{22}\} < \infty$, *where*

$$F_{21} = \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=j}^{\infty} a_{k,j}^q u_k^q\right)^{\frac{1}{q}},$$

$$F_{22} = \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=j}^{\infty} u_k^q\right)^{\frac{1}{q}}.$$

(iii) (see [13]). *If* $0 < q < p < \infty$, $p > 1$ *and the entries of the matrix* $(a_{k,i})$ *satisfy condition* (2), *then inequality* (4) *holds if and only if* $F_3 = \max\{F_{31}, F_{32}\} < \infty$, *where*

$$F_{31} = \left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} a_{n,i}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{k=n}^{\infty} u_k^q\right)^{\frac{q}{p-q}} u_n^q\right)^{\frac{p-q}{pq}},$$

$$F_{32} = \left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} a_{k,n}^q u_k^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_n^{-p'}\right)^{\frac{p-q}{pq}}.$$

Moreover, $C \approx F_i$, $i = 1,2,3$, *where* C *is the best constant in* (4).

THEOREM N. (see [12]). *(i) If* $1 < p \leqslant q < \infty$, *then inequality*

$$\left(\sum_{k=1}^{\infty} u_k^q \left|\sum_{i=k}^{\infty} f_i\right|^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}, \tag{6}$$

holds if and only if $N_1 < \infty$, *where*

$$N_1 = \sup_{j\geqslant 1}\left(\sum_{i=j}^{\infty} v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=1}^{j} u_k^q\right)^{\frac{1}{q}}.$$

(ii) If $0 < q < p < \infty$ *and* $p > 1$, *then inequality* (6) *holds if and only if* $N_2 < \infty$, *where*

$$N_2 = \left[\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} u_k^q\right)^{\frac{p}{p-q}}\left(\sum_{i=n}^{\infty} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_n^{-p'}\right]^{\frac{p-q}{pq}},$$

$$N_2 \approx \widetilde{N}_2 = \left[\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} u_k^q\right)^{\frac{q}{p-q}}\left(\sum_{i=n}^{\infty} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_n^q\right]^{\frac{p-q}{pq}}.$$

Moreover, $C \approx N_i$, $i = 1,2$, *where* C *is the best constant in* (6).

ASSUMPTION A. (see [16]). There exist $d \geqslant 1$, a sequence of positive numbers $\{\omega_k\}_{k=1}^{\infty}$ and a non-negative matrix $(b_{i,j})$, where $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j such that the inequalities

$$\frac{1}{d}(a_{i,k} + b_{k,j}\omega_i) \leqslant a_{i,j} \leqslant d(a_{i,k} + b_{k,j}\omega_i) \tag{7}$$

or, equivalently, the relation $a_{i,j} \approx a_{i,k} + b_{k,j}\omega_i$ hold for all $i \geqslant k \geqslant j \geqslant 1$.

THEOREM Q. *(i)* (see [22]). *If* $0 < q < p < \infty$, $p > 1$ *and the entries of the matrix* $(a_{i,k})$ *satisfy Assumption A, then the inequality*

$$\left(\sum_{k=1}^{\infty} u_k^q \left|\sum_{i=k}^{\infty} a_{i,k} f_i\right|^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p,v}, \tag{8}$$

holds if and only if $Q_1 = \max\{Q_{11}, Q_{12}\} < \infty$, *where*

$$Q_{11} = \left(\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n} b_{n,j}^q u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{k=n}^{\infty} \omega_k^{p'} v_k^{-p'}\right)^{\frac{p(q-1)}{p-q}} \omega_n^{p'} v_n^{-p'}\right)^{\frac{p-q}{pq}},$$

$$Q_{12} = \left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} u_k^q\right)^{\frac{q}{p-q}}\left(\sum_{i=n}^{\infty} a_{i,n}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_n^q\right)^{\frac{p-q}{pq}}.$$

(ii) (see [21]). *If* $1 < p \leqslant q < \infty$ *and the entries of the matrix* $(a_{k,i})$ *satisfy Asumption A, then the inequality* (8) *holds if and only if* $Q_2 = \max\{Q_{21}, Q_{22}\} < \infty$, *where*

$$Q_{21} = \sup_{j \geqslant 1}\left(\sum_{i=1}^{j} u_i^q\right)^{\frac{1}{q}}\left(\sum_{k=j}^{\infty} a_{k,j}^{p'} v_k^{-p'}\right)^{\frac{1}{p'}},$$

$$Q_{22} = \sup_{j \geqslant 1}\left(\sum_{i=1}^{j} b_{j,i}^q u_i^q\right)^{\frac{1}{q}}\left(\sum_{k=j}^{\infty} \omega_k^{p'} v_k^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $C \approx Q_i$, $i = 1,2$ *where* C *is the best constant in* (8).

REMARK 1. Note that the parameter ratio in Theorem F (iii) and Theorem Q (i) from works [13] and [22] was $1 < q < p < \infty$. However, it is not difficult to see that these results can be extended to the case $0 < q < p < \infty$, $p > 1$, which we will use in Section 4 of this article.

We also need the reverse Hölder inequality for $p>1$:

$$\left(\sum_{i=1}^{\infty}(d_iz_i)^p\right)^{\frac{1}{p}}=\sup_{h\geqslant 0}\left(\sum_{i=1}^{\infty}d_ih_i\right)\left(\sum_{i=1}^{\infty}h_i^{p'}z_i^{-p'}\right)^{-\frac{1}{p'}}. \tag{9}$$

LEMMA 1. (see [2]). *Let $r>0$ and a_k be a non-negative sequence. Then*

$$\left(\sum_{i=1}^{j}a_i\right)^{r}\approx\sum_{k=1}^{j}a_k\left(\sum_{i=1}^{k}a_i\right)^{r-1},\quad j\geqslant 1.$$

If $\sum\limits_k a_k<\infty$ and $1\leqslant j\leqslant m\leqslant\infty$

$$\left(\sum_{i=j}^{m}a_i\right)^{r}\approx\sum_{k=j}^{m}a_k\left(\sum_{i=k}^{m}a_i\right)^{r-1}.$$

Convention: The symbol $E\ll F$ means $E\leqslant CF$ with some constant C, depending on the parameters p, q and s. Moreover, the notation $E\approx F$ means $E\ll F\ll E$.

3. The main results

THEOREM 1. *Let $\min\{p,s\}\leqslant\max\{p,s\}\leqslant q<\infty$ and $p,s\in(0,1]$. Let the entries of the matrix $(a_{n,j})$ satisfy condition (2). Then inequality (1) holds if and only if $A=\max\{A_1,A_2\}<\infty$, where*

$$A_1=\sup_{j\geqslant 1}\left(\sum_{k=j}^{\infty}a_{k,j}^{q}u_k^{q}\right)^{\frac{1}{q}}v_j^{-1}\sup_{m\leqslant j}w_m^{-1},$$

$$A_2=\sup_{j\geqslant 1}v_j^{-1}\sup_{m\geqslant j}\left(\sum_{k=m}^{\infty}a_{k,j}^{q}u_k^{q}\right)^{\frac{1}{q}}w_m^{-1}.$$

Moreover, $C\approx A$, where C is the best constant in (1).

Proof. Let $f\geqslant 0$ and $g\geqslant 0$ be such that $\sum\limits_{i=1}^{\infty}|v_if_i|^p<\infty$ and $\sum\limits_{i=1}^{\infty}|w_ig_i|^s<\infty$. Suppose that C is the best constant in the inequality (1), then we have

$$C\approx\sup_{g\geqslant 0}\sup_{f\geqslant 0}\left(\sum_{n=1}^{\infty}u_n^q\left(\sum_{i=1}^{n}g_i\right)^{q}\left(\sum_{j=1}^{n}a_{n,j}f_j\right)^{q}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{\infty}|v_if_i|^p\right)^{-\frac{1}{p}}\left(\sum_{i=1}^{\infty}|w_ig_i|^s\right)^{-\frac{1}{s}}. \tag{10}$$

Let's denote $B_n^q := u_n^q \left(\sum_{i=1}^{n} g_i\right)^q$, $\|f\|_{p,v} := \left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{1}{p}}$ and $\|g\|_{s,w} := \left(\sum_{i=1}^{\infty} |w_i g_i|^s\right)^{\frac{1}{s}}$. Then applying Theorem F (i) to (10), we obtain

$$C \approx \sup_{g\geqslant 0} \|g\|_{s,w}^{-1} \sup_{f\geqslant 0} \left(\sum_{n=1}^{\infty} B_n^q \left(\sum_{j=1}^{n} a_{n,j} f_j\right)^q\right)^{\frac{1}{q}} \|f\|_{p,v}^{-1} \tag{11}$$

$$\approx \sup_{g\geqslant 0} \|g\|_{s,w}^{-1} \sup_{j\geqslant 1} \left(\sum_{k=j}^{\infty} a_{k,j}^q B_k^q\right)^{\frac{1}{q}} v_j^{-1}$$

$$= \sup_{j\geqslant 1} v_j^{-1} \sup_{g\geqslant 0} \left(\sum_{k=j}^{\infty} a_{k,j}^q u_k^q \left(\sum_{i=1}^{k} g_i\right)^q\right)^{\frac{1}{q}} \|g\|_{s,w}^{-1}.$$

Let $\sigma_{k,j} = 1$ for $k \geqslant j$ and $\sigma_{k,j} = 0$ for $k < j$, then rewrite last estimate as follows

$$C \approx \sup_{j\geqslant 1} v_j^{-1} \sup_{g\geqslant 0} \left(\sum_{k=1}^{\infty} \sigma_{k,j} a_{k,j}^q u_k^q \left(\sum_{i=1}^{k} g_i\right)^q\right)^{\frac{1}{q}} \|g\|_{s,w}^{-1}.$$

By applying Theorem A (i) we have

$$C \approx \sup_{j\geqslant 1} v_j^{-1} \sup_{m\geqslant 1} \left(\sum_{k=m}^{\infty} \sigma_{k,j} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} w_m^{-1}.$$

We consider the supremum with respect to m, dividing the range into two intervals.

$$C \approx \sup_{j\geqslant 1} v_j^{-1} \max\left\{ \sup_{1\leqslant m\leqslant j} \left(\sum_{k=m}^{\infty} \sigma_{k,j} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} w_m^{-1}, \sup_{m\geqslant j} \left(\sum_{k=m}^{\infty} \sigma_{k,j} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} w_m^{-1} \right\}$$

$$= \sup_{j\geqslant 1} v_j^{-1} \max\left\{ \left(\sum_{k=j}^{\infty} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} \sup_{1\leqslant m\leqslant j} w_m^{-1}, \sup_{m\geqslant j} \left(\sum_{k=m}^{\infty} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} w_m^{-1} \right\}.$$

Therefore, $C \approx A$. We confirmed that the order of taking the supremum with respect to the sequences f and g in (10) gives the same result in both cases. Proof of Theorem 1 is complete. □

REMARK 2. Theorem 1 means that the inequality (1) holds for both cases $p \leqslant s \leqslant q < \infty$, $0 < p,s \leqslant 1$ and $s < p \leqslant q < \infty$, $0 < p,s \leqslant 1$.

THEOREM 2. *Let* $0 < \min\{p,s\} \leqslant 1 < \max\{p,s\} \leqslant q < \infty$. *Let the elements of the matrix* $(a_{n,j})$ *satisfy condition* (2).

(a) If $0 < s \leqslant 1 < p \leqslant q < \infty$, then inequality (1) holds if and only if $D = \max\{D_1, D_2, D_3\} < \infty$, where

$$D_1 = \sup_{n\geqslant 1} w_n^{-1} \sup_{j\geqslant n} \left(\sum_{k=j}^{\infty} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}},$$

$$D_2 = \sup_{n\geqslant 1} w_n^{-1} \sup_{j\leqslant n} \left(\sum_{k=n}^{\infty} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}},$$

$$D_3 = \sup_{n\geqslant 1} w_n^{-1} \sup_{j\geqslant n} \left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} u_k^q\right)^{\frac{1}{q}}.$$

(b) If $0 < p \leqslant 1 < s \leqslant q < \infty$, then inequality (1) holds if and only if $\mathscr{D} < \infty$, where

$$\mathscr{D} = \sup_{n\geqslant 1} \left(\sum_{i=1}^{n} w_i^{s'}\right)^{\frac{1}{s'}} \sup_{j\leqslant n} \left(\sum_{k=n}^{\infty} a_{k,j}^q u_k^q\right)^{\frac{1}{q}} v_j^{-1}.$$

Moreover, $C \approx D$ in case (a) and $C \approx \mathscr{D}$ in case (b), where C is the best constant in (1).

Proof. Case (a). The proof of Theorem 2 can be started similarly as Theorem 1. We apply Theorem F (ii) to (11), then we get

$$C \approx \sup_{g\geqslant 0} \|g\|_{s,w}^{-1} \max\left\{ \sup_{j\geqslant 1} \left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} a_{k,j}^q B_k^q\right)^{\frac{1}{q}}, \right.$$

$$\left. \sup_{j\geqslant 1} \left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} B_k^q\right)^{\frac{1}{q}} \right\}.$$

By changing the order of the supremum and inserting $\sigma_{k,j}$, we obtain

$$C \approx \max\left\{ \sup_{j\geqslant 1} \left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}} \sup_{g\geqslant 0} \left(\sum_{k=1}^{\infty} \sigma_{k,j} a_{k,j}^q u_k^q \left(\sum_{n=1}^{k} g_n\right)^q\right)^{\frac{1}{q}} \|g\|_{s,w}^{-1}, \right.$$

$$\left. \sup_{j\geqslant 1} \left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \sup_{g\geqslant 0} \left(\sum_{k=1}^{\infty} \sigma_{k,j} u_k^q \left(\sum_{n=1}^{k} g_n\right)^q\right)^{\frac{1}{q}} \|g\|_{s,w}^{-1} \right\}.$$

By Theorem A (i), we get

$$C \approx \max\left\{ \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}} \sup_{n\geqslant 1}\left(\sum_{k=n}^{\infty} \sigma_{k,j} a_{k,j}^{q} u_k^{q}\right)^{\frac{1}{q}} w_n^{-1}, \right.$$

$$\left. \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \sup_{n\geqslant 1}\left(\sum_{k=n}^{\infty} \sigma_{k,j} u_k^{q}\right)^{\frac{1}{q}} w_n^{-1} \right\} = \max\{I_1, I_2\}. \tag{12}$$

First, we estimate I_1. We consider the supremum with respect to n, dividing the range into two intervals.

$$I_1 \approx \max\left\{ \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} a_{k,j}^{q} u_k^{q}\right)^{\frac{1}{q}} \sup_{1\leqslant n\leqslant j} w_n^{-1}, \right.$$

$$\left. \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}} \sup_{n\geqslant j}\left(\sum_{k=n}^{\infty} a_{k,j}^{q} u_k^{q}\right)^{\frac{1}{q}} w_n^{-1} \right\}.$$

By changing the order of the supremum, we have

$$I_1 \approx \max\{D_1, D_2\}. \tag{13}$$

Now we estimate I_2 using the previous proof for I_1.

$$I_2 \approx \max\left\{ \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} u_k^{q}\right)^{\frac{1}{q}} \sup_{1\leqslant n\leqslant j} w_n^{-1}, \right.$$

$$\left. \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \sup_{n\geqslant j}\left(\sum_{k=n}^{\infty} u_k^{q}\right)^{\frac{1}{q}} w_n^{-1} \right\}$$

By changing the order of the supremum, we have

$$I_2 \approx \max\left\{ \sup_{n\geqslant 1} w_n^{-1} \sup_{j\geqslant n}\left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} u_k^{q}\right)^{\frac{1}{q}}, \right.$$

$$\left. \sup_{n\geqslant 1}\left(\sum_{i=1}^{n} a_{n,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=n}^{\infty} u_k^{q}\right)^{\frac{1}{q}} w_n^{-1} \right\} = D_3 \tag{14}$$

Therefore, (12), (13) and (14) give that $C \approx D.$.

Case (b) can be proved similarly by exploiting Theorem F (i) and Theorem A (ii). Proof of Theorem 2 is complete. □

Let

$$A_k := \left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}, \quad \bar{W}_k := \max_{1\leqslant i\leqslant k} w_i^{-\frac{sq}{s-q}}.$$

Since sequences $\{A_k\}_k$ and $\{\bar{W}_k\}_k$ are non-decreasing, we define

$$\Delta\bar{W}_n = \bar{W}_n - \bar{W}_{n-1}, \quad \Delta A_k = A_k - A_{k-1},$$

where $A_0 = 0$ and $\bar{W}_0 = 0$.

THEOREM 3. *Let $1 < \min\{p,s\} \leqslant q < \max\{p,s\} < \infty$. Let the elements of the matrix $(a_{n,j})$ satisfy condition (2).*

(a) If $1 < s \leqslant q < p < \infty$, then inequality (1) holds if and only if $M = \max\{M_1, M_2, M_3\} < \infty$, where

$$M_1 = \sup_{j\geqslant 1}\left(\sum_{m=1}^{j} w_m^{-s'}\right)^{\frac{1}{s'}}\left(\sum_{k=j}^{\infty}\left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{n=k}^{\infty} u_n^q\right)^{\frac{q}{p-q}} u_k^q\right)^{\frac{p-q}{pq}},$$

$$M_2 = \sup_{j\geqslant 1}\left(\sum_{m=1}^{j} w_m^{-s'}\right)^{\frac{1}{s'}}\left(\sum_{k=1}^{j}\left(\sum_{n=j}^{\infty} a_{n,k}^{q} u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}},$$

$$M_3 = \sup_{j\geqslant 1}\left(\sum_{m=1}^{j} w_m^{-s'}\right)^{\frac{1}{s'}}\left(\sum_{k=j}^{\infty}\left(\sum_{n=k}^{\infty} a_{n,k}^{q} u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}}.$$

(b) If $1 < p \leqslant q < s < \infty$, then inequality (1) holds if and only if $\mathscr{M} = \max\{M_4, M_5, M_6, M_7\} < \infty$, where

$$M_4 = \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} a_{n,j}^{q} u_n^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{m=1}^{j} w_m^{-s'}\right)^{\frac{1}{s'}},$$

$$M_5 = \sup_{j\geqslant 1}\left(\sum_{m=1}^{j} v_m^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{n=j}^{\infty}\left(\sum_{k=n}^{\infty} a_{k,j}^{q} u_k^q\right)^{\frac{s}{s-q}}\left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} w_n^{-s'}\right)^{\frac{s-q}{sq}},$$

$$M_6 = \sup_{j\geqslant 1}\left(\sum_{k=1}^{j} a_{j,k}^{p'} v_k^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{n=j}^{\infty} u_n^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{j} w_i^{-s'}\right)^{\frac{1}{s'}},$$

$$M_7 = \sup_{j\geqslant 1}\left(\sum_{m=1}^{j} a_{j,m}^{p'} v_m^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{n=j}^{\infty}\left(\sum_{k=n}^{\infty} u_k^q\right)^{\frac{s}{s-q}}\left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} w_n^{-s'}\right)^{\frac{s-q}{sq}}.$$

Moreover, $C \approx M$ in case (a) and $C \approx \mathscr{M}$ in case (b) where C is the best constant in (1).

Proof. Case (a). Let $\Delta f_k = f_k - f_{k-1}$ and $\Delta^+ g_k = g_k - g_{k+1}$. If $i > j$, then $\sum\limits_{k=i}^{j} = 0$. Let's write the Abel transformation for $n \geqslant m \geqslant 1$ as follows

$$\sum_{k=m}^{n} f_k \Delta^+ g_k = \sum_{k=m+1}^{n} g_k \Delta f_k + f_m g_m - f_n g_{n+1}.$$

Taking into account the relation $a_k^{\alpha-1}(a_k - a_{k+1}) \approx a_k^{\alpha} - a_{k+1}^{\alpha}$ for the positive decreasing sequences $\{a_k\}$ and $\alpha > 0$, we have

$$\left(\sum_{j=k}^{\infty} B_j^q\right)^{\frac{q}{p-q}} B_k^q = \left(\sum_{j=k}^{\infty} B_j^q\right)^{\frac{q}{p-q}} \left(\sum_{j=k}^{\infty} B_j^q - \sum_{j=k+1}^{\infty} B_j^q\right) \approx \Delta^+ \left(\sum_{j=k}^{\infty} B_j^q\right)^{\frac{p}{p-q}},$$

then we obtain

$$\sum_{k=1}^{\infty} A_k \left(\sum_{j=k}^{\infty} B_j^q\right)^{\frac{q}{p-q}} B_k^q \approx \sum_{k=1}^{\infty} A_k \Delta^+ \left(\sum_{j=k}^{\infty} B_j^q\right)^{\frac{p}{p-q}}. \tag{15}$$

By using Abel transformation for $m \geqslant 1$, we obtain

$$\sum_{k=1}^{m} A_k \Delta^+ \left(\sum_{j=k}^{m} B_j^q\right)^{\frac{p}{p-q}} = \sum_{k=1}^{m} \Delta A_k \left(\sum_{j=k}^{m} B_j^q\right)^{\frac{p}{p-q}}$$

where $A_0 = 0$ and $\sum\limits_{j=m+1}^{m} = 0$. By passing to the limit as $m \longrightarrow \infty$ and according to (15), we obtain

$$\sum_{k=1}^{\infty} A_k \left(\sum_{j=k}^{\infty} B_j^q\right)^{\frac{q}{p-q}} B_k^q \approx \sum_{k=1}^{\infty} \Delta A_k \left(\sum_{j=k}^{\infty} B_j^q\right)^{\frac{p}{p-q}}. \tag{16}$$

Then we apply Theorem F (iii) to (11), we get

$$\begin{aligned} C \approx \max \Bigg\{ &\sup_{g \geqslant 0} \|g\|_{s,w}^{-1} \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=k}^{\infty} B_j^q\right)^{\frac{q}{p-q}} B_k^q\right)^{\frac{p-q}{pq}}, \\ &\sup_{g \geqslant 0} \|g\|_{s,w}^{-1} \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} \left(\sum_{j=k}^{\infty} a_{j,k}^q B_j^q\right)^{\frac{p}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}} \Bigg\} \\ = \max\{&I_1, I_2\}. \end{aligned} \tag{17}$$

Now we estimate I_1. From (16), we have

$$I_1 = \sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left(\sum_{k=1}^{\infty}A_k\left(\sum_{j=k}^{\infty}B_j^q\right)^{\frac{q}{p-q}}B_k^q\right)^{\frac{p-q}{pq}}$$

$$\approx \sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left(\sum_{k=1}^{\infty}\Delta A_k\left(\sum_{j=k}^{\infty}B_j^q\right)^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}}$$

$$= \sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left[\left(\sum_{k=1}^{\infty}\left((\Delta A_k)^{\frac{p-q}{p}}\sum_{n=k}^{\infty}u_n^q\left(\sum_{i=1}^{n}g_i\right)^q\right)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right]^{\frac{1}{q}}.$$

Since $p>q$ in the case (a), it follows that $\frac{p}{p-q}>1$. Thus, we apply the reverse Hölder inequality (9) into the last estimate.

$$I_1 \approx \sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left[\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-1}\sum_{k=1}^{\infty}(\Delta A_k)^{\frac{p-q}{p}}\sum_{n=k}^{\infty}u_n^q\left(\sum_{i=1}^{n}g_i\right)^q h_k\right]^{\frac{1}{q}}.$$

By changing the order of the supremum and the sums, and then applying Theorem A (ii), we obtain

$$I_1 \approx \sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-\frac{1}{q}}\sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left(\sum_{n=1}^{\infty}u_n^q\left(\sum_{i=1}^{n}g_i\right)^q\sum_{k=1}^{n}(\Delta A_k)^{\frac{p-q}{p}}h_k\right)^{\frac{1}{q}}$$

$$\approx \sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-\frac{1}{q}}\sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty}u_n^q\sum_{k=1}^{n}(\Delta A_k)^{\frac{p-q}{p}}h_k\right)^{\frac{1}{q}}\left(\sum_{i=1}^{j}w_i^{-s'}\right)^{\frac{1}{s'}}$$

$$= \sup_{j\geqslant 1}\left(\sum_{i=1}^{j}w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-1}\sum_{n=1}^{\infty}\sigma_{n,j}u_n^q\sum_{k=1}^{n}(\Delta A_k)^{\frac{p-q}{p}}h_k\right)^{\frac{1}{q}}.$$

By changing the order of sums and applying (9) again, we found that

$$I_1 \approx \sup_{j\geqslant 1}\left(\sum_{i=1}^{j}w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-1}\sum_{k=1}^{\infty}h_k(\Delta A_k)^{\frac{p-q}{p}}\sum_{n=k}^{\infty}\sigma_{n,j}u_n^q\right)^{\frac{1}{q}}$$

$$= \sup_{j\geqslant 1}\left(\sum_{i=1}^{j}w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sum_{k=1}^{\infty}\Delta A_k\left(\sum_{n=k}^{\infty}\sigma_{n,j}u_n^q\right)^{\frac{p}{p-q}}\right)^{\frac{p-q}{pq}}.$$

By applying Lemma 1 into the last expression and then changing the order of the sums, we have

$$I_1 \approx \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sum_{k=1}^{\infty}\Delta A_k\sum_{n=k}^{\infty}\sigma_{n,j}u_n^q\left(\sum_{z=n}^{\infty}\sigma_{z,j}u_z^q\right)^{\frac{q}{p-q}}\right)^{\frac{p-q}{pq}}$$

$$=\sup_{j\geqslant 1}\left(\sum_{i=1}^{j} w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sum_{n=1}^{\infty}\sigma_{n,j}u_n^q\left(\sum_{z=n}^{\infty}\sigma_{z,j}u_z^q\right)^{\frac{q}{p-q}}\sum_{k=1}^{n}\Delta A_k\right)^{\frac{p-q}{pq}}.$$

Here, noting that $A_0=0$, we have

$$I_1 \approx \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sum_{n=j}^{\infty}u_n^q\left(\sum_{z=n}^{\infty}u_z^q\right)^{\frac{q}{p-q}}A_n\right)^{\frac{p-q}{pq}} = M_1. \tag{18}$$

Now, let us estimate I_2 in the same way we estimated I_1.

$$I_2=\sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\left(\sum_{j=k}^{\infty}a_{j,k}^qB_j^q\right)^{\frac{p}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}$$

$$=\sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left[\left(\sum_{k=1}^{\infty}\left(\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{(q-1)}v_k^{\frac{p-q}{1-p}}\sum_{j=k}^{\infty}a_{j,k}^qB_j^q\right)^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right]^{\frac{1}{q}}.$$

By applying (9) and then replacing notation B_j back, we obtain

$$I_2\approx\sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left[\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-1}\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{(q-1)}v_k^{\frac{p-q}{1-p}}\sum_{j=k}^{\infty}a_{j,k}^qB_j^qh_k\right]^{\frac{1}{q}}$$

$$=\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-\frac{1}{q}}\sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{(q-1)}v_k^{\frac{p-q}{1-p}}\sum_{j=k}^{\infty}a_{j,k}^qu_j^q\left(\sum_{i=1}^{j}g_i\right)^qh_k\right)^{\frac{1}{q}}.$$

Changing the order the sums, then we can estimate as follows according to Theorem A (ii):

$$I_2\approx\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-\frac{1}{q}}\sup_{g\geqslant 0}\|g\|_{s,w}^{-1}\left(\sum_{j=1}^{\infty}u_j^q\left(\sum_{i=1}^{j}g_i\right)^q\sum_{k=1}^{j}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{(q-1)}v_k^{\frac{p-q}{1-p}}a_{j,k}^qh_k\right)^{\frac{1}{q}}$$

$$\approx\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-\frac{1}{q}}\sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty}u_n^q\sum_{k=1}^{n}a_{n,k}^qh_k\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{(q-1)}v_k^{\frac{p-q}{1-p}}\right)^{\frac{1}{q}}\left(\sum_{m=1}^{j}w_i^{-s'}\right)^{\frac{1}{s'}}.$$

We put the value $\sigma_{n,j}$, then by switching the order of the supremum and the sums, we find

$$I_2 \approx \sup_{j\geqslant 1}\left(\sum_{m=1}^{j} w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-1}\sum_{n=1}^{\infty}\sigma_{n,j}u_n^q\sum_{k=1}^{n}a_{n,k}^q h_k\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{(q-1)}v_k^{\frac{p-q}{1-p}}\right)^{\frac{1}{q}}$$

$$=\sup_{j\geqslant 1}\left(\sum_{m=1}^{j} w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sup_{h\geqslant 0}\|h\|_{\frac{p}{q}}^{-1}\sum_{k=1}^{\infty}h_k\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{(q-1)}v_k^{\frac{p-q}{1-p}}\sum_{n=k}^{\infty}\sigma_{n,j}a_{n,k}^q u_n^q\right)^{\frac{1}{q}}.$$

By (9) we get the following estimate

$$I_2 \approx \sup_{j\geqslant 1}\left(\sum_{m=1}^{j} w_i^{-s'}\right)^{\frac{1}{s'}}\left(\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\sigma_{n,j}a_{n,k}^q u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}$$

$$\approx \sup_{j\geqslant 1}\left(\sum_{m=1}^{j} w_i^{-s'}\right)^{\frac{1}{s'}}\left(\left(\sum_{k=1}^{j}+\sum_{k=j}^{\infty}\right)\left(\sum_{n=k}^{\infty}\sigma_{n,j}a_{n,k}^q u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}.$$

Thus, $I_2 \approx \max\{I_{21}, I_{22}\}$, where

$$I_{21} = \left(\sum_{k=1}^{j}\left(\sum_{n=k}^{\infty}\sigma_{n,j}a_{n,k}^q u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}$$

$$= \left(\sum_{k=1}^{j}\left(\sum_{n=j}^{\infty}a_{n,k}^q u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}} \tag{19}$$

and

$$I_{22} = \left(\sum_{k=j}^{\infty}\left(\sum_{n=k}^{\infty}\sigma_{n,j}a_{n,k}^q u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}$$

$$= \left(\sum_{k=j}^{\infty}\left(\sum_{n=k}^{\infty}a_{n,k}^q u_n^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_k^{-p'}\right)^{\frac{p-q}{pq}}. \tag{20}$$

In the view of (19) and (20), we have

$$I_2 \approx \max\{M_2, M_3\}. \tag{21}$$

From (17), (18) and (21), we get

$$C \approx \max\{M_1, M_2, M_3\}.$$

Case (b). As noted earlier, to validate our results, changing the order of the supremum in (11) yields the same results. Assume that C is the best constant in (1), then

$$C = \sup_{f \geqslant 0} \|f\|_{p,v}^{-1} \sup_{g \geqslant 0} \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{j=1}^{n} a_{n,j} f_j \right)^q \left(\sum_{i=1}^{n} g_i \right)^q \right)^{\frac{1}{q}} \|g\|_{s,w}^{-1}, \tag{22}$$

by Theorem A (iii), we obtain

$$C \approx \sup_{f \geqslant 0} \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} u_k^q \left(\sum_{j=1}^{k} a_{k,j} f_j \right) \right)^{\frac{s}{s-q}} \left(\sum_{i=1}^{n} w_i^{-s'} \right)^{\frac{s(q-1)}{s-q}} w_n^{-s'} \right)^{\frac{s-q}{sq}} \|f\|_{p,v}^{-1}$$

$$= \sup_{f \geqslant 0} \left[\left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} u_k^q \left(\sum_{j=1}^{k} a_{k,j} f_j \right)^q \left(\sum_{i=1}^{n} w_i^{-s'} \right)^{(q-1)} w_n^{\frac{s-q}{1-s}} \right)^{\frac{s}{s-q}} \right)^{\frac{s-q}{s}} \right]^{\frac{1}{q}} \|f\|_{p,v}^{-1}.$$

Since $\frac{s}{s-q} > 1$ in case b, then applying (9) and changing the order of the sums, we have

$$C \approx \sup_{f \geqslant 0} \|f\|_{p,v}^{-1} \left[\sup_{h \geqslant 0} \left(\sum_{n=1}^{\infty} h_n^{\frac{s}{q}} \right)^{-\frac{q}{s}} \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} u_k^q \left(\sum_{j=1}^{k} a_{k,j} f_j \right)^q \left(\sum_{i=1}^{n} w_i^{-s'} \right)^{(q-1)} w_n^{\frac{s-q}{1-s}} h_n \right]^{\frac{1}{q}}$$

$$= \sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-\frac{1}{q}} \sup_{f \geqslant 0} \left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{j=1}^{k} a_{k,j} f_j \right)^q \sum_{n=1}^{k} \left(\sum_{i=1}^{n} w_i^{-s'} \right)^{(q-1)} w_n^{\frac{s-q}{1-s}} h_n \right)^{\frac{1}{q}} \|f\|_{p,v}^{-1}.$$

By using Theorem F (ii) and then setting the value $\sigma_{k,t}$, we obtain

$$C \approx \max\left\{ \sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-\frac{1}{q}} \sup_{t \geqslant 1} \left(\sum_{i=1}^{t} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=t}^{\infty} a_{k,t}^q u_k^q \sum_{n=1}^{k} \left(\sum_{i=1}^{n} w_i^{-s'} \right)^{(q-1)} w_n^{\frac{s-q}{1-s}} h_n \right)^{\frac{1}{q}}, \right.$$

$$\left. \sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-\frac{1}{q}} \sup_{t \geqslant 1} \left(\sum_{i=1}^{t} a_{t,i}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=t}^{\infty} u_k^q \sum_{n=1}^{k} \left(\sum_{i=1}^{n} w_i^{-s'} \right)^{(q-1)} w_n^{\frac{s-q}{1-s}} h_n \right)^{\frac{1}{q}} \right\}$$

$$\approx \max\left\{ \sup_{t \geqslant 1} \left(\sum_{i=1}^{t} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-1} \sum_{k=1}^{\infty} \sigma_{k,t} a_{k,t}^q u_k^q \sum_{n=1}^{k} \left(\sum_{i=1}^{n} w_i^{-s'} \right)^{(q-1)} w_n^{\frac{s-q}{1-s}} h_n \right)^{\frac{1}{q}}, \right.$$

$$\left. \sup_{t \geqslant 1} \left(\sum_{i=1}^{t} a_{t,i}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-1} \sum_{k=1}^{\infty} \sigma_{k,t} u_k^q \sum_{n=1}^{k} \left(\sum_{i=1}^{n} w_i^{-s'} \right)^{(q-1)} w_n^{\frac{s-q}{1-s}} h_n \right)^{\frac{1}{q}} \right\}.$$

We change the order of sum and find that

$$C \approx \max\left\{ \sup_{t\geqslant 1}\left(\sum_{i=1}^{t} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sup_{h\geqslant 0}\|h\|_{\frac{s}{q}}^{-1}\sum_{n=1}^{\infty} h_n \left(\sum_{i=1}^{n} w_i^{-s'}\right)^{(q-1)} w_n^{\frac{s-q}{1-s}}\sum_{k=n}^{\infty}\sigma_{k,t}a_{k,t}^q u_k^q\right)^{\frac{1}{q}}, \right.$$
$$\left. \sup_{t\geqslant 1}\left(\sum_{i=1}^{t} a_{t,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sup_{h\geqslant 0}\|h\|_{\frac{s}{q}}^{-1}\sum_{n=1}^{\infty} h_n \left(\sum_{i=1}^{n} w_i^{-s'}\right)^{(q-1)} w_n^{\frac{s-q}{1-s}}\sum_{k=n}^{\infty}\sigma_{k,t} u_k^q\right)^{\frac{1}{q}} \right\}.$$

Applying the reverse Hölder inequality to the previous expression again, we get

$$C \approx \max\left\{ \sup_{t\geqslant 1}\left(\sum_{i=1}^{t} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} w_n^{-s'}\left(\sum_{k=n}^{\infty}\sigma_{k,t}a_{k,t}^q u_k^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{sq}}, \right.$$
$$\left. \sup_{t\geqslant 1}\left(\sum_{i=1}^{t} a_{t,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} w_n^{-s'}\left(\sum_{k=n}^{\infty}\sigma_{k,t} u_k^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{sq}} \right\}$$
$$= \max\{I_1, I_2\}. \tag{23}$$

First, we estimate I_1. And rewrite I_1 as follows:

$$I_1 \approx \sup_{t\geqslant 1}\left(\sum_{i=1}^{t} v_i^{-p'}\right)^{\frac{1}{p'}} \left[\left(\sum_{n=1}^{t}+\sum_{n=t}^{\infty}\right) w_n^{-s'}\left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}}\left(\sum_{k=n}^{\infty}\sigma_{k,t}a_{k,t}^q u_k^q\right)^{\frac{s}{s-q}}\right]^{\frac{s-q}{sq}}$$
$$\approx \sup_{t\geqslant 1}\left(\sum_{i=1}^{t} v_i^{-p'}\right)^{\frac{1}{p'}} \max\left\{\left(\sum_{n=1}^{t} w_n^{-s'}\left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}}\left(\sum_{k=n}^{\infty}\sigma_{k,t}a_{k,t}^q u_k^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{sq}}, \right.$$
$$\left. \left(\sum_{n=t}^{\infty} w_n^{-s'}\left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}}\left(\sum_{k=n}^{\infty}\sigma_{k,t}a_{k,t}^q u_k^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{sq}}\right\}$$
$$= \sup_{t\geqslant 1}\left(\sum_{i=1}^{t} v_i^{-p'}\right)^{\frac{1}{p'}} \max\{I_{11}, I_{12}\}. \tag{24}$$

Let us calculate I_{11} and I_{12} separately.

$$I_{11} = \left(\sum_{k=t}^{\infty} a_{k,t}^q u_k^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{t} w_n^{-s'} \left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}}\right)^{\frac{s-q}{sq}}$$

$$\approx \left(\sum_{n=1}^{t} w_n^{-s'}\right)^{\frac{1}{s'}} \left(\sum_{k=t}^{\infty} a_{k,t}^q u_k^q\right)^{\frac{1}{q}} \tag{25}$$

and

$$I_{12} = \left(\sum_{n=t}^{\infty} w_n^{-s'} \left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} \left(\sum_{k=n}^{\infty} a_{k,t}^q u_k^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{sq}}. \tag{26}$$

From (24), (25) and (26) implies that

$$I_1 \approx \max\{M_4, M_5\}. \tag{27}$$

Let's estimate I_2 as above.

$$I_2 \approx \sup_{t\geqslant 1} \left(\sum_{i=1}^{t} a_{t,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left[\left(\sum_{n=1}^{t} + \sum_{n=t}^{\infty}\right) w_n^{-s'} \left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} \left(\sum_{k=n}^{\infty} \sigma_{k,t} u_k^q\right)^{\frac{s}{s-q}}\right]^{\frac{s-q}{sq}}$$

$$\approx \sup_{t\geqslant 1} \left(\sum_{i=1}^{t} a_{t,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \max\left\{\left(\sum_{n=1}^{t} w_n^{-s'} \left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} \left(\sum_{k=n}^{\infty} \sigma_{k,t} u_k^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{sq}},\right.$$

$$\left.\left(\sum_{n=t}^{\infty} w_n^{-s'} \left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} \left(\sum_{k=n}^{\infty} \sigma_{k,t} u_k^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{sq}}\right\} \approx \sup_{t\geqslant 1} \left(\sum_{i=1}^{t} a_{t,i}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}}$$

$$\times \max\left\{\left(\sum_{n=1}^{t} w_n^{-s'}\right)^{\frac{1}{s'}} \left(\sum_{k=t}^{\infty} u_k^q\right)^{\frac{1}{q}}, \left(\sum_{n=t}^{\infty} w_n^{-s'} \left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} \left(\sum_{k=n}^{\infty} u_k^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{sq}}\right\}$$

$$= \max\{M_6, M_7\}.$$

Therefore, the latter with (23) and (27) give that $C \approx \mathscr{M}$. Proof of Theorem 3 is complete. □

THEOREM 4. *Let* $\min\{p,s\} \leqslant q < \max\{p,s\} < \infty$, $0 < \min\{p,s\} \leqslant 1$, $\max\{p,s\} > 1$. *Let the elements of the matrix* $(a_{n,j})$ *satisfy condition* (2).

(a) If $s \leqslant q < p < \infty$, $0 < s \leqslant 1$ *and* $p > 1$, *then inequality* (1) *holds if and only if* $B = \max\{B_1, B_2, B_3\} < \infty$, *where*

$$B_1 = \sup_{j\geqslant 1}\left(\sum_{k=j}^{\infty}\left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{n=k}^{\infty} u_n^q\right)^{\frac{q}{p-q}} u_k^q\right)^{\frac{p-q}{pq}} w_j^{-1},$$

$$B_2 = \sup_{j\geqslant 1}\left(\sum_{k=1}^{j}\left(\sum_{n=j}^{\infty} a_{n,k}^{q} u_n^{q}\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}} w_j^{-1},$$

$$B_3 = \sup_{j\geqslant 1}\left(\sum_{k=j}^{\infty}\left(\sum_{n=k}^{\infty} a_{n,k}^{q} u_n^{q}\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{k} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}} w_j^{-1}.$$

(b) If If $p \leqslant q < s < \infty$, $0 < p \leqslant 1$ *and* $s > 1$, *then inequality* (1) *holds if and only if* $\mathscr{B} = \max\{B_4, B_5\} < \infty$, *where*

$$B_4 = \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty} a_{n,j}^{q} u_n^{q}\right)^{\frac{1}{q}}\left(\sum_{m=1}^{j} w_m^{-s'}\right)^{\frac{1}{s'}} v_j^{-1},$$

$$B_5 = \sup_{j\geqslant 1}\left(\sum_{n=j}^{\infty}\left(\sum_{k=n}^{\infty} a_{k,j}^{q} u_k^{q}\right)^{\frac{s}{s-q}}\left(\sum_{i=1}^{n} w_i^{-s'}\right)^{\frac{s(q-1)}{s-q}} w_n^{-s'}\right)^{\frac{s-q}{sq}} v_j^{-1}.$$

Moreover, $C \approx B$ *in case (a) and* $C \approx \mathscr{B}$ *in case (b) where* C *is the best constant in* (1).

We can prove Theorem 4 in the same way as Theorem 3, using Theorem F (iii), Theorem F (i) and Theorem A (i).

THEOREM 5. *Let* $0 < q < s \leqslant 1 < p < \infty$, $r_1 = \frac{qp}{p-q}$ *and* $r_2 = \frac{qs}{s-q}$. *Let the elements of the matrix* $(a_{n,i})$ *satisfy condition* (2).

(a) If $0 < q < s \leqslant r_1 < \infty$, *then inequality* (1) *holds if and only if* $K = \max\{K_1, K_2, K_3, K_4\} < \infty$, *where*

$$K_1 = \sup_{j\geqslant 1}\left(\sum_{k=j}^{\infty} \Delta A_k \left(\sum_{n=k}^{\infty} u_n^q\right)^{\frac{r_1}{q}}\right)^{\frac{1}{r_1}} \max_{1\leqslant i\leqslant j} w_i^{-1},$$

$$K_2 = \sup_{j \geqslant 1} \left(\sum_{i=1}^{j} a_{j,i}^{p\prime} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} \Delta \bar{W}_k \left(\sum_{n=k}^{\infty} u_n^q \right)^{\frac{r_2}{q}} \right)^{\frac{1}{r_2}},$$

$$K_3 = \sup_{j \geqslant 1} \left(\sum_{k=j}^{\infty} \left(\sum_{z=k}^{\infty} a_{z,k}^q u_z^q \right)^{\frac{r_1}{q}} \left(\sum_{n=1}^{k} v_n^{-p'} \right)^{\frac{r_1}{q'}} v_k^{-p'} \right)^{\frac{1}{r_1}} \max_{1 \leqslant i \leqslant j} w_i^{-1},$$

$$K_4 = \sup_{j \geqslant 1} \left(\sum_{i=1}^{j} v_i^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{k=j}^{\infty} \left(\sum_{z=k}^{\infty} a_{z,j}^q u_k^q \right)^{\frac{r_2}{q}} \Delta \bar{W}_k \right)^{\frac{1}{r_2}},$$

(b) If $0 < q < r_1 < s < \infty$, then inequality (1) holds if and only if $\bar{K} = \max\{\bar{K}_1, \bar{K}_2, \bar{K}_3, \bar{K}_4\} < \infty$, where

$$\bar{K}_1 = \left[\sum_{n=1}^{\infty} \Delta A_n \left(\sum_{z=n}^{\infty} u_z^q \right)^{\frac{r_1}{q}} \left(\sum_{i=n}^{\infty} \Delta A_i \left(\sum_{z=i}^{\infty} u_z^q \right)^{\frac{r_1}{q}} \right)^{\frac{r_1}{s-r_1}} \max_{1 \leqslant k \leqslant n} w_k^{-\frac{sr_1}{s-r_1}} \right]^{\frac{s-r_1}{sr_1}},$$

$$\bar{K}_2 = \left[\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_{n,i}^{p'} v_i^{-p'} \right)^{\frac{r_2(p-1)}{p-r_2}} \left(\sum_{k=n}^{\infty} \Delta \bar{W}_k \left(\sum_{m=k}^{\infty} u_m^q \right)^{\frac{r_2}{q}} \right)^{\frac{r_2}{p-r_2}} \left(\sum_{z=n}^{\infty} u_z^q \right)^{\frac{r_2}{q}} \Delta \bar{W}_n \right]^{\frac{p-r_2}{pr_2}},$$

$$\bar{K}_3 = \left[\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \left(\sum_{z=k}^{\infty} a_{z,k}^q u_z^q \right)^{\frac{r_1}{q}} \left(\sum_{m=1}^{k} v_m^{-p'} \right)^{\frac{r_1}{q'}} v_k^{-p'} \right)^{\frac{r_1}{s-r_1}} \left(\sum_{z=n}^{\infty} a_{z,n}^q u_z^q \right)^{\frac{r_1}{q}} \right.$$
$$\left. \times \left(\sum_{m=1}^{n} v_m^{-p'} \right)^{\frac{r_1}{q'}} v_n^{-p'} \max_{1 \leqslant i \leqslant n} w_i^{-\frac{sr_1}{s-r_1}} \right]^{\frac{s-r_1}{sr_1}},$$

$$\bar{K}_4 = \left[\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} v_i^{-p'} \right)^{\frac{p(r_2-1)}{p-r_2}} \left(\sum_{k=n}^{\infty} \left(\sum_{z=k}^{\infty} a_{z,n}^q u_z^q \right)^{\frac{r_2}{q}} \Delta \bar{W}_k \right)^{\frac{p}{p-r_2}} v_n^{-p'} \right]^{\frac{p-r_2}{pr_2}}.$$

Moreover, $C \approx K$ in case (a) and $C \approx \bar{K}$ in case (b), where C is the best constant in (1)

Proof. In this case, we apply Theorem A (iv) to (22), then we get

$$C \approx \sup_{f \geqslant 0} \|f\|_{p,v}^{-1} \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{j=1}^{n} a_{n,j} f_j \right)^q \left(\sum_{i=n}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^q \right)^{\frac{q}{s-q}} \max_{1 \leqslant i \leqslant n} w_i^{-\frac{sq}{s-q}} \right)^{\frac{s-q}{sq}}. \tag{28}$$

Since $\bar{W}_0 = 0$ and $U_n := \sum_{i=n}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^q$ is positive decreasing sequence, we use Abel transformation for (28), then

$$C \approx \sup_{f \geqslant 0} \|f\|_{p,v}^{-1} \left(\sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^q \right)^{\frac{s}{s-q}} \Delta \bar{W}_n \right)^{\frac{s-q}{sq}}.$$

Since $\frac{s}{s-q} > 1$, it follows from reverse Hölder inequality that

$$C \approx \sup_{f \geqslant 0} \|f\|_{p,v}^{-1} \sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-\frac{1}{q}} \left(\sum_{n=1}^{\infty} h_n \left[\Delta \bar{W}_n \right]^{\frac{s-q}{s}} \sum_{i=n}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^q \right)^{\frac{1}{q}}.$$

From then on we will use the notation $\left[\Delta \bar{W}_n \right]^{\frac{s-q}{s}} = \Delta \bar{W}_n^{\frac{s-q}{s}}$. Changing the order of the supremum and the sums, and then using Theorem F (iii), we get

$$\begin{aligned} C &\approx \sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-\frac{1}{q}} \sup_{f \geqslant 0} \|f\|_{p,v}^{-1} \left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^q \sum_{n=1}^{i} h_n \Delta \bar{W}_n^{\frac{s-q}{s}} \right)^{\frac{1}{q}} \\ &\approx \max \Bigg\{ \sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-\frac{1}{q}} \Bigg(\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_{n,i}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \\ &\times \left(\sum_{k=n}^{\infty} u_k^q \sum_{z=1}^{k} h_z \Delta \bar{W}_z^{\frac{s-q}{s}} \right)^{\frac{q}{p-q}} u_n^q \sum_{z=1}^{n} h_z \Delta \bar{W}_z^{\frac{s-q}{s}} \Bigg)^{\frac{p-q}{pq}}, \\ &\sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-\frac{1}{q}} \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} a_{k,n}^q u_k^q \sum_{z=1}^{k} h_z \Delta \bar{W}_z^{\frac{s-q}{s}} \right)^{\frac{p}{p-q}} \left(\sum_{i=1}^{n} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} v_n^{-p'} \right)^{\frac{p-q}{pq}} \Bigg\} \\ &= \max\{I_1, I_2\}. \end{aligned} \tag{29}$$

First, we estimate I_1. As above, by Abel transformation, we get

$$I_1 = \sup_{h \geqslant 0} \|h\|_{\frac{s}{q}}^{-\frac{1}{q}} \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} u_k^q \sum_{z=1}^{k} h_z \Delta \bar{W}_z^{\frac{s-q}{s}} \right)^{\frac{p}{p-q}} \Delta A_n \right)^{\frac{p-q}{pq}}.$$

According to

$$\sum_{k=n}^{\infty} u_k^q \sum_{z=1}^{k} h_z \Delta \bar{W}_z^{\frac{s-q}{s}} \approx \sum_{k=n}^{\infty} u_k^q \sum_{z=1}^{n} h_z \Delta \bar{W}_z^{\frac{s-q}{s}} + \sum_{z=n}^{\infty} h_z \Delta \bar{W}_z^{\frac{s-q}{s}} \sum_{k=z}^{\infty} u_k^q$$

we estimate I_1 as follows

$$I_1^q \approx \sup_{h\geqslant 0}\|h\|_{\frac{s}{q}}^{-1}\left[\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}u_k^q\right)^{\frac{p}{p-q}}\left(\sum_{z=1}^{n}h_z\Delta\bar{W}_z^{\frac{s-q}{s}}\right)^{\frac{p}{p-q}}\Delta A_n\right]^{\frac{p-q}{p}}$$
$$+\sup_{h\geqslant 0}\|h\|_{\frac{s}{q}}^{-1}\left[\sum_{n=1}^{\infty}\left(\sum_{z=n}^{\infty}h_z\Delta\bar{W}_z^{\frac{s-q}{s}}\sum_{k=z}^{\infty}u_k^q\right)^{\frac{p}{p-q}}\Delta A_n\right]^{\frac{p-q}{p}}$$
$$\approx \max\{I_{11}^q, I_{12}^q\}. \tag{30}$$

Now we consider I_2. Since

$$\sum_{k=n}^{\infty}a_{k,n}^q u_k^q\sum_{z=1}^{k}h_z\Delta\bar{W}_z^{\frac{s-q}{s}} \approx \sum_{z=1}^{n}h_z\Delta\bar{W}_z^{\frac{s-q}{s}}\sum_{k=n}^{\infty}a_{k,n}^q u_k^q+\sum_{z=n}^{\infty}h_z\Delta\bar{W}_z^{\frac{s-q}{s}}\sum_{k=z}^{\infty}a_{k,n}^q u_k^q,$$

we can write I_2 as follows

$$I_2^q \approx \max\left\{\sup_{h\geqslant 0}\|h\|_{\frac{s}{q}}^{-1}\left[\sum_{n=1}^{\infty}\left(\sum_{z=1}^{n}h_z\Delta\bar{W}_z^{\frac{s-q}{s}}\right)^{\frac{p}{p-q}}\left(\sum_{k=n}^{\infty}a_{k,n}^q u_k^q\right)^{\frac{p}{p-q}}V_n\right]^{\frac{p-q}{p}},\right.$$
$$\left.\sup_{h\geqslant 0}\|h\|_{\frac{s}{q}}^{-1}\left[\sum_{n=1}^{\infty}\left(\sum_{z=n}^{\infty}h_z\Delta\bar{W}_z^{\frac{s-q}{s}}\sum_{k=z}^{\infty}a_{k,n}^q u_k^q\right)^{\frac{p}{p-q}}V_n\right]^{\frac{p-q}{p}}\right\} = \max\{I_{21}^q, I_{22}^q\}, \tag{31}$$

where $V_n = \left(\sum_{i=1}^{n}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}v_n^{-p'}$.

There are two cases for the parameters relation that need to be considered in the further estimation of the obtained quantities:

$$\text{case (a):}\quad 1<\frac{s}{q}\leqslant\frac{p}{p-q}\Leftrightarrow q<s\leqslant r_1,$$

$$\text{case (b):}\quad 1<\frac{p}{p-q}<\frac{s}{q}\Leftrightarrow q<r_1<s.$$

Case (a). By applying Theorem A (ii) to I_{11}, we obtain

$$I_{11}^q \approx \sup_{j\geqslant 1}\left(\sum_{i=1}^{j}\Delta\bar{W}_i\right)^{\frac{s-q}{s}}\left(\sum_{k=j}^{\infty}\left(\sum_{m=k}^{\infty}u_m^q\right)^{\frac{p}{p-q}}\Delta A_k\right)^{\frac{p-q}{p}}.$$

Since $\bar{W}_0 = 0$, the first sum is equal to $\bar{W}_j$. Then

$$I_{11} \approx \sup_{j\geqslant 1}\left(\sum_{k=j}^{\infty}\Delta A_k\left(\sum_{m=k}^{\infty}u_m^q\right)^{\frac{r_1}{q}}\right)^{\frac{1}{r_1}}\max_{1\leqslant i\leqslant j}w_i^{-1} = K_1. \tag{32}$$

Now we estimate I_{12}. From Theorem N (i), it follows that

$$I_{12}^q \approx \sup_{j\geqslant 1}\left(\sum_{k=j}^{\infty}\Delta \bar{W}_k\left(\sum_{m=k}^{\infty}u_m^q\right)^{\frac{s}{s-q}}\right)^{\frac{s-q}{s}}\left(\sum_{i=1}^{j}\Delta A_k\right)^{\frac{p-q}{p}},$$

yields that

$$I_{12} \approx \sup_{j\geqslant 1}\left(\sum_{i=1}^{j}a_{j,i}^{p'}v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{k=j}^{\infty}\left(\sum_{m=k}^{\infty}u_m^q\right)^{\frac{r_2}{q}}\Delta \bar{W}_k\right)^{\frac{1}{r_2}} = K_2. \tag{33}$$

Now we consider the estimations I_{21} and I_{22}. According to Theorem A (ii), we obtain that

$$I_{21}^q \approx \sup_{j\geqslant 1}\left(\sum_{k=1}^{j}\Delta \bar{W}_k\right)^{\frac{s-q}{s}}\left(\sum_{i=j}^{\infty}\left(\sum_{z=i}^{\infty}a_{z,i}^q u_z^q\right)^{\frac{p}{p-q}}V_i\right)^{\frac{p-q}{p}}.$$

Therefore

$$I_{21} \approx \sup_{j\geqslant 1}\left(\sum_{i=j}^{\infty}\left(\sum_{z=i}^{\infty}a_{z,i}^q u_z^q\right)^{\frac{r_1}{q}}\left(\sum_{n=1}^{i}v_n^{-p'}\right)^{\frac{r_1}{q'}}v_i^{-p'}\right)^{\frac{1}{r_1}}\max_{1\leqslant k\leqslant j} w_k^{-1} = K_3. \tag{34}$$

In I_{22}, there exists an operator with the matrix $(b_{z,n})$, where $b_{z,n} = \sum_{k=z}^{\infty} a_{k,n}^q u_k^q$, $z \geqslant n$. From $1 \leqslant n \leqslant i \leqslant z$ it follows that $1 \leqslant n \leqslant i \leqslant k$. Due to the condition (2), we have

$$b_{z,n} \approx \sum_{k=z}^{\infty}a_{k,i}^q u_k^q + a_{i,n}^q\sum_{k=z}^{\infty}u_k^q = b_{z,i} + \bar{a}_{i,n}\bar{u}_z,$$

it implies that the entries of the matrix $(b_{z,n})$ satisfy assumption (10). By using Theorem Q (ii), we get

$$I_{22}^q \approx \max\left\{\sup_{n\geqslant 1}\left(\sum_{j=1}^{n}V_j\right)^{\frac{p-q}{p}}\left(\sum_{i=n}^{\infty}b_{i,n}^{\frac{s}{s-q}}\Delta \bar{W}_i\right)^{\frac{s-q}{s}},\right.$$

$$\left.\sup_{n\geqslant 1}\left(\sum_{j=1}^{n}\bar{a}_{n,j}^{\frac{p}{p-q}}V_j\right)^{\frac{p-q}{p}}\left(\sum_{i=n}^{\infty}\bar{u}_i^{\frac{s}{s-q}}\Delta \bar{W}_i\right)^{\frac{s-q}{s}}\right\} = \max\{I_{211}^q, I_{212}^q\}. \tag{35}$$

After replacing the notations, we obtain

$$I_{211} \approx \sup_{n\geqslant 1}\left(\sum_{j=1}^{n}v_j^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{i=n}^{\infty}\left(\sum_{k=i}^{\infty}a_{k,n}^q u_k^q\right)^{\frac{r_2}{q}}\Delta \bar{W}_i\right)^{\frac{1}{r_2}} = K_4, \tag{36}$$

$$I_{212} \approx \sup_{n\geqslant 1}\left(\sum_{j=1}^{n} a_{n,j}^{r_1}\left(\sum_{m=1}^{j} v_m^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_j^{-p'}\right)^{\frac{1}{r_1}}\left(\sum_{i=n}^{\infty}\left(\sum_{k=i}^{\infty} u_k^q\right)^{\frac{r_2}{q}}\Delta\bar{W}_i\right)^{\frac{1}{r_2}} = K_5. \quad (37)$$

In case $0<q<s\leqslant r_1$, from (29)–(37), we conclude that

$$C \approx \max\{K_1,K_2,K_3,K_4,K_5\}$$

where C is the best constant in (1). Since $r_1 = p' + \frac{p'p(q-1)}{p-q}$, we can compare expressions K_2 and K_5 as follows:

$$K_5 = \sup_{n\geqslant 1}\left(\sum_{j=1}^{n} a_{n,j}^{p'}\left(\sum_{m=1}^{j} a_{n,j}^{p'} v_m^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_j^{-p'}\right)^{\frac{1}{r_1}}\left(\sum_{i=n}^{\infty}\left(\sum_{k=i}^{\infty} u_k^q\right)^{\frac{r_2}{q}}\Delta\bar{W}_i\right)^{\frac{1}{r_2}},$$

by taking into account the condition (2) and Lemma 1, we proceed

$$K_5 \leqslant \sup_{n\geqslant 1}\left(\sum_{j=1}^{n} a_{n,j}^{p'}\left(\sum_{m=1}^{j} a_{n,m}^{p'} v_m^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_j^{-p'}\right)^{\frac{1}{r_1}}\left(\sum_{i=n}^{\infty}\left(\sum_{k=i}^{\infty} u_k^q\right)^{\frac{r_2}{q}}\Delta\bar{W}_i\right)^{\frac{1}{r_2}}$$

$$= \sup_{n\geqslant 1}\left(\sum_{j=1}^{n} a_{n,j}^{p'} v_j^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{i=n}^{\infty}\left(\sum_{k=i}^{\infty} u_k^q\right)^{\frac{r_2}{q}}\Delta\bar{W}_i\right)^{\frac{1}{r_2}} = K_2.$$

Therefore

$$C \approx \max\{K_1,K_2,K_3,K_4\}.$$

Case (b). We use Theorem A (iii) and Theorem N (ii) to I_{11} and I_{12}, respectively.

$$I_{11}^q \approx \left(\sum_{n=1}^{\infty}\left(\sum_{i=n}^{\infty}\Delta A_i\left(\sum_{m=i}^{\infty} u_m^q\right)^{\frac{p}{p-q}}\right)^{\frac{pq}{sp-sq-pq}}\right.$$

$$\left.\times\left(\sum_{k=1}^{n}\Delta\bar{W}_k\right)^{\frac{p(s-q)}{sp-sq-pq}}\Delta A_n\left(\sum_{m=n}^{\infty} u_m^q\right)^{\frac{p}{p-q}}\right)^{\frac{sp-sq-pq}{ps}}.$$

$$I_{12}^q \approx \left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\Delta A_i\right)^{\frac{s(p-q)}{sp-sq-pq}}\left(\sum_{k=n}^{\infty}\left(\sum_{m=k}^{\infty} u_m^q\right)^{\frac{s}{s-q}}\Delta\bar{W}_k\right)^{\frac{sq}{sp-sq-pq}}\right.$$

$$\left.\times\left(\sum_{m=n}^{\infty} u_m^q\right)^{\frac{s}{s-q}}\Delta\bar{W}_n\right)^{\frac{sp-sq-pq}{ps}},$$

Hence

$$I_{11} \approx \left(\sum_{n=1}^{\infty} \Delta A_n \left(\sum_{m=n}^{\infty} u_m^q \right)^{\frac{r_1}{q}} \left(\sum_{i=n}^{\infty} \Delta A_i \left(\sum_{m=i}^{\infty} u_m^q \right)^{\frac{r_1}{q}} \right)^{\frac{r_1}{s-r_1}} \max_{1\leqslant k\leqslant n} w_k^{-\frac{sr_1}{s-r_1}} \right)^{\frac{s-r_1}{sr_1}} = \bar{K}_1, \tag{38}$$

$$I_{12} \approx \left(\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} a_{n,i}^{p'} v_i^{-p'} \right)^{\frac{r_2(p-1)}{p-r_2}} \left(\sum_{k=n}^{\infty} \left(\sum_{m=k}^{\infty} u_m^q \right)^{\frac{r_2}{q}} \Delta \bar{W}_k \right)^{\frac{r_2}{p-r_2}} \left(\sum_{m=n}^{\infty} u_m^q \right)^{\frac{r_2}{q}} \Delta \bar{W}_n \right)^{\frac{p-r_2}{pr_2}} = \bar{K}_2. \tag{39}$$

Now we estimate I_{21}. from Theorem A (iii), it follows that

$$I_{21} \approx \left[\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \left(\sum_{z=k}^{\infty} a_{z,k}^q u_z^q \right)^{\frac{p}{p-q}} V_k \right)^{\frac{pq}{sp-sq-pq}} \left(\sum_{i=1}^{n} \Delta \bar{W}_i \right)^{\frac{p(s-q)}{sp-sq-pq}} \times \left(\sum_{z=n}^{\infty} a_{z,n}^q u_z^q \right)^{\frac{p}{p-q}} V_n \right]^{\frac{sp-sq-pq}{psq}}$$

$$= \left[\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \left(\sum_{z=k}^{\infty} a_{z,k}^q u_z^q \right)^{\frac{r_1}{q}} V_k \right)^{\frac{r_1}{s-r_1}} \left(\sum_{z=n}^{\infty} a_{z,n}^q u_z^q \right)^{\frac{r_1}{q}} V_n \max_{1\leqslant i\leqslant n} w_i^{-\frac{sr_1}{s-r_1}} \right]^{\frac{s-r_1}{sr_1}} = \bar{K}_3. \tag{40}$$

We use Theorem Q (i) for I_{22}, then we get

$$I_{22}^q \approx \max \left\{ \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} \bar{a}_{i,j}^{\frac{p}{p-q}} V_j \right)^{\frac{s(p-q)}{sp-sq-pq}} \left(\sum_{k=i}^{\infty} \bar{u}_k^{\frac{s}{s-q}} \Delta \bar{W}_k \right)^{\frac{sq}{sp-sq-pq}} \bar{u}_i^{\frac{s}{s-q}} \Delta \bar{W}_i \right]^{\frac{sp-sq-pq}{ps}}, \right.$$

$$\left. \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} V_j \right)^{\frac{pq}{sp-sq-pq}} \left(\sum_{k=i}^{\infty} b_{k,i}^{\frac{s}{s-q}} \Delta \bar{W}_k \right)^{\frac{p(s-q)}{sp-sq-pq}} V_i \right]^{\frac{sp-sq-pq}{ps}} \right\} = \max\{I_{221}^q, I_{222}^q\}. \tag{41}$$

After replacing the notations, we rewrite the estimates as follows

$$I_{221} \approx \left[\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} a_{i,j}^{r_1} V_j \right)^{\frac{r_2(p-q)}{q(p-r_2)}} \left(\sum_{k=i}^{\infty} \left(\sum_{k=i}^{\infty} u_k^q \right)^{\frac{r_2}{q}} \Delta \bar{W}_k \right)^{\frac{r_2}{p-r_2}} \left(\sum_{k=i}^{\infty} u_k \right)^{\frac{r_2}{q}} \Delta \bar{W}_i \right]^{\frac{(p-r_2)}{pr_2}} = \bar{K}_5, \tag{42}$$

$$I_{222} \approx \left[\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} v_j^{-p'}\right)^{\frac{p(r_2-1)}{p-r_2}}\left(\sum_{k=i}^{\infty}\left(\sum_{m=k}^{\infty} a_{m,i}^q u_m^q\right)^{\frac{r_2}{q}}\Delta\bar{W}_k\right)^{\frac{p}{p-r_2}} v_i^{-p'}\right]^{\frac{p-r_2}{pr_2}} = \bar{K}_4. \quad (43)$$

Similarly as above, we can calculate $\max\{\bar{K}_2,\bar{K}_5\} = \bar{K}_2$ for case $0 < q < r_1 < s$. So, the latter with (29)–(31) and (38)–(43) give that

$$C \approx \max\{\bar{K}_1,\bar{K}_2,\bar{K}_3,\bar{K}_4\}.$$

Proof of Theorem 4 is complete. □

Acknowledgement. We sincerely thank the reviewer for their valuable suggestions, which have improved the final version of this paper. This work was financially supported by the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22684768).

REFERENCES

[1] M. I. AGUILAR CANESTRO, P. ORTEGA SALVADOR AND C. RAMIREZ TORREBLANCA, *Weighted bilinear Hardy inequalities*, J. Math. Anal. Appl., **387**, (2012), 320–334.

[2] K. F. ANDERSEN AND H. HEINIG, *Weighted norm inequalities for certain integral operators*, SIAM J. Math., **14**, (1983), 834–844.

[3] G. BENNETT, *Some elementary inequalities*, Quart. J. Math. Oxford Ser., **38**, 2 (1987), 401–425.

[4] G. BENNETT, *Some elementary inequalities II*, Quart. J. Math. Oxford Ser., **39**, 2 (1989), 385–400.

[5] G. BENNETT, *Some elementary inequalities III*, Quart. J. Math. Oxford Ser., **42**, 2 (1991), 149–174.

[6] M. SH. BRAVERMAN AND V. D. STEPANOV, *On the discrete Hardy inequality*, Bull. London Math. Soc., **26**, (1994), 283–287.

[7] H. HEINIG, *Weighted norm inequalities for certain integral operators II*, Proc. Amer. Math. Soc., **95** (1985), 387–395.

[8] P. JAIN, S. KANJILAL, G. E. SHAMBILOVA AND V. D. STEPANOV, *Bilinear weighted Hardy-type inequalities in discrete and q-calculus*, Math. Ineq. Appl., **23**, 4 (2020), 1279–1310, https://doi.org/10.7153/mia-2020-23-96.

[9] A. KALYBAY, A. TEMIRKHANOVA AND N. ZHANGABERGENOVA, *On iterated discrete Hardy type inequalities for a class of matrix operators*, Anal. Math., **49**, 1 (2022), 137–150, https://doi.org/10.1007/s10476-022-0182-2.

[10] M. KŘEPELA, *Iterating bilinear Hardy inequalities*, Proc. Edinb. Math. Soc., **60**, 4 (2017), 955–971.

[11] M. KŘEPELA, *Bilinear weighted Hardy inequality for non-increasing functions*, Publicacions Matematiques, **61**, 1 (2017), 3–50.

[12] A. KUFNER, L. MALIGRANDA AND L-E. PERSSON, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[13] R. OINAROV, C. A. OKPOTI AND L.-E. PERSSON, *Weighted inequalities of Hardy type for matrix operators: the case $q < p$*, Math. Inequal. Appl., **10**, 4 (2007), 843–861.

[14] R. OINAROV, L.-E. PERSSON AND A. TEMIRKHANOVA, *Weighted inequalities for class of matrix operators: the case $p \leqslant q$*, Math. Ineq. Appl., **12**, 4 (2009), 891–903.

[15] R. OINAROV AND S. KH. SHALGYNBAEVA, *Weighted additive estimate of a class of matrix operators*, Izvestiya NAN RK, serial Phys.-Mat., **7**, 1 (2004), 39–49, (in Russian).

[16] R. OINAROV AND Z. TASPAGANBETOVA, *Criteria of boundedness and compactness of a class of matrix operators*, J. Ineq. Appl., **2012**, 53 (2012), https://doi.org/10.1186/1029-242X-2012-53.

[17] B. K. OMARBAYEVA, L.-E. PERSSON AND A. M. TEMIRKHANOVA, *Weighted iterated discrete Hardy-type inequalities*, Math. Ineq. Appl., **23**, 3 (2020), 943–959, doi:10.7153/mia-2020-23-73.

[18] S. SHAIMARDAN AND S. SHALGYNBAEVA, *Hardy-type inequalities for matrix operators*, Bulletin of the Karaganda University – Math. Ser., **88**, 4 (2017), 63–72.

[19] V. D. STEPANOV AND G. E. SHAMBILOVA, *Multidimensional bilinear Hardy inequalities*, Sib. math. journal., **61**, 4 (2020), 913–93, (in Russian).

[20] V. D. STEPANOV AND G. E. SHAMBILOVA, *On two-dimensional bilinear inequalities with rectangular Hardy operators in weighted Lebesgue spaces*, Proceedings of Steklov Mathematical Institute, **312**, (2021), 251–258, (in Russian).

[21] ZH. TASPAGANBETOVA AND A. TEMIRKHANOVA, *Boundedness and compactness criteria of a certain class of matrix operators*, Math. Journal., **11**, 2 (40) (2011), 73–85.

[22] ZH. TASPAGANBETOVA AND A. TEMIRKHANOVA, *Criteria on boundedness of matrix operators in weighted spaces of sequences and their applications*, Ann. Funct. Anal. **2**, 1 (2011), 114–127.

[23] N. ZHANGABERGENOVA AND A. M. TEMIRKHANOVA, *Iterated discrete Hardy-type inequalities*, Eurasian Mathematical Journal, **14**, 1 (2023), 81–95, doi:https://doi.org/10.32523/2077-9879-2023-14-1-81-95.

[24] N. ZHANGABERGENOVA, *Weighted estimates for a class of matrix operators*, Math. Ineq. Appl., **26**, 3 (2023), 627–644, doi:10.7153/mia-2023-26-38.

(Received December 1, 2024)

Nazerke Zhangabergenova
Department of Fundamental Mathematics
Faculty of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
Satpayev str. 2, Nur-Sultan, 010000 Kazakhstan
e-mail: zhanabergenova.ns@gmail.com

Ainur Temirkhanova
Department of Fundamental Mathematics
Faculty of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
Satpayev str. 2, Nur-Sultan, 010000 Kazakhstan
e-mail: ainura-t@yandex.kz