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STRONG LAWS FOR NONSTATIONARY GARCH MODELS

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Abstract. This paper studies the basic limit theories for the volatilities in nonstationary GARCH(1,1) models. Those include the classical results such as the Marcinkiewicz-Zygmund strong law of large numbers and the Hartman-Wintner law of the iterated logarithm. The main results precisely characterize asymptotic behaviors of the volatilities in nonstationary GARCH(1,1) models, and provide more insight into this top. Some numerical simulations are provided to verify the validity of theoretical results.

1. Introduction

Generalized autoregressive conditional heteroscedastic (GARCH) models, the extension of autoregressive conditional heteroscedastic models, are treated as the benchmark model to capture conditional volatilities, and now have been especially popular in econometric modeling, statistical inference and financial data analyzing, help econometricians estimate the variance of particular point of a sequence. GARCH models were found by Bollerslev (1986), who extended the initial models in Engle (1982). From then on, GARCH models have attracted more attention, and numerous works have been done. One can refer to Nelson (1990), Lumsdaine (1996), Ling and Li (1997), Hall and Yao (2003), Chan and Ling (2006), Francq et al. (2012, 2013). A comprehensive account of GARCH models is available in Francq and Zakoïan (2010).

A GARCH(1,1) model is defined as

$$\begin{aligned} y_t &= \eta_t \sqrt{h_t}, \quad t = 1,2,\cdots, \\ h_t &= \omega + \alpha y_{t-1}^2 + \beta h_{t-1}, \end{aligned} \tag{1.1}$$

with initial values y_0 and h_0, $\omega > 0$, $\alpha > 0$, and $\beta \geqslant 0$. Both y_0 and h_0 are nonnegative random variables and independent of $\{\eta_t, t \geqslant 0\}$, where $\{\eta_t, t \geqslant 0\}$ is a sequence of independent and identically distributed random variables with η_1^2 is nondegenerate.

Let η be a random variable with the same distribution as η_t, denote

$$\mu = E\log(\beta + \alpha\eta^2).$$

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Then, μ is called the top Lyapunov exponent associated to model (1.1). Nelson (1990) proved that the necessary and sufficient condition is $\mu < 0$ for the existence of strict stationary solution to GARCH(1,1) model. For more associated results of GARCH models in the case $\mu < 0$, we can go through the related articles. For the probabilistic properties, see Bibi, Aknouche (2009), Posedel (2005), Bougerol and Picard (1992a), Berkes (2003). For asymptotic behavior, see Lee and Shin (2004), and Carrasco and Chen (2002). For strict stationarity properties, see Bougerol and Picard (1992b), Francq and Zakoïan (2012).

For the associated results of GARCH models in the case $\mu \geqslant 0$, one can refer to Nelson (1990), Kleibergen and van Dijk (1993), Francq and Zakoïan (2013), Linton et al. (2010), Li et al. (2014), and Hong and Hwang (2016). For more details, Nelson (1990) first studied dynamic behavior of h_t in (1.1) when $\mu \geqslant 0$, and showed that h_t tends to infinity almost surely. Linton et al. (2010) estimated a nonstationary semi-strong GARCH(1,1) model with heavy-tailed error, showed that h_t is divergent, and renormalized to converge in distribution to a nondegenerate limit. Francq and Zakoïan (2013) studied the inference in nonstationary asymmtric GARCH models which includes model (1.1), and proved that

$$\rho^t h_t \to \infty \text{ a.s.} \tag{1.2}$$

as $t \to \infty$ for any $\rho > e^{-\mu}$ when $\mu > 0$. Li et al. (2014) characterized asymptotic behaviors of the volatilities in nonstationary GARCH(1,1) models by showing that the volatility converges in distribution to a non-degenerate limit after suitable renormalization. Hong and Hwang (2016) extended the result of Li et al. (2014).

Due to the works in the nonstationary case, in particular to the work of Li et al. (2014), a question is nature to rise, whether can we give more precise convergence rate than (1.2) after suitable renormalization? This paper will give the positive answer to the question, and study the basic limit theories, such as the Marcinkiewicz-Zygmund strong law of large numbers and the Hartman-Wintner law of the iterated logarithm, for the volatilities in nonstationary GARCH(1,1) models. These classical results will provide more insight into this top.

2. Main results

We first recall the Marcinkiewicz-Zygmund strong law of large numbers, the Hartman-Wintner law of the iterated logarithm and some automatic results. Let $\{Y_t, t \geqslant 1\}$ be a sequence of independent random variables with common distribution as Y. For $1 \leqslant p < 2$, the Marcinkiewicz-Zygmund strong law of large numbers states that

$$t^{-1/p} \sum_{j=1}^{t} Y_j \to 0 \text{ a.s.} \tag{2.1}$$

if and only if $EY = 0$ and $E|Y|^p < \infty$. By (2.1), it is easy to show that

$$t^{-1/p} \max_{1 \leqslant k \leqslant t} |\sum_{j=1}^{k} Y_j| \to 0 \text{ a.s.} \tag{2.2}$$

Note that

$$\sum_{j=1}^{t-1} Y_j \leqslant \max_{1\leqslant k\leqslant t-1} \sum_{j=1}^{k} Y_{t-j} \leqslant \sum_{j=1}^{t-1} Y_j + \max_{1\leqslant k\leqslant t-1} \left| \sum_{j=1}^{k} Y_j \right|.$$

By (2.1) and (2.2),

$$t^{-1/p} \max_{1\leqslant k\leqslant t-1} \sum_{j=1}^{k} Y_{t-j} \to 0 \text{ a.s.} \tag{2.3}$$

Meanwhile, $E|Y|^p < \infty$ implies that

$$t^{-1/p} Y_t \to 0 \text{ a.s.} \tag{2.4}$$

The Hartman-Wintner law of the iterated logarithm (see Hartman and Wintner, 1941) states that if $EY = 0$ and $\sigma^2 = EY^2 \in (0,\infty)$, then

$$\limsup_{t\to\infty} \frac{\sum_{j=1}^{t} Y_j}{\sigma\sqrt{2t\log\log t}} = 1 \text{ a.s.}, \quad \liminf_{t\to\infty} \frac{\sum_{j=1}^{t} Y_j}{\sigma\sqrt{2t\log\log t}} = -1 \text{ a.s.}, \tag{2.5}$$

and the cluster set of the sequence $\{(\sigma\sqrt{2t\log\log t})^{-1}\sum_{j=1}^{t} Y_j, t \geqslant 3\}$ is $[-1,1]$ with probability one. Furthermore, by the Strassen's strong invariance principle (see, Corollary of Theorem 3 in Strassen, 1964),

$$\limsup_{t\to\infty} \frac{\max_{1\leqslant k\leqslant t-1}\sum_{j=1}^{k} Y_{t-j}}{\sigma\sqrt{2t\log\log t}} = 1 \text{ a.s.}, \quad \liminf_{t\to\infty} \frac{\max_{1\leqslant k\leqslant t-1}\sum_{j=1}^{k} Y_{t-j}}{\sigma\sqrt{2t\log\log t}} = 0 \text{ a.s.}, \tag{2.6}$$

and the cluster set of the sequence $\{(\sigma\sqrt{2t\log\log t})^{-1}\max_{1\leqslant k\leqslant t-1}\sum_{j=1}^{k} Y_{t-j}, t \geqslant 3\}$ is $[0,1]$ with probability one. Conversely, if (2.5) holds for some $0 < \sigma < \infty$, Strassen (1966) showed that $EY = 0$ and $EY^2 = \sigma^2$.

Now we state the main results. Some preliminary lemmas and the proofs of the main results will be detailed in the next section.

THEOREM 2.1. *Let $1 \leqslant p < 2$, $\{\eta, \eta_t, t \geqslant 0\}$ be a sequence of independent and identically distributed random variables in (1.1) with $\mu = E\log(\beta + \alpha\eta^2) \in [0,\infty)$. Assume $E|\log(\beta+\alpha\eta^2)|^p < \infty$, then*

$$\left(\frac{h_t}{e^{\mu t}}\right)^{t^{-1/p}} \to 1 \ \ a.s. \tag{2.7}$$

COROLLARY 2.1. *Let $\{\eta, \eta_t, t \geqslant 0\}$ be a sequence of independent and identically distributed random variables in (1.1) with $\mu = E\log(\beta + \alpha\eta^2) \in [0,\infty)$. Then*

$$\rho^t h_t \to \begin{cases} 0 & a.s., \quad if\ 0 < \rho < e^{-\mu}, \\ \infty & a.s., \quad if\ \rho > e^{-\mu}. \end{cases} \tag{2.8}$$

And when $\mu > 0$,

$$0 = \liminf_{t\to\infty} e^{-\mu t} h_t < \limsup_{t\to\infty} e^{-\mu t} h_t = \infty \ \ a.s. \tag{2.9}$$

THEOREM 2.2. *Let* $\{\eta, \eta_t, t \geqslant 0\}$ *be a sequence of independent and identically distributed random variables in (1.1) with* $\mu = E\log(\beta + \alpha\eta^2) \in [0,\infty)$. *Assume that* $\sigma^2 = Var[\log(\beta + \alpha\eta^2)] \in (0,\infty)$, *then when* $\mu > 0$,

$$\limsup_{t\to\infty}\left(\frac{h_t}{e^{\mu t}}\right)^{(\sigma\sqrt{2t\log\log t})^{-1}} = e \ \ a.s., \ \liminf_{t\to\infty}\left(\frac{h_t}{e^{\mu t}}\right)^{(\sigma\sqrt{2t\log\log t})^{-1}} = e^{-1} \ \ a.s., \tag{2.10}$$

and the cluster set of the sequence $\{(h_t/e^{\mu t})^{(\sigma\sqrt{2t\log\log t})^{-1}}, t \geqslant 3\}$ *is* $[e^{-1}, e]$ *with probability one. When* $\mu = 0$,

$$\limsup_{t\to\infty} h_t^{(\sigma\sqrt{2t\log\log t})^{-1}} = e \ \ a.s., \ \liminf_{t\to\infty} h_t^{(\sigma\sqrt{2t\log\log t})^{-1}} = 1 \ \ a.s., \tag{2.11}$$

and the cluster set of the sequence $\{h_t^{(\sigma\sqrt{2t\log\log t})^{-1}}, t \geqslant 3\}$ *is* $[1,e]$ *with probability one.*

REMARK 2.1. Theorem 2.2 shows that the dynamic behavior of h_t at the boundary (i.e., $\mu = 0$) is sharply different from that in the explosive region (i.e., $\mu > 0$), but Theorem 2.1 does not.

REMARK 2.2. Corollary 2.1 gives more details than (1.8). In particular, (2.9) shows the limit behavior of $\rho^t h_t$ in the critical case $\rho = e^{-\mu}$ when $\mu > 0$.

3. Lemmas and proofs of main results

By (1.1), it is easy to get the expression of h_t as

$$h_t = h_0 \prod_{j=0}^{t-1}(\beta + \alpha\eta_j^2) + \omega\left\{1 + \sum_{k=1}^{t-1}\prod_{j=1}^{k}(\beta + \alpha\eta_{t-j}^2)\right\}, \ \ t \geqslant 1, \tag{3.1}$$

where set $\sum_{k=1}^{0} x_k = 0$ for any sequence $\{x_n, n \geqslant 1\}$.

To prove the main results, the following lemmas are needed.

LEMMA 3.1. *Assume that* $0 < \mu < \infty$, *then the series*

$$\sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\frac{1}{\beta + \alpha\eta_j^2}$$

converges almost surely.

Proof. Using the Kolmogorov strong law of large numbers, i.e., using (2.1) by taking $p = 1$ and $Y_j = \log(\beta + \alpha\eta_j) - \mu$,

$$\left\{\prod_{j=0}^{k-1}\frac{1}{\beta + \alpha\eta_j^2}\right\}^{1/k} = \exp\left\{-\frac{1}{k}\sum_{j=0}^{k-1}\log(\beta + \alpha\eta_j^2)\right\} \to e^{-\mu} < 1 \ \text{ a.s.},$$

which follows the desired result by Cauchy root test for series. $\square$

LEMMA 3.2. *Let* $0 < a_{1n}, \cdots, a_{mn} < \infty$, $0 < b_n \to 0$, *where* $m \geqslant 1$. *Assume that* $a_{in}^{b_n} \to 1$ *for all* $i = 1, \cdots, m$, *then*

$$\left(\sum_{i=1}^{m} a_{in}\right)^{b_n} \to 1.$$

Proof. For all $\varepsilon \in (0,1)$, there exists a natural number N, such that $1-\varepsilon < a_{in}^{b_n} < 1+\varepsilon$, and so $(1-\varepsilon)^{b_n^{-1}} < a_{in} < (1+\varepsilon)^{b_n^{-1}}$ for all $n > N$ and $i = 1, \cdots, m$. Therefore,

$$m^{b_n}(1-\varepsilon) < \left(\sum_{i=1}^{m} a_{in}\right)^{b_n} < m^{b_n}(1+\varepsilon),$$

which implies the desired result by the fact $m^{b_n} \to 1$ and the arbitrariness of ε. $\square$

Proof of Theorem 2.1. We first prove the case $\mu > 0$. By (3.1),

$$h_t = \left\{ h_0 + \omega \sum_{k=1}^{t} \prod_{j=0}^{k-1} \frac{1}{\beta + \alpha \eta_j^2} \right\} \prod_{j=0}^{t-1} (\beta + \alpha \eta_j^2),$$

and hence,

$$\left(\frac{h_t}{e^{\mu t}}\right)^{t^{-1/p}} = \left\{ h_0 + \omega \sum_{k=1}^{t} \prod_{j=0}^{k-1} \frac{1}{\beta + \alpha \eta_j^2} \right\}^{t^{-1/p}} \left\{ \frac{\prod_{j=0}^{t-1}(\beta + \alpha \eta_j^2)}{e^{\mu t}} \right\}^{t^{-1/p}}. \tag{3.2}$$

Consider the first factor in (3.2). Set

$$S_t = \sum_{k=1}^{t} \prod_{j=0}^{k-1} \frac{1}{\beta + \alpha \eta_j^2}.$$

Note that S_t is nondecreasing, and $S_\infty = \lim_{t\to\infty} S_t$ is finite almost surely by Lemma 3.1, hence

$$1 \leftarrow (h_0 + \omega S_1)^{t^{-1/p}} \leqslant (h_0 + \omega S_t)^{t^{-1/p}} \leqslant (h_0 + \omega S_\infty)^{t^{-1/p}} \to 1 \ \text{ a.s.} \tag{3.3}$$

Consider the second factor in (3.2). Using (2.1) by taking $Y_j = \log(\beta + \alpha \eta_j^2) - \mu$,

$$\begin{aligned} \left\{ \frac{\prod_{j=0}^{t-1}(\beta + \alpha \eta_j^2)}{e^{\mu t}} \right\}^{t^{-1/p}} &= \exp\left\{ \frac{\sum_{j=0}^{t-1} \log(\beta + \alpha \eta_j^2) - t\mu}{t^{1/p}} \right\} \\ &\to \exp(0) = 1 \ \text{ a.s.,} \end{aligned}$$

which and (3.2), (3.3) imply that (2.7) holds.

We now prove the case $\mu = 0$. Set

$$M_t = \max_{1\leqslant k\leqslant t-1}\prod_{j=1}^{k}(\beta+\alpha\eta_{t-j}^2). \tag{3.4}$$

We have

$$h_t^{t^{-1/p}} = \left(\frac{h_t}{M_t}\right)^{t^{-1/p}} M_t^{t^{-1/p}}. \tag{3.5}$$

Consider the first factor in (3.5), by (3.1),

$$\frac{h_t}{M_t} = \frac{\omega}{M_t} + \omega\frac{\sum_{k=1}^{t-1}\prod_{j=1}^{k}(\beta+\alpha\eta_{t-j}^2)}{M_t} + \frac{\prod_{j=1}^{t-1}(\beta+\alpha\eta_j^2)}{M_t}h_0(\beta+\alpha\eta_0^2).$$

Note that $M_t > \beta+\alpha\eta_{t-1}^2 > 0$ a.s, and

$$1 \leqslant \frac{\sum_{k=1}^{t-1}\prod_{j=1}^{k}(\beta+\alpha\eta_{t-j}^2)}{M_t} \leqslant t, \quad 0 \leqslant \frac{\prod_{j=1}^{t-1}(\beta+\alpha\eta_j^2)}{M_t} \leqslant 1.$$

Therefore,

$$\omega \leqslant \frac{h_t}{M_t} \leqslant \frac{\omega}{\beta+\alpha\eta_{t-1}^2} + \omega t + h_0(\beta+\alpha\eta_0^2).$$

Note that $\omega^{t^{-1/p}} \to 1$, $(\omega t)^{t^{-1/p}} \to 1$, $(h_0(\beta+\alpha\eta_0^2))^{t^{-1/p}} \to 1$ a.s., and using (2.4) by taking $Y_{t-1} = \log(\beta+\alpha\eta_{t-1}^2)$,

$$\left(\frac{\omega}{\beta+\alpha\eta_{t-1}^2}\right)^{t^{-1/p}} = \exp\left\{t^{-1/p}\left(\log\omega - \log(\beta+\alpha\eta_{t-1}^2)\right)\right\} \to e^0 = 1 \text{ a.s.}$$

Hence, by Lemma 3.2,

$$1 \leftarrow \omega^{t^{-1/p}} \leqslant \left(\frac{h_t}{M_t}\right)^{t^{-1/p}} \leqslant \left\{\frac{\omega}{\beta+\alpha\eta_{t-1}^2} + \omega t + h_0(\beta+\alpha\eta_0^2)\right\}^{t^{-1/p}} \to 1 \text{ a.s.} \tag{3.6}$$

Consider the second factor in (3.5),

$$\begin{aligned} M_t^{t^{-1/p}} &= \exp\left\{t^{-1/p}\log M_t\right\} \\ &= \exp\left\{t^{-1/p}\log\left[\max_{1\leqslant k\leqslant t-1}\prod_{j=1}^{k}(\beta+\alpha\eta_{t-j}^2)\right]\right\} \\ &= \exp\left\{t^{-1/p}\max_{1\leqslant k\leqslant t-1}\sum_{j=1}^{k}\log(\beta+\alpha\eta_{t-j}^2)\right\}. \end{aligned}$$

Then using (2.3) by taking $Y_j = \log(\beta + \alpha\eta_j^2)$,

$$M_t^{t^{-1/p}} \to 1 \text{ a.s.},$$

which and (3.5), (3.6) imply that (2.7) holds. The proof is completed. □

Proof of Corollary 2.1. Taking $p = 1$, then (2.7) is equivalent to

$$P\left\{h_t \notin (e^{(\mu-\varepsilon)t}, e^{(\mu+\varepsilon)t}), \text{infinitely often}\right\} = 0, \quad \forall\, \varepsilon > 0,$$

which follows that (2.8) holds. By the Chung-Fuchs theorem (see, for instance, Chow and Teicher, 1997),

$$\liminf_{t\to\infty} \sum_{j=0}^{t-1} [\log(\beta + \alpha\eta_j^2) - \mu] = -\infty \ \ a.s., \quad \limsup_{t\to\infty} \sum_{j=0}^{t-1} [\log(\beta + \alpha\eta_j^2) - \mu] = \infty \ \ a.s.$$

By (3.1),

$$\frac{h_t}{e^{\mu t}} = \left\{h_0 + \omega \sum_{k=1}^{t} \prod_{j=0}^{k-1} \frac{1}{\beta + \alpha\eta_j^2}\right\} \exp\left\{\sum_{j=0}^{t-1} [\log(\beta + \alpha\eta_j^2) - \mu]\right\}.$$

By Lemma 3.1, (2.9) follows from the above two formulas. The proof is completed. □

Proof of Theorem 2.2. Set $a_t = \sigma\sqrt{2t \log\log t}$. We first prove the case $\mu > 0$. By (3.1),

$$\left(\frac{h_t}{e^{\mu t}}\right)^{a_t^{-1}} = \left\{h_0 + \omega \sum_{k=1}^{t} \prod_{j=0}^{k-1} \frac{1}{\beta + \alpha\eta_j^2}\right\}^{a_t^{-1}} \left\{\frac{\prod_{j=0}^{t-1}(\beta + \alpha\eta_j^2)}{e^{\mu t}}\right\}^{a_t^{-1}}. \tag{3.7}$$

Consider the first factor in (3.7). Similar as (3.3),

$$1 \leftarrow (h_0 + \omega S_1)^{a_t^{-1}} \leqslant (h_0 + \omega S_t)^{a_t^{-1}} \leqslant (h_0 + \omega S_\infty)^{a_t^{-1}} \to 1 \ \text{ a.s.} \tag{3.8}$$

Consider the second factor in (3.7). Using (2.5) by taking $Y_j = \log(\beta + \alpha\eta_j^2)$,

$$\limsup_{t\to\infty} \left\{\frac{\prod_{j=0}^{t-1}(\beta + \alpha\eta_j^2)}{e^{\mu t}}\right\}^{a_t^{-1}} = \exp\left\{\limsup_{t\to\infty} \frac{\sum_{j=0}^{t-1} \log(\beta + \alpha\eta_j^2) - t\mu}{a_t}\right\} = e \ \text{ a.s.},$$

and

$$\liminf_{t\to\infty} \left\{\frac{\prod_{i=0}^{t-1}(\beta + \alpha\eta_i^2)}{e^{\mu t}}\right\}^{a_t^{-1}} = \exp\left\{\liminf_{t\to\infty} \frac{\sum_{j=0}^{t-1} \log(\beta + \alpha\eta_j^2) - t\mu}{a_t}\right\} = e^{-1} \ \text{ a.s.},$$

which and (3.7), (3.8) imply that (2.10) holds. Meanwhile, the cluster set of the sequence $\{(h_t/e^{\mu t})^{(\sigma\sqrt{2t\log\log t})^{-1}}, t \geqslant 3\}$ is $[e^{-1}, e]$ with probability one.

We now prove the case $\mu = 0$. Set M_t as (3.4), we have

$$h_t^{a_t^{-1}} = \left(\frac{h_t}{M_t}\right)^{a_t^{-1}} M_t^{a_t^{-1}}. \tag{3.9}$$

Consider the first factor in (3.9), similar as (3.6),

$$1 \leftarrow \omega^{a_t^{-1}} \leqslant \left(\frac{h_t}{M_t}\right)^{a_t^{-1}} \leqslant \left\{\frac{\omega}{\beta+\alpha\eta_{t-1}^2} + \omega t + h_0(\beta+\alpha\eta_0^2)\right\}^{a_t^{-1}} \\ \to 1 \ \text{ a.s.} \tag{3.10}$$

Consider the second factor in (3.9),

$$\begin{aligned} M_t^{a_t^{-1}} &= \exp\left\{\frac{1}{a_t}\log M_t\right\} \\ &= \exp\left\{\frac{1}{a_t}\log\left[\max_{1\leqslant k\leqslant t-1}\prod_{j=1}^{k}(\beta+\alpha\eta_{t-j}^2)\right]\right\} \\ &= \exp\left\{\frac{1}{a_t}\max_{1\leqslant k\leqslant t-1}\sum_{j=1}^{k}\log(\beta+\alpha\eta_{t-j}^2)\right\}. \end{aligned}$$

Using (2.6) by taking $Y_j = \log(\beta+\alpha\eta_j^2)$, we have

$$\limsup_{t\to\infty} M_t^{a_t^{-1}} = e \ \text{ a.s.}, \quad \liminf_{t\to\infty} M_t^{a_t^{-1}} = 1 \ \text{ a.s.},$$

which and (3.9), (3.10) imply that (2.11) holds. Meanwhile the cluster set of the sequence $\{h_t^{(\sigma\sqrt{2t\log\log t})^{-1}}, t \geqslant 3\}$ is $[1,e]$ with probability one. The proof is completed. □

4. Simulation

In this section, we will present a simulation study based on the model (1.1) by R software and show the limit behavior of the paths of $(h_t/e^{\mu t})^{t^{-1/p}}$.

Let $\beta > 0$, and $\eta \sim U(-1,1)$, then $\mu = E\log(\beta+\eta^2) = \log(1+\beta) - 2 + 2\sqrt{\beta}\arctan(1/\sqrt{\beta})$. Set

$$\log(1+\beta) - 2 + 2\sqrt{\beta}\arctan(1/\sqrt{\beta}) = 0,$$

then the unique solution is $\beta_0 \approx 0.706461201$.

Take the initial values $y_0 = h_0 = 1$, and the parameters $\omega = \alpha = 1$, we give one path of $(h_t/e^{\mu t})^{t^{-1/p}}$ for each of the two case: $\mu = 0$ ($\beta = \beta_0$, correspondingly) and $\mu = 0.839451791$ ($\beta = 2$, correspondingly), with size $T = 800$, by taking $p = 1, 1.4, 1.8$.

From Figure 1 and Figure 2, we can see that the path of $(h_t/e^{\mu t})^{t^{-1/p}}$ approaches to 1 as t increases. These simulation results agree with the theoretical results.

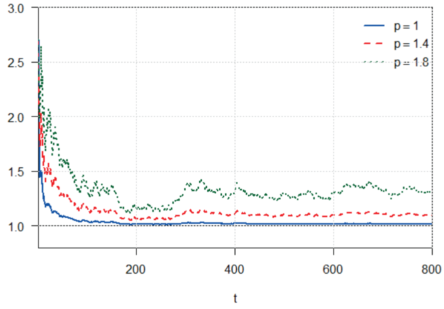

Figure 1: $\mu = 0$.

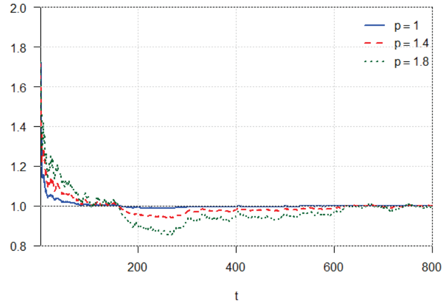

Figure 2: $\mu > 0$.

Conclusions

In this paper, we have precisely characterized the dynamic behavior of h_t in a nonstationary GARCH(1,1) model, via the Marcinkiewicz-Zygmund strong law of large numbers and the Hartman-Wintner law of the iterated logarithm. The Hartman-Wintner law of the iterated logarithm shows that the dynamic behavior of h_t at the boundary is sharply different from that in the explosive region, but the Marcinkiewicz-Zygmund strong law of large numbers does not.

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