Journal of Mathematical Inequalities
Volume 19, Number 3 (2025), 875–892 doi:10.7153/jmi-2025-19-56

SOME NEW NUMERICAL RADIUS INEQUALITIES VIA AN IMPROVED VERSION OF KATO'S INEQUALITY

YONGHUI REN AND MOHAMED AMINE IGHACHANE*

(*Communicated by M. Sababheh*)

Abstract. In this paper, we present new refinements of Kato's inequality. These refinements are then applied to derive improved upper bounds for specific numerical radius and norm inequalities. Our findings strengthen and extend several well-known numerical radius inequalities, providing more precise estimates than those previously established. In particular, we introduce two distinct refinement approaches based on improved versions of Young's inequality and a generalized Buzano's inequality. These results yield sharper bounds and wider applicability to bounded linear operators. Furthermore, we demonstrate the utility of these refinements through applications to triangle inequalities and Furuta-type inequalities, illustrating the robustness and versatility of our methods.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ represent the $\mathbb{C}^*$-algebra consisting of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ equipped with an inner product $\langle , \rangle$. An operator $\mathscr{T} \in \mathbb{B}(\mathscr{H})$ is called positive, denoted $\mathscr{T} \geqslant 0$, if it satisfies $\langle \mathscr{T}x, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. The set of all positive operators is indicated by $\mathscr{B}(\mathscr{H})^+$. For an operator $\mathscr{T} \in \mathbb{B}(\mathscr{H})$, we denote $|\mathscr{T}| := (\mathscr{T}^*\mathscr{T})^{\frac{1}{2}}$. The Schwarz inequality applied to positive operators states that if $\mathscr{T}$ is a positive operator in $\mathbb{B}(\mathscr{H})$, then for any $x, y \in \mathscr{H}$,

$$|\langle \mathscr{T}x, y\rangle|^2 \leqslant \langle \mathscr{T}x, x\rangle\langle \mathscr{T}y, y\rangle. \tag{1.1}$$

In 1952, Kato [11] proposed an associated inequality to (1.1), known as the mixed Schwarz inequality, which states

$$|\langle \mathscr{T}x, y\rangle|^2 \leqslant \left\langle |\mathscr{T}|^{2\theta}x, x\right\rangle \left\langle |\mathscr{T}^*|^{2(1-\theta)}y, y\right\rangle, \tag{1.2}$$

for every $\mathscr{T} \in \mathbb{B}(\mathscr{H})$, $0 \leqslant \theta \leqslant 1$ and $x, y \in \mathscr{H}$. Furuta [8] presented a generalized form of Kato's inequality (1.2) as follows:

$$|\langle \mathscr{T}|\mathscr{T}|^{\theta+\gamma-1}x, y\rangle|^2 \leqslant \sqrt{\langle |\mathscr{T}|^{2\theta}x, x\rangle \left\langle |\mathscr{T}^*|^{2\gamma}y, y\right\rangle}, \tag{1.3}$$

Mathematics subject classification (2020): Primary 47A30; Secondary 47A12, 47B15.
Keywords and phrases: Kato's inequality, numerical radius, operator norm, triangle inequality.
* Corresponding author.

Paper JMI-19-56

where $x,y \in \mathscr{H}$ and $\theta,\gamma \in (0,1)$ with $\theta+\gamma \geqslant 1$. Another extension of the mixed Schwarz inequality, established by Kittaneh [12], states that for $x,y \in \mathscr{H}$,

$$|\langle \mathscr{T}x,y\rangle|^2 \leqslant \|f(|\mathscr{T}|)x\|\|g(|\mathscr{T}^*|)y\|, \tag{1.4}$$

here f and g are non-negative continuous functions defined on $[0,\infty)$, with the property that $f(\alpha)g(\alpha)=\alpha$ for all $\alpha \in [0,\infty)$.

One of the significant scalar quantities associated with an operator $\mathscr{T} \in \mathbb{B}(\mathscr{H})$, are the standard operator norm and the numerical radius, defined as

$$\|\mathscr{T}\| = \sup_{\|x\|=1} \|\mathscr{T}x\| \text{ and } \omega(\mathscr{T}) = \sup_{\|x\|=1} |\langle \mathscr{T}x,x\rangle|.$$

The equality $\omega(\mathscr{T}) = \|\mathscr{T}\|$ holds true if $\mathscr{T}$ is normal. Further, the numerical radius $\omega(\mathscr{T})$ establishes a norm on $\mathbb{B}(\mathscr{H})$, that is equivalent to the operator norm $\|.\|$. Furthermore, it follows that:

$$\frac{1}{2}\|\mathscr{T}\| \leqslant \omega(\mathscr{T}) \leqslant \|\mathscr{T}\|, \tag{1.5}$$

the first inequality becomes an equality between the operator norm and the numerical radius under the condition that $\mathscr{T}$ is square-zero ($i.e.$, $\mathscr{T}^2=0$). As demonstrated in [2, 9, 10], operators and matrices have been crucial in deriving more precise and stringent relationships between the operator norm and the numerical radius. An improved version of the second inequality in (1.5) has been provided in [13]. It says that for $\mathscr{T} \in \mathbb{B}(\mathscr{H})$,

$$\omega(\mathscr{T}) \leqslant \frac{1}{2}\||\mathscr{T}|+|\mathscr{T}^*|\| \leqslant \frac{1}{2}\left(\|\mathscr{T}\|+\|\mathscr{T}^2\|^{\frac{1}{2}}\right). \tag{1.6}$$

Two years later, Kittaneh [14] proved his celebrated two-sided inequalities

$$\frac{1}{4}\|\mathscr{T}\mathscr{T}^*+\mathscr{T}^*\mathscr{T}\| \leqslant \omega^2(\mathscr{T}) \leqslant \frac{1}{2}\|\mathscr{T}\mathscr{T}^*+\mathscr{T}^*\mathscr{T}\|. \tag{1.7}$$

Using Kato's inequality (1.2), El-Haddad and Kittaneh [7] showed a generalization of the second inequality in (1.7). For $\theta \in (0,1)$ and $s \geqslant 1$, we have

$$\omega^{2s}(\mathscr{T}) \leqslant \|\theta|\mathscr{T}|^{2s}+(1-\theta)|\mathscr{T}^*|^{2s}\| \tag{1.8}$$

and

$$\omega^{s}(\mathscr{T}) \leqslant \frac{1}{2}\||\mathscr{T}|^{2s\theta}+|\mathscr{T}^*|^{2s(1-\theta)}\|. \tag{1.9}$$

For some recent developments on the numerical radius, the reader is encouraged to consult the following papers: [6, 17, 18, 19].

Additionally, we will incorporate concepts related to convexity into our discussion. Recall that if $\psi:[0,\infty)\to[0,\infty)$, then ψ is referred to as doubly convex if ψ is convex in the conventional sense and

$$\psi\left(a^\theta b^{1-\theta}\right) \leqslant \psi^\theta(a)\psi^{1-\theta}(b);\ \ a,b \geqslant 0,\ \ 0 \leqslant \theta \leqslant 1. \tag{1.10}$$

The functions $\psi(t) = \sinh t$ and $\psi(t) = \cosh t$ are examples of functions that are doubly convex on on $[0,\infty)$.

In this work, further refinements of the previously mentioned inequalities are presented. We propose new upper bounds for Kato's inequality. Utilizing these bounds, we refine several classical numerical radius inequalities and the triangle inequality.

The structure of the paper is organized to progressively build upon foundational operator inequalities toward more refined numerical estimates. Following a brief overview of preliminary concepts and key results from the literature, the second section introduces novel generalizations of the mixed Cauchy–Schwarz inequality, developed through enhanced forms of Young's inequality and a generalized version of Buzano's inequality. These formulations not only extend earlier contributions but also serve as pivotal tools for subsequent refinements. Furthermore, in the third section, we apply the derived inequalities to establish improved upper bounds for the numerical radius and operator norms. These results significantly sharpen known estimates and highlight the broader applicability of our approach to bounded linear operators on Hilbert spaces.

2. Some new improvements of Kato's inequality

This section is structured into two parts. The initial part focuses at presenting some new refinements of Kato's inequality via some recent improvement of Young's inequality and other related consequences. In the second part, by the famous generalized Buzano's inequality, we present a new refinement of the mixed Schwarz inequality, which generalizes the obtained results in [15].

2.1. Refined Kato's inequality via Young's inequality

To accomplish our objective in this paper, we require the following four lemmas. The first Lemma is a refined version of the classical Young's inequality, which can easily be derived from Theorem 2.2 in [3].

LEMMA 2.1. *Consider a and b as two positive numbers and J be a set such that $(0,1) \subset J \subset \mathbb{R}$, and let h be a mapping $h : J \to \mathbb{R}^+$ such that $h(\mu) + h(1-\mu) = 1$ for $\mu \in (0,1)$. Then for all $n \geqslant 1$, we have*

$$a^{h(\mu)} b^{h(1-\mu)} \leqslant U_{(n,h)}(\mu,a,b) \leqslant h(\mu)a + h(1-\mu)b, \tag{2.1}$$

where $U_{(n,h)}(\mu,a,b) := \left(h(\mu)a^{\frac{1}{n}} + h(1-\mu)b^{\frac{1}{n}}\right)^n$. Moreover, we have $\left(U_{(n,h)}(\mu,a,b)\right)_{n\geqslant 1}$ is a non-increasing sequence satisfying

$$\lim_{n\to+\infty} U_{(n,h)}(\mu,a,b) = a^{h(\mu)} b^{h(1-\mu)}.$$

The second lemma concerns operators and is referred to as McCarthy's inequality.

LEMMA 2.2. *Let $\mathscr{T} \in \mathbb{B}(\mathscr{H})$ be self-adjoint with spectrum in the interval J. If $\psi : J \to \mathbb{R}$ is convex, then*

$$\psi(\langle \mathscr{T}x,x\rangle) \leqslant \langle \psi(\mathscr{T})x,x\rangle \tag{2.2}$$

for all $x \in \mathscr{H}$, *with* $\|x\| = 1$. *Furthermore, when* f *is concave, then the inequality is reversed.*

The third lemma deals with operators in $\mathbb{B}(\mathscr{H})$, which was demonstrated by Kittaneh in [12].

LEMMA 2.3. *Let* $\mathscr{T}, \mathscr{A}, \mathscr{B} \in \mathbb{B}(\mathscr{H})$ *be such that* $\mathscr{A}, \mathscr{B} \geqslant O$. *Then*

$$\begin{bmatrix} \mathscr{A} & \mathscr{T}^* \\ \mathscr{T} & \mathscr{B} \end{bmatrix} \geqslant O \Leftrightarrow |\langle \mathscr{T}x, y\rangle|^2 \leqslant \langle \mathscr{A}x, x\rangle \langle \mathscr{B}y, y\rangle \quad \textit{for all } x, y \in \mathscr{H}. \tag{2.3}$$

The fourth and last lemma is available in [12].

LEMMA 2.4. *Let* $\mathscr{T} \in \mathbb{B}(\mathscr{H})$ *and let* f *and* g *be non-negative continuous functions defined on* $[0,\infty)$ *satisfying* $f(\alpha)g(\alpha) = \alpha$ *for all* $\alpha \in [0,\infty)$. *Then*

$$\begin{bmatrix} f^2(|\mathscr{T}|) & \mathscr{T}^* \\ \mathscr{T} & g^2(|\mathscr{T}^*|) \end{bmatrix} \geqslant O.$$

For $\mathscr{T} \in \mathbb{B}(\mathscr{H})$, $\mathscr{X}, \mathscr{Y} \in \mathbb{B}^+(\mathscr{H})$, $\mu \in [0,1]$ and h is a function as in Lemma 2.1, we consider the following sequence:

$$\mathscr{U}_{(n,h)}(\mu, \mathscr{X}, \mathscr{Y}) := \Bigg[h(\mu) \sqrt[2n]{|\langle \mathscr{T}x, y\rangle| \sqrt{\langle \mathscr{X}x, x\rangle \langle \mathscr{Y}y, y\rangle}} + h(1-\mu) \sqrt[2n]{\langle \mathscr{X}x, x\rangle \langle \mathscr{Y}y, y\rangle} \Bigg]^{2n}.$$

We have $\left(\mathscr{U}_{(n,h)}(\mu, \mathscr{X}, \mathscr{Y})\right)_{n \geqslant 1}$ is a nonincreasing sequence satisfying:

$$\lim_{n \to +\infty} \mathscr{U}_{(n,h)}(\mu, \mathscr{X}, \mathscr{Y}) = \left(|\langle \mathscr{T}x, y\rangle| \sqrt{\langle \mathscr{X}x, x\rangle \langle \mathscr{Y}y, y\rangle}\right)^{h(\mu)} . \left(\langle \mathscr{X}x, x\rangle \langle \mathscr{Y}y, y\rangle\right)^{h(1-\mu)}.$$

We are now prepared to present our first main result regarding Kato's inequality, which refines Lemma 2.3.

THEOREM 2.1. *Let* $\mathscr{T}, \mathscr{A}, \mathscr{B} \in \mathbb{B}(\mathscr{H})$ *be such that* $\mathscr{A}, \mathscr{B} \geqslant 0$. *Then*

$$\begin{bmatrix} \mathscr{A} & \mathscr{T}^* \\ \mathscr{T} & \mathscr{B} \end{bmatrix} \geqslant O \Leftrightarrow |\langle \mathscr{T}x, y\rangle|^2 \leqslant \mathscr{U}_{(n,h)}(\mu, \mathscr{A}, \mathscr{B}) \textit{ for all } n \in \mathbb{N}^* \textit{and for all } x, y \in \mathscr{H}. \tag{2.4}$$

Proof. We assume that $\begin{bmatrix} \mathscr{A} & \mathscr{T}^* \\ \mathscr{T} & \mathscr{B} \end{bmatrix} \geqslant 0$. Then by Lemma 2.3, we have

$$|\langle \mathscr{T}x, y\rangle| \leqslant \sqrt{\langle \mathscr{A}x, x\rangle \langle \mathscr{B}y, y\rangle}. \tag{2.5}$$

Now, we have

$$
\begin{aligned}
|\langle \mathscr{T}x,y\rangle| &= |\langle \mathscr{T}x,y\rangle|^{h(\mu)+h(1-\mu)} \\
&\leqslant |\langle \mathscr{T}x,y\rangle|^{h(\mu)} \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}^{h(1-\mu)} \quad \text{(by inequality (2.5))} \\
&\leqslant |\langle \mathscr{T}x,y\rangle|^{\frac{h(\mu)}{2}} \sqrt[4]{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}^{h(\mu)} \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}^{h(1-\mu)} \\
&\qquad\qquad \text{(by inequality (2.5))} \\
&= \left(\sqrt{|\langle \mathscr{T}x,y\rangle| \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}}\right)^{h(\mu)} \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}^{2h(1-\mu)} \\
&\leqslant \left(h(\mu) \sqrt[2n]{|\langle \mathscr{T}x,y\rangle| \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}} + h(1-\mu) \sqrt[2n]{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}\right)^{n} \\
&= \sqrt{\mathscr{U}_{(n,h)}(\mu,\mathscr{A},\mathscr{B})}.
\end{aligned}
$$

The last inequality is derived from Lemma 2.1.

For the converse, we assume that

$$|\langle \mathscr{T}x,y\rangle|^2 \leqslant \mathscr{U}_{(n,h)}(\mu,\mathscr{A},\mathscr{B}) \text{ for all } n \in \mathbb{N}^*. \tag{2.6}$$

Then by taking the limit as $n \to +\infty$ in inequality (2.6), we get

$$
\begin{aligned}
|\langle \mathscr{T}x,y\rangle|^2 &\leqslant \left(|\langle \mathscr{T}x,y\rangle| \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}\right)^{h(\mu)} \cdot \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}^{h(1-\mu)} \\
&= |\langle \mathscr{T}x,y\rangle|^{h(\mu)} \cdot \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}^{2-h(\mu)},
\end{aligned}
$$

which is equivalent to

$$|\langle \mathscr{T}x,y\rangle| \leqslant \sqrt{\langle \mathscr{A}x,x\rangle \langle \mathscr{B}y,y\rangle}.$$

Then we conclude from Lemma 2.3 that $\begin{bmatrix} \mathscr{A} & \mathscr{T}^* \\ \mathscr{T} & \mathscr{B} \end{bmatrix} \geqslant 0.$ □

For the rest of this paper, we assume that f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(\alpha)g(\alpha) = \alpha$ for all $\alpha \in [0,\infty)$, and $\psi : [0,\infty) \to [0,\infty)$ is an increasing doubly convex function.

Combining Lemma 2.4 with Theorem 2.1, the following result is obtained.

THEOREM 2.2. *Let $\mathscr{T} \in \mathbb{B}(\mathscr{H})$. Then for all $x,y \in \mathscr{H}$,*

$$|\langle \mathscr{T}x,y\rangle|^2 \leqslant \mathscr{U}_{(n,h)}(\mu, f^2(|\mathscr{T}|), g^2(|\mathscr{T}^*|)) \text{ for all } n \in \mathbb{N}^*. \tag{2.7}$$

Specifically, for any $0 \leqslant \theta \leqslant 1$,

$$|\langle \mathscr{T}x,y\rangle|^2 \leqslant \mathscr{U}_{(n,h)}(\mu, |\mathscr{T}|^{2\theta}, |\mathscr{T}^*|^{2(1-\theta)}) \text{ for all } n \in \mathbb{N}^*. \tag{2.8}$$

THEOREM 2.3. *Let $\mathscr{T} \in \mathbb{B}(\mathscr{H})$. Then the following holds*

$$\psi(|\langle \mathscr{T}x,y\rangle|) \leqslant \mathscr{U}_{(n,h,\psi)}\left(\mu, \psi\left(f^2(|\mathscr{T}|)\right), \psi\left(g^2(|\mathscr{T}^*|)\right)\right) \text{ for all } n \in \mathbb{N}^*. \tag{2.9}$$

In particular, for any $0 \leqslant \theta \leqslant 1$,

$$\psi(|\langle \mathscr{T}x, y\rangle|) \leqslant \mathscr{U}_{(n,h,\psi)}\left(\mu, \psi\left(|\mathscr{T}|^{2\theta}\right), \psi\left(|\mathscr{T}^*|^{2(1-\theta)}\right)\right) \text{ for all } n \in \mathbb{N}^*, \tag{2.10}$$

where

$$\mathscr{U}_{(n,h,\psi)}(\mu, \mathscr{X}, \mathscr{Y}) := \left[h(\mu)\sqrt[2n]{\psi(|\langle \mathscr{T}x, y\rangle|)\sqrt{\langle \mathscr{X}x, x\rangle\langle \mathscr{Y}y, y\rangle}} + h(1-\mu)\sqrt[2n]{\langle \mathscr{X}x, x\rangle\langle \mathscr{Y}y, y\rangle}\right]^n.$$

Proof. By Theorem 2.2, we have

$$\begin{aligned} |\langle \mathscr{T}x, y\rangle| &\leqslant \sqrt{\mathscr{U}_{(n,h)}(\mu, f^2(|\mathscr{T}|), g^2(|\mathscr{T}^*|))} \\ &= \sum_{k=0}^{n}\binom{k}{n} h^k(\mu)h^{n-k}(1-\mu)a^{\frac{k}{n}}b^{\frac{n-k}{n}}, \end{aligned}$$

where

$$a = \sqrt{|\langle \mathscr{T}x, y\rangle|\sqrt{\langle f^2(|\mathscr{T}|)x, x\rangle\langle g^2(|\mathscr{T}^*|)y, y\rangle}}$$

and

$$b = \sqrt{\langle f^2(|\mathscr{T}|)x, x\rangle\langle g^2(|\mathscr{T}^*|)y, y\rangle}.$$

The fact that ψ is an increasing doubly convex function implies that

$$\begin{aligned} \psi(|\langle \mathscr{T}x, y\rangle|) &\leqslant \psi\left(\sum_{k=0}^{n}\binom{k}{m} h^k(\mu)h^{n-k}(1-\mu)a^{\frac{k}{n}}b^{\frac{n-k}{n}}\right) \\ &\leqslant \sum_{k=0}^{n}\binom{k}{m} h^k(\mu)h^{n-k}(1-\mu)\psi\left(a^{\frac{k}{n}}b^{\frac{n-k}{n}}\right) \\ &\leqslant \sum_{k=0}^{n}\binom{k}{m} h^k(\mu)h^{n-k}(1-\mu)\psi^{\frac{k}{n}}(a)\psi^{\frac{n-k}{n}}(b), \end{aligned} \tag{2.11}$$

where the second inequality is derived from Jensen's inequality. Given that ψ is an increasing doubly convex function, and by applying Lemma 2.2, we obtain

$$\psi(a) \leqslant \sqrt{\psi(|\langle \mathscr{T}x, y\rangle|)\sqrt{\langle \psi(f^2(|\mathscr{T}|))x, x\rangle\langle \psi(g^2(|\mathscr{T}^*|))y, y\rangle}} \tag{2.12}$$

and

$$\psi(b) \leqslant \sqrt{\langle \psi(f^2(|\mathscr{T}|))x, x\rangle\langle \psi(g^2(|\mathscr{T}^*|))y, y\rangle}. \tag{2.13}$$

Combining inequalities (2.11), (2.12) ad (2.13), we get the desired result. □

THEOREM 2.4. *Let $\mathscr{S},\mathscr{T},\mathscr{X},\mathscr{Y}\in\mathbb{B}(\mathscr{H})$ and let h be a function defined as in lemma 2.1. Then*

$$|\langle\mathscr{S}\mathscr{T}\mathscr{X}\mathscr{Y}x,y\rangle|^2\leqslant\mathscr{V}_{(n,h)}\left(\mu,\mathscr{Y}^*\mathscr{X}^*|\mathscr{T}|\mathscr{X}\mathscr{Y},\mathscr{S}|\mathscr{T}^*|\mathscr{S}^*\right)\ \text{for all } n\in\mathbb{N}^*, \tag{2.14}$$

where

$$\begin{aligned}&\mathscr{V}_{(n,h)}\left(\mu,\mathscr{Y}^*\mathscr{X}^*|\mathscr{T}^*|\mathscr{X}\mathscr{Y},\mathscr{S}|\mathscr{T}|\mathscr{S}^*\right)\\ &:=\Bigg[h(\mu)\sqrt[2n]{\langle\mathscr{S}\mathscr{T}\mathscr{X}\mathscr{Y}x,y\rangle\sqrt{\langle\mathscr{Z}x,x\rangle\langle\mathscr{S}|\mathscr{T}|\mathscr{S}^*y,y\rangle}}\\ &\quad+h(1-\mu)\sqrt[2n]{\langle\mathscr{Z}x,x\rangle\langle\mathscr{S}|\mathscr{T}|\mathscr{S}^*y,y\rangle}\Bigg]^{2n},\end{aligned}$$

where $\mathscr{Z}=\mathscr{Y}^*\mathscr{X}^*|\mathscr{T}|\mathscr{X}\mathscr{Y}$.

Proof. By Theorem 2.2, for $f(t)=g(t)=\sqrt{t}$, we have

$$|\langle\mathscr{T}x,y\rangle|^2\leqslant\mathscr{U}_{(n,h)}(\mu,|\mathscr{T}|,|\mathscr{T}^*|)\ \text{for all } n\in\mathbb{N}^*. \tag{2.15}$$

By taking $x=\mathscr{X}\mathscr{Y}x$ and $y=\mathscr{S}^*y$ in inequality (2.15), we get the desired result. □

Due to Theorem 2.4, we have the following refinement of Furuta's inequality (1.3).

THEOREM 2.5. *Let $\mathscr{T}\in\mathbb{B}(\mathscr{H})$ and $\theta\geqslant-1,\ \gamma\geqslant2$. Then for all $x,y\in\mathscr{H}$,*

$$|\langle\mathscr{T}|\mathscr{T}|^{\theta+\gamma-1}x,y\rangle|^2\leqslant\mathscr{U}_{(n,h)}\left(\mu,|\mathscr{T}^*|^{2\theta+\gamma},|\mathscr{T}|\right)\ \text{for all } n\in\mathbb{N}^*. \tag{2.16}$$

Proof. First, let $\mathscr{T}=U|\mathscr{T}|$ represent the polar decomposition of $\mathscr{T}$. By setting $\mathscr{S}=U$, $\mathscr{T}=|\mathscr{T}|^\gamma$, $\mathscr{X}=I$, and $\mathscr{Y}=|\mathscr{T}|^\theta$ in Theorem 2.4, we obtain the desired result. □

2.2. Refined Kato's inequality via generalized Buzano's inequality

In this subsection, by adopting some ideas from [15], we extend some recent results in [15] by the famous generalized Buzano's inequality. First we need the following lemmas. The first lemma presents a generalized version of Buzano's inequality [5].

LEMMA 2.5. ([1]) *Let $x,y,z\in\mathscr{H}$ and $\tau\in[0,1]$. Then we have*

$$|\langle x,z\rangle\langle z,y\rangle|\leqslant\|z\|^2\left(\frac{1+\tau}{2}||x||||y||+\frac{1-\tau}{2}|\langle x,y\rangle|\right). \tag{2.17}$$

The second lemma is the following, which is equivalent to Lemma 2.3.

LEMMA 2.6. ([4], Proposition 1.3.2) *Let* $\mathscr{T}, \mathscr{A}, \mathscr{B} \in \mathbb{B}(\mathscr{H})$ *be such that* $\mathscr{A}, \mathscr{B} \geqslant O$. *Then* $\begin{bmatrix} \mathscr{A} & \mathscr{T}^* \\ \mathscr{T} & \mathscr{B} \end{bmatrix} \geqslant O$ *if and only* $\mathscr{T} = \mathscr{A}^{\frac{1}{2}} K \mathscr{B}^{\frac{1}{2}}$ *for some contraction* K.

LEMMA 2.7. *Let* $\mathscr{T} \in \mathbb{B}(\mathscr{H})$ *be represented by the polar decomposition* $\mathscr{T} = V|\mathscr{T}|$. *Then for all vectors* $x, y \in \mathscr{H}$ *and* $\tau \in [0,1]$,

$$|\langle \mathscr{T}x, y\rangle| \leqslant \|\mathscr{T}\| \left(\frac{1-\tau}{2} |\langle x, V^* y\rangle| + \frac{1+\tau}{2} \|x\| \, \|V^* y\| \right). \tag{2.18}$$

Proof. Consider $\mathscr{T} \in \mathbb{B}(\mathscr{H})$ as a positive contraction. Thus, $0 \leqslant \mathscr{T} \leqslant I$. Given $\mathscr{T}^2 \leqslant \mathscr{T}$, the following inequality holds for all $x, y \in \mathscr{H}$,

$$\begin{aligned} \|\mathscr{T}x\|^2 |\langle \mathscr{T}x, y\rangle| &= \langle \mathscr{T}x, \mathscr{T}x\rangle |\langle \mathscr{T}x, y\rangle| \\ &= \left\langle \mathscr{T}^2 x, x\right\rangle |\langle \mathscr{T}x, y\rangle| \\ &\leqslant \langle \mathscr{T}x, x\rangle |\langle \mathscr{T}x, y\rangle| \\ &\leqslant \|\mathscr{T}x\|^2 \left(\frac{1-\tau}{2} |\langle x, y\rangle| + \frac{1+\tau}{2} \|x\| \|y\| \right), \end{aligned}$$

where the last inequality follows by applying the Lemma 2.5. So,

$$|\langle \mathscr{T}x, y\rangle| \leqslant \left(\frac{1-\tau}{2} |\langle x, y\rangle| + \frac{1+\tau}{2} \|x\| \|y\| \right),$$

for any positive contraction $\mathscr{T}$. Considering an arbitrary positive operator $\mathscr{T} \in \mathbb{B}(\mathscr{H})$, substituting $\mathscr{T}$ with $\frac{\mathscr{T}}{\|\mathscr{T}\|}$, yields the desired inequality in Lemma 2.7. Alternatively, for an arbitrary operator $\mathscr{T} \in \mathbb{B}(\mathscr{H})$, we utilize the polar decomposition $\mathscr{T} = V|\mathscr{T}|$ to complete the proof of the lemma. □

Our primary theorem in this subsection is as follows.

THEOREM 2.6. *Let* $\mathscr{T}, \mathscr{A}, \mathscr{B} \in \mathbb{B}(\mathscr{H})$ *with* $\mathscr{A}, \mathscr{B} \geqslant O$ *and* $\tau \in [0,1]$. *Then* $\begin{bmatrix} \mathscr{A} & \mathscr{T}^* \\ \mathscr{T} & \mathscr{B} \end{bmatrix} \geqslant O$ *if and only if for all* $x, y \in \mathscr{H}$ *and for a certain partial isometry* $V \in \mathbb{B}(\mathscr{H})$, *we have*

$$|\langle \mathscr{T}x, y\rangle| \leqslant \frac{1-\tau}{2} \left\langle \mathscr{B}^{\frac{1}{2}} V \mathscr{A}^{\frac{1}{2}} x, y \right\rangle + \frac{1+\tau}{2} \sqrt{\langle \mathscr{A}x, x\rangle \langle \mathscr{B}y.y\rangle}. \tag{2.19}$$

Proof. Assume that $\begin{bmatrix} \mathscr{A} & \mathscr{T}^* \\ \mathscr{T} & \mathscr{B} \end{bmatrix} \geqslant O$. According to Lemma 2.3, there exists a contraction K such that $\mathscr{T} = \mathscr{B}^{\frac{1}{2}} K \mathscr{A}^{\frac{1}{2}}$. Let $K = V|K|$ denote the polar decomposition of

K. Then

$$\begin{aligned}
|\langle \mathscr{T}x,y\rangle| &= \left|\left\langle \mathscr{B}^{\frac{1}{2}}K\mathscr{A}^{\frac{1}{2}}x,y\right\rangle\right| \quad \text{(by Lemma 2.6)}\\
&\leqslant \|K\|\left(\frac{1-\tau}{2}\left|\left\langle \mathscr{B}^{\frac{1}{2}}V\mathscr{A}^{\frac{1}{2}}x,y\right\rangle\right| + \frac{1+\tau}{2}\left\|V^*\mathscr{B}^{\frac{1}{2}}y\right\|\left\|\mathscr{A}^{\frac{1}{2}}x\right\|\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.7)}\\
&\leqslant \frac{1-\tau}{2}\left|\left\langle \mathscr{B}^{\frac{1}{2}}VA^{\frac{1}{2}}x,y\right\rangle\right| + \frac{1+\tau}{2}\left\|V^*\mathscr{B}^{\frac{1}{2}}y\right\|\left\|\mathscr{A}^{\frac{1}{2}}x\right\| \quad (\text{since } \|K\|\leqslant 1)\\
&= \frac{1-\tau}{2}\left|\left\langle \mathscr{B}^{\frac{1}{2}}V\mathscr{A}^{\frac{1}{2}}x,y\right\rangle\right| + \frac{1+\tau}{2}\sqrt{\left\langle \mathscr{B}^{\frac{1}{2}}VV^*\mathscr{B}^{\frac{1}{2}}y,y\right\rangle\langle Ax,x\rangle}\\
&\leqslant \frac{1-\tau}{2}\left|\left\langle \mathscr{B}^{\frac{1}{2}}V\mathscr{A}^{\frac{1}{2}}x,y\right\rangle\right| + \frac{1+\tau}{2}\sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle} \quad (\text{since } VV^*\leqslant I)
\end{aligned}$$

for all vectors $x,y \in \mathscr{H}$. For the converse, if for all $x,y \in \mathscr{H}$, the inequality

$$|\langle \mathscr{T}x,y\rangle| \leqslant \frac{1-\tau}{2}\left|\left\langle \mathscr{B}^{\frac{1}{2}}V\mathscr{A}^{\frac{1}{2}}x,y\right\rangle\right| + \frac{1+\tau}{2}\sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle},$$

holds for some partial isometry $V \in \mathbb{B}(\mathscr{H})$, then by using the well-known Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left|\left\langle \mathscr{B}^{\frac{1}{2}}V\mathscr{A}^{\frac{1}{2}}x,y\right\rangle\right| &= \left|\left\langle V\mathscr{A}^{\frac{1}{2}}x,\mathscr{B}^{\frac{1}{2}}y\right\rangle\right|\\
&\leqslant \left\|V\mathscr{A}^{\frac{1}{2}}x\right\|\left\|\mathscr{B}^{\frac{1}{2}}y\right\|\\
&= \sqrt{\left\langle V\mathscr{A}^{\frac{1}{2}}x,V\mathscr{A}^{\frac{1}{2}}x\right\rangle\left\langle \mathscr{B}^{\frac{1}{2}}y,\mathscr{B}^{\frac{1}{2}}y\right\rangle}\\
&= \sqrt{\left\langle \mathscr{A}^{\frac{1}{2}}V^*V\mathscr{A}^{\frac{1}{2}}x,x\right\rangle\langle \mathscr{B}y,y\rangle}\\
&\leqslant \sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle} \quad (\text{by the fact that } V^*V\leqslant I).
\end{aligned}$$

Then we have

$$\begin{aligned}
|\langle \mathscr{T}x,y\rangle| &\leqslant \frac{1-\tau}{2}\left\langle \mathscr{B}^{\frac{1}{2}}V\mathscr{A}^{\frac{1}{2}}x,y\right\rangle + \frac{1+\tau}{2}\sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle}\\
&\leqslant \frac{1-\tau}{2}\left\langle \mathscr{B}^{\frac{1}{2}}V\mathscr{A}^{\frac{1}{2}}x,y\right\rangle + \frac{1+\tau}{2}\sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle}\\
&\leqslant \frac{1-\tau}{2}\sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle} + \frac{1+\tau}{2}\sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle}\\
&\leqslant \left(\frac{1-\tau}{2}+\frac{1+\tau}{2}\right)\sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle}\\
&= \sqrt{\langle \mathscr{A}x,x\rangle\langle \mathscr{B}y,y\rangle}.
\end{aligned}$$

Now the result is a consequence of Lemma 2.3. □

REMARK 2.1. A special case of the above theorem is Theorem 2.1, which is already proved in [15] for $\tau=0$.

Based on Theorem 2.6 and Lemma 2.4, we refine inequality (1.4) as follows.

THEOREM 2.7. *Let $\mathscr{T}\in\mathbb{B}(\mathscr{H})$ and $\tau\in[0,1]$. Then for all $x,y\in\mathscr{H}$, a partial isometry $V\in\mathbb{B}(\mathscr{H})$ exists such that*

$$|\langle\mathscr{T}x,y\rangle|\leqslant\frac{1-\tau}{2}|\langle f(|\mathscr{T}|)Vg(|\mathscr{T}^*|)x,y\rangle|+\frac{1+\tau}{2}\sqrt{\langle g^2(|\mathscr{T}^*|)x,x\rangle\langle f^2(|\mathscr{T}|)y,y\rangle}. \tag{2.20}$$

Specifically, for any $0\leqslant\theta\leqslant1$,

$$|\langle\mathscr{T}x,y\rangle|\leqslant\frac{1-\tau}{2}\left|\left\langle|\mathscr{T}|^{\theta}V|\mathscr{T}^*|^{1-\theta}x,y\right\rangle\right|+\frac{1+\tau}{2}\sqrt{\left\langle|\mathscr{T}^*|^{2(1-\theta)}x,x\right\rangle\langle|\mathscr{T}|^{2\theta}y,y\rangle}. \tag{2.21}$$

In our next theorem, based on the double convexity properties presented in inequality (1.10), we obtain an extension of the above theorem as follows.

THEOREM 2.8. *Let $\mathscr{T}\in\mathbb{B}(\mathscr{H})$ and $\tau\in[0,1]$. Then a partial isometry $V\in\mathbb{B}(\mathscr{H})$ exists such that for all unit vectors $x,y\in\mathscr{H}$,*

$$\begin{aligned}\psi(|\langle\mathscr{T}x,y\rangle|)\leqslant&\frac{1-\tau}{2}\psi(|\langle f(|\mathscr{T}|)Vg(|\mathscr{T}^*|)x,y\rangle|)\\&+\frac{1+\tau}{2}\sqrt{\langle\psi(g^2(|\mathscr{T}^*|))x,x\rangle\langle\psi(f^2(|\mathscr{T}|))y,y\rangle}.\end{aligned}$$

In particular, for any $0\leqslant\theta\leqslant1$ and $s\geqslant1$,

$$|\langle\mathscr{T}x,y\rangle|^s\leqslant\frac{1-\tau}{2}\left|\left\langle|\mathscr{T}|^{\theta}V|\mathscr{T}^*|^{1-\theta}x,y\right\rangle\right|^s+\frac{1+\tau}{2}\sqrt{\left\langle|\mathscr{T}^*|^{2s(1-\theta)}x,x\right\rangle\langle|\mathscr{T}|^{2s\theta}y,y\rangle}. \tag{2.22}$$

Proof. Let $x,y\in\mathscr{H}$ with $\|x\|=\|y\|=1$. Beginning with the first inequality in Theorem 2.7, the property that ψ is an increasing doubly convex function implies that

$$\begin{aligned}&\psi(|\langle\mathscr{T}x,y\rangle|)\\&\leqslant\psi\left(\frac{1-\tau}{2}|\langle f(|\mathscr{T}|)Vg(|\mathscr{T}^*|)x,y\rangle|+\frac{1+\tau}{2}\sqrt{\langle g^2(|\mathscr{T}^*|)x,x\rangle\langle f^2(|\mathscr{T}|)y,y\rangle}\right)\\&\leqslant\frac{1-\tau}{2}(\psi(|\langle f(|\mathscr{T}|)Vg(|\mathscr{T}^*|)x,y\rangle|))+\frac{1+\tau}{2}\left(\psi\left(\sqrt{\langle g^2(|\mathscr{T}^*|)x,x\rangle\langle f^2(|\mathscr{T}|)y,y\rangle}\right)\right)\\&\leqslant\frac{1-\tau}{2}(\psi(|\langle f(|\mathscr{T}|)Vg(|\mathscr{T}^*|)x,y\rangle|))+\frac{1+\tau}{2}\sqrt{\langle\psi(g^2(|\mathscr{T}^*|))x,x\rangle\langle\psi(f^2(|\mathscr{T}|))y,y\rangle}.\end{aligned}$$

where the second inequality is a consequence of the convexity of the function ψ, the third inequality is established using inequality (1.10), and the last inequality follows from Lemma 2.2. □

We conclude this section with the following alternative form of Furuta's inequality (1.3).

THEOREM 2.9. *Let* $\mathscr{S},\mathscr{T},\mathscr{X},\mathscr{Y}\in\mathbb{B}(\mathscr{H})$ *and* $\tau\in[0,1]$. *Then for all* $x,y\in\mathscr{H}$,

$$\begin{aligned}|\langle\mathscr{S}\mathscr{T}\mathscr{X}\mathscr{Y}x,y\rangle| &\leqslant \frac{1-\tau}{2}\left|\left\langle\mathscr{S}|\mathscr{T}|^{\frac{1}{2}}V\,|\mathscr{T}^*|^{\frac{1}{2}}\,\mathscr{X}\mathscr{Y}x,y\right\rangle\right|\\ &\quad+\frac{1+\tau}{2}\sqrt{\langle\mathscr{Y}^*\mathscr{X}^*\,|\mathscr{T}^*|\,\mathscr{X}\mathscr{Y}x,x\rangle\,\langle\mathscr{S}|\mathscr{T}|\mathscr{S}^*y,y\rangle}\end{aligned}$$

for some partial isometry $V\in\mathbb{B}(\mathscr{H})$.

Proof. By Theorem 2.6, for any $\mathscr{A}\in\mathbb{B}(\mathscr{H})$, there is a partial isometry V such that

$$|\langle\mathscr{A}x,y\rangle|\leqslant\frac{1-\tau}{2}\left|\left\langle|\mathscr{A}|^{\frac{1}{2}}V\,|\mathscr{A}^*|^{\frac{1}{2}}\,x,y\right\rangle\right|+\frac{1+\tau}{2}\sqrt{\langle|\mathscr{A}^*|x,x\rangle\,\langle|\mathscr{A}|y,y\rangle}. \tag{2.23}$$

If we replace x by $\mathscr{X}\mathscr{Y}x,\mathscr{A}$ by $\mathscr{T}$, and y by $\mathscr{S}^*y$, in inequality (2.23), we obtain the desired result. □

As a consequence of the theorem above, we derive the following generalized refinement of Furuta's inequality (1.3).

THEOREM 2.10. *Let* $\mathscr{T}\in\mathbb{B}(\mathscr{H})$ *and* $\theta\geqslant-1$ $\gamma\geqslant 2$, *and* $\tau\in[0,1]$. *Then for all* $x,y\in\mathscr{H}$,

$$\begin{aligned}|\langle\mathscr{T}|\mathscr{T}|^{\theta+\gamma-1}x,y\rangle|^2 &\leqslant\frac{1-\tau}{2}\left|\left\langle\mathscr{T}|\mathscr{T}|^{\frac{\gamma}{2}-1}V\,|\mathscr{T}|^{\frac{\gamma}{2}+\theta}\,x,y\right\rangle\right|\\ &\quad+\frac{1+\tau}{2}\sqrt{\langle|\mathscr{T}|^{2\theta+\gamma}x,x\rangle\,\langle|\mathscr{T}^*|^{\gamma}y,y\rangle}.\end{aligned}$$

3. Applications to numerical radius and norm inequalities

The objective of this section is to improve several inequalities mentioned in the introduction in a novel way by using inequalities given in Section 2. Furthermore, in the end, we present a new refinement of the triangle inequality for norms of operators.

We begin this section with the following improvement of inequality (1.9).

THEOREM 3.1. *Let* $\mathscr{T}\in\mathbb{B}(\mathscr{H})$. *Then we have*

$$\psi(\omega(\mathscr{T}))\leqslant\left[h(\mu)\sqrt[n]{\sqrt{\psi(\omega(\mathscr{T}))u}}+h(1-\mu)\sqrt[n]{u}\right]^n \quad \textit{for all } n\in\mathbb{N}^*, \tag{3.1}$$

where $u=\frac{1}{2}\left\|\psi\left(g^2(|\mathscr{T}^*|)\right)+\psi\left(f^2(|\mathscr{T}|)\right)\right\|$. *In particular, for any* $0\leqslant\theta\leqslant1$ *and* $s\geqslant1$,

$$\omega^s(\mathscr{T})\leqslant\left[h(\mu)\sqrt[n]{\sqrt{\omega^s(\mathscr{T})}v}+h(1-\mu)\sqrt[n]{v}\right]^n \quad \textit{for all } n\in\mathbb{N}^*, \tag{3.2}$$

where $v=\frac{1}{2}\left\||\mathscr{T}|^{2s\theta}+|\mathscr{T}^*|^{2s(1-\theta)}\right\|$.

Proof. By Theorem 2.3, we have for any $x \in \mathscr{H}$,

$$\psi\left(|\langle \mathscr{T}x,x\rangle|\right) \leqslant \mathscr{U}_{(n,h,\psi)}\left(\mu, \psi\left(f^2(|\mathscr{T}|)\right), \psi\left(g^2(|\mathscr{T}^*|)\right)\right) = \left[h(\mu)\sqrt[n]{\sqrt{\psi\left(|\langle \mathscr{T}x,x\rangle|\right)}a} + h(1-\mu)\sqrt[n]{a}\right]^n, \tag{3.3}$$

where $a = \sqrt{\langle \psi\left(f^2(|\mathscr{T}|)\right)x,x\rangle \langle \psi\left(g^2(|\mathscr{T}^*|\right)x,x\rangle}$. Now, by the arithmetic-geometric mean inequality, we have

$$\sqrt{\langle \psi\left(f^2(|\mathscr{T}|)\right)x,x\rangle \langle \psi\left(g^2(|\mathscr{T}^*|\right)y,y\rangle} \leqslant \frac{1}{2}\left[\left\langle \psi\left(f^2(|\mathscr{T}|)\right)x,x\right\rangle \left\langle \psi\left(g^2(|\mathscr{T}^*|\right)x,x\right\rangle\right] = \frac{1}{2}\left[\left\langle\left(\psi\left(f^2(|\mathscr{T}|)\right) + \psi\left(g^2(|\mathscr{T}^*|)\right)\right)x,x\right\rangle\right]. \tag{3.4}$$

Combining inequalities (3.3) and (3.4) and taking the supremum over all unit vectors $x \in \mathscr{H}$, we obtain the desired result. □

A non trivial refinement of inequality (1.8) is considered in the following result.

THEOREM 3.2. *Let $\mathscr{T} \in \mathbb{B}(\mathscr{H})$, $0 \leqslant \theta \leqslant 1$ and $s \geqslant 1$. Then for all $x,y \in \mathscr{H}$,*

$$\omega^{2s}(\mathscr{T}) \leqslant \left[h(\mu)\sqrt[n]{\sqrt{\omega^s(\mathscr{T})}u} + h(1-\mu)\sqrt[n]{u}\right]^n \quad \textit{for all } n \in \mathbb{N}^*,$$

where $u = \left\|\theta|\mathscr{T}|^{2s} + (1-\theta)|\mathscr{T}^*|^{2s}\right\|$.

Proof. By using Theorem 2.2, we have

$$|\langle \mathscr{T}x,x\rangle|^2 \leqslant \mathscr{U}_{(n,h)}(\mu, |\mathscr{T}|^{2\theta}, |\mathscr{T}^*|^{2(1-\theta)}) = \left[h(\mu)\sqrt[n]{\sqrt{|\langle \mathscr{T}x,x\rangle|}a} + h(1-\mu)\sqrt[n]{a}\right]^n, \tag{3.5}$$

where $a = \sqrt{\langle |\mathscr{T}|^{2\theta}x,x\rangle \left\langle |\mathscr{T}^*|^{2(1-\theta)}x,x\right\rangle}$. Now, by employing the convexity property of the function $t^s, s \geqslant 1$, we get

$$|\langle \mathscr{T}x,x\rangle|^{2s} \leqslant \left[h(\mu)\sqrt[n]{\left(\sqrt{\psi\left(|\langle \mathscr{T}x,x\rangle|\right)}a\right)^s} + h(1-\mu)\sqrt[n]{a^s}\right]^n.$$

Appling Lemma 2.2 and Young's inequality, we get

$$a^s = \left(\sqrt{\langle |\mathscr{T}|^{2\theta}x,x\rangle \left\langle |\mathscr{T}^*|^{2(1-\theta)}x,x\right\rangle}\right)^s \leqslant \sqrt{\langle |\mathscr{T}|^{2s}x,x\rangle^{\theta} \langle |\mathscr{T}^*|^{2s}x,x\rangle^{(1-\theta)}} \leqslant \sqrt{\langle \theta|\mathscr{T}|^{2s} + (1-\theta)|\mathscr{T}^*|^{2s}x,x\rangle}. \tag{3.6}$$

Combining inequalities (3.5) and (3.6) and taking the supremum over all vectors $x \in \mathscr{H}$, with $\|x\| = 1$, we achieve the desired result. □

In the following theorem, we introduce a further improvement of inequality (1.8).

THEOREM 3.3. *Let $\mathscr{T} \in \mathbb{B}(\mathscr{H})$ and $\tau \in [0,1]$. Then, a partial isometry V exists such that*

$$\psi(\omega(\mathscr{T})) \leqslant \frac{1-\tau}{2}\psi\left(\omega\left(f(|\mathscr{T}|)Vg\left(|\mathscr{T}^*|\right)\right)\right) + \frac{1-\tau}{4}\left\|\psi\left(g^2\left(|\mathscr{T}^*|\right)\right) + \psi\left(f^2(|\mathscr{T}|)\right)\right\|. \tag{3.7}$$

In particular, for any $0 \leqslant \theta \leqslant 1$ and $s \geqslant 1$,

$$\omega^s(\mathscr{T}) \leqslant \frac{1-\tau}{2}\omega^s\left(|\mathscr{T}|^\theta V|\mathscr{T}^*|^{1-\theta}\right) + \frac{1-\tau}{4}\left\||\mathscr{T}^*|^{2s(1-\theta)} + |\mathscr{T}|^{2s\theta}\right\|. \tag{3.8}$$

Proof. By Theorem 2.8, we have for $x \in \mathscr{H}$, and some partial isometry V, that

$$\begin{aligned}\psi(|\langle\mathscr{T}x,x\rangle|) &\leqslant \frac{1-\tau}{2}\psi\left(|\langle f(|\mathscr{T}|)Vg\left(|\mathscr{T}^*|\right)x,x\rangle|\right)\\ &\quad + \frac{1+\tau}{2}\sqrt{\langle\psi\left(g^2\left(|\mathscr{T}^*|\right)\right)x,x\rangle\langle\psi\left(f^2(|\mathscr{T}|)\right)x,x\rangle}\\ &\leqslant \frac{1-\tau}{2}\psi\left(|\langle f(|\mathscr{T}|)Vg\left(|\mathscr{T}^*|\right)x,x\rangle|\right)\\ &\quad + \frac{1+\tau}{4}\sqrt{\langle\psi\left(f^2(|\mathscr{T}|)\right)+\psi\left(g^2\left(|\mathscr{T}^*|\right)\right)x,x\rangle}.\end{aligned}$$

Taking the supremum over all vectors $x \in \mathscr{H}$, with $\|x\| = 1$, and noting that ψ is increasing function, we achieve the desired result. □

In the following theorem, we propose an additional refinement of inequality (1.9).

THEOREM 3.4. *Let $\mathscr{T} \in \mathbb{B}(\mathscr{H})$, $0 \leqslant \theta \leqslant 1$, $\tau \in [0,1]$ and $s \geqslant 1$. Then*

$$\omega^{2s}(\mathscr{T}) \leqslant \frac{1-\tau}{2}\left(\omega^{2s}\left(|\mathscr{T}|^\theta U|\mathscr{T}^*|^{1-\theta}\right)\right) + \frac{1+\tau}{2}\left(\left\|(1-\theta)|\mathscr{T}^*|^{2s} + \theta|\mathscr{T}|^{2s}\right\|\right) \tag{3.9}$$

for some partial isometry $V \in \mathbb{B}(\mathscr{H})$.

Proof. Let $x \in \mathscr{H}$. Then

$$\begin{aligned}&|\langle\mathscr{T}x,x\rangle|^s\\ &\leqslant \frac{1-\tau}{2}\left|\left\langle|\mathscr{T}|^\theta V|\mathscr{T}^*|^{1-\theta}x,x\right\rangle\right|^s\\ &\quad + \frac{1+\tau}{2}\sqrt{\left\langle|\mathscr{T}^*|^{2s(1-\theta)}x,x\right\rangle\langle|\mathscr{T}|^{2s\theta}x,x\rangle} \quad \text{(by inequality (2.22))}\\ &\leqslant \frac{1-\tau}{2}\left(\left|\left\langle|\mathscr{T}|^\theta V|\mathscr{T}^*|^{1-\theta}x,x\right\rangle\right|^s\right)\\ &\quad + \frac{1+\tau}{2}\sqrt{\left\langle|\mathscr{T}^*|^{2s}x,x\right\rangle^{1-\theta}\langle|\mathscr{T}|^{2s}x,x\rangle^\theta} \quad \text{(by Lemma 2.2)}\\ &\leqslant \frac{1-\tau}{2}\left(\left|\left\langle|\mathscr{T}|^\theta V|\mathscr{T}^*|^{1-\theta}x,x\right\rangle\right|^s\right) + \frac{1+\tau}{2}\sqrt{\left\langle\left((1-\theta)|\mathscr{T}^*|^{2s}+\theta|\mathscr{T}|^{2s}\right)x,x\right\rangle},\end{aligned}$$

where the last inequality follows by Young's inequality. This implies that

$$\omega^s(\mathscr{T}) \leqslant \left(\frac{1-\tau}{2} \omega^s \left(|\mathscr{T}|^{\theta} V |\mathscr{T}^*|^{1-\theta} \right) + \frac{1+\tau}{2} \left\| (1-\theta)|\mathscr{T}^*|^{2s} + \theta|\mathscr{T}|^{2s} \right\|^{\frac{1}{2}} \right). \tag{3.10}$$

Square both sides of the inequality above, and utilize the convexity of the function $\xi(\lambda) = \lambda^2$, to deduce that

$$\begin{aligned}\omega^{2s}(\mathscr{T}) &\leqslant \left(\frac{1-\tau}{2} \omega^s \left(|\mathscr{T}|^{\theta} V |\mathscr{T}^*|^{1-\theta} \right) + \frac{1+\tau}{2} \left\| (1-\theta)|\mathscr{T}^*|^{2s} + \theta|\mathscr{T}|^{2s} \right\|^{\frac{1}{2}} \right)^2 \\ &\leqslant \frac{1-\tau}{2} \omega^{2s} \left(|\mathscr{T}|^{\theta} U |\mathscr{T}^*|^{1-\theta} \right) + \frac{1+\tau}{2} \left(\left\| (1-\theta)|\mathscr{T}^*|^{2s} + \theta|\mathscr{T}|^{2r} \right\| \right).\end{aligned}$$

This completes the proof. □

The triangle inequality will be refined in the following theorems.

THEOREM 3.5. *Let $\mathscr{X}, \mathscr{Y} \in \mathbb{B}(\mathscr{H})$. Then for all $n \in \mathbb{N}^*$ and $\mu \in (0,1)$, we have*

$$\begin{aligned} &\|\mathscr{X} + \mathscr{Y}\| \\ &\leqslant \left[h(\mu) \sqrt[2n]{\|\mathscr{X}\| \sqrt{\|f^2(|\mathscr{X}|)\| \|g^2(|\mathscr{X}^*|)\|}} \right. \\ &\qquad \left. + h(1-\mu) \sqrt[2n]{\|f^2(|\mathscr{X}|)\| \|g^2(|\mathscr{X}^*|)\|} \right]^n \\ &\quad + \left[h(\mu) \sqrt[2n]{\|\mathscr{Y}\| \sqrt{\|f^2(|\mathscr{Y}|)\| \|g^2(|\mathscr{Y}^*|)\|}} \right. \\ &\qquad \left. + h(1-\mu) \sqrt[2n]{\|f^2(|\mathscr{Y}|)\| \|g^2(|\mathscr{Y}^*|)\|} \right]^n \end{aligned}$$

and

$$\begin{aligned} &\|\mathscr{X} + \mathscr{Y}\| \\ &\leqslant \left[h(\mu) \sqrt[2n]{\|\mathscr{X}\| \sqrt{\||\mathscr{X}|^{2\theta}\| \||\mathscr{X}^*|^{2(1-\theta)}\|}} \right. \\ &\qquad \left. + h(1-\mu) \sqrt[2n]{\||\mathscr{X}|^{2\theta}\| \||\mathscr{X}^*|^{2(1-\theta)}\|} \right]^n \\ &\quad + \left[h(\mu) \sqrt[2n]{\|\mathscr{Y}\| \sqrt{\||Y|^{2\theta}\| \||\mathscr{Y}^*|^{2(1-\theta)}\|}} \right. \\ &\qquad \left. + h(1-\mu) \sqrt[2n]{\||\mathscr{Y}|^{2\theta}\| \||\mathscr{Y}^*|^{2(1-\theta)}\|} \right]^n. \end{aligned}$$

Proof. By using the triangle inequality and Theorem 2.2, we have

$$\begin{aligned}
&|\langle(\mathscr{X}+\mathscr{Y})x,y\rangle| \leqslant |\langle\mathscr{X}x,y\rangle| + |\langle\mathscr{Y}x,y\rangle| \\
&\leqslant \sqrt{\mathscr{U}_{(n,h)}(\mu, f^2(|\mathscr{X}|), g^2(|X^*|)} + \sqrt{\mathscr{U}_{(n,h)}(\mu, f^2(|\mathscr{Y}|), g^2(|\mathscr{Y}^*|)} \\
&\leqslant \left[h(\mu)\sqrt[2n]{\langle\mathscr{X}x,y\rangle\sqrt{\langle f^2(|\mathscr{X}|)x,x\rangle\langle g^2(|\mathscr{X}^*|)y,y\rangle}}\right. \\
&\quad \left. + h(1-\mu)\sqrt[2n]{\langle f^2(|\mathscr{X}|)x,x\rangle\langle g^2(|\mathscr{X}^*|)y,y\rangle}\right]^n \\
&\quad + \left[h(\mu)\sqrt[2n]{\langle\mathscr{Y}x,y\rangle\sqrt{\langle f^2(|\mathscr{Y}|)x,x\rangle\langle g^2(|\mathscr{Y}^*|)y,y\rangle}}\right. \\
&\quad \left. + h(1-\mu)\sqrt[2n]{\langle f^2(|\mathscr{Y}|)x,x\rangle\langle g^2(|\mathscr{Y}^*|)y,y\rangle}\right]^n.
\end{aligned}$$

Taking the supremum over all unit vectors $x,y \in \mathscr{H}$, we get the desired result. □

We adapt the approach in [15] to establish the following refinement of the triangle inequality.

THEOREM 3.6. *Let $\mathscr{X},\mathscr{Y} \in \mathbb{B}(\mathscr{H})$ and $\tau \in [0,1]$. Then, there exist partial isometries $U,V \in \mathbb{B}(\mathscr{H})$, such that*

$$\begin{aligned}
\|\mathscr{X}+\mathscr{Y}\| &\leqslant \frac{1-\tau}{2}\left(\|f(|\mathscr{X}|)Ug(|\mathscr{X}^*|)\| + \|f(|\mathscr{Y}|)Vg(|T^*|)\|\right) \\
&\quad + \frac{1+\tau}{4}\left(\left\|g^2(|\mathscr{X}^*|)+g^2(|\mathscr{Y}^*|)\right\| + \left\|f^2(|\mathscr{X}|)+f^2(|\mathscr{Y}|)\right\|\right).
\end{aligned}$$

Specifically, for any $0 \leqslant \theta \leqslant 1$,

$$\begin{aligned}
\|\mathscr{X}+\mathscr{Y}\| &\leqslant \frac{1-\tau}{2}\left(\left\||\mathscr{X}|^{\theta}U|\mathscr{X}^*|^{1-\theta}\right\| + \left\||\mathscr{Y}|^{\theta}U|\mathscr{Y}^*|^{1-\theta}\right\|\right) \\
&\quad + \frac{1+\tau}{4}\left(\left\||\mathscr{X}^*|^{2(1-\theta)}+|\mathscr{Y}^*|^{2(1-\theta)}\right\| + \left\||X|^{2\theta}+|\mathscr{Y}|^{2\theta}\right\|\right).
\end{aligned}$$

Proof. By employing the triangle inequality and Theorem 2.7, we obtain for all unit vectors $x,y \in \mathscr{H}$,

$$\begin{aligned}
&|\langle(\mathscr{X}+\mathscr{Y})x,y\rangle| \\
&= |\langle\mathscr{X}x,y\rangle + \langle\mathscr{Y}x,y\rangle| \\
&\leqslant |\langle\mathscr{X}x,y\rangle| + |\langle\mathscr{Y}x,y\rangle| \\
&\leqslant \frac{1-\tau}{2}|\langle f(|\mathscr{X}|)Ug(|\mathscr{X}^*|)x,y\rangle| + \frac{1+\tau}{2}\sqrt{\langle g^2(|\mathscr{X}^*|)x,x\rangle\langle f^2(|\mathscr{X}|)y,y\rangle} \\
&\quad + \frac{1-\tau}{2}\left(|\langle f(|\mathscr{Y}|)Vg(|\mathscr{Y}^*|)x,y\rangle| + \frac{1+\tau}{2}\sqrt{\langle g^2(|Y^*|)x,x\rangle\langle f^2(|\mathscr{Y}|)y,y\rangle}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Theorem 2.7)}
\end{aligned}$$

$$\leqslant \frac{1-\tau}{2}\left(|\langle f(|\mathscr{X}|)Ug(|\mathscr{X}^*|)x,y\rangle| + |\langle f(|\mathscr{Y}|)Vg(|\mathscr{Y}^*|)x,y\rangle|\right)$$
$$+\frac{1+\tau}{4}\left(\left\langle\left(g^2(|\mathscr{X}^*|)+g^2(|\mathscr{Y}^*|)\right)x,x\right\rangle + \left\langle\left(f^2(|\mathscr{X}|)+f^2(|\mathscr{Y}|)\right)y,y\right\rangle\right).$$

Hence,

$$|\langle(\mathscr{X}+\mathscr{Y})x,y\rangle|$$
$$\leqslant \frac{1-\tau}{2}\left(|\langle f(|\mathscr{X}|)Ug(|\mathscr{X}^*|)x,y\rangle| + |\langle f(|\mathscr{Y}|)Vg(|\mathscr{Y}^*|)x,y\rangle|\right)$$
$$+\frac{1+\tau}{4}\left(\left\langle\left(g^2(|\mathscr{X}^*|)+g^2(|\mathscr{Y}^*|)\right)x,x\right\rangle + \left\langle\left(f^2(|\mathscr{X}|)+f^2(|\mathscr{Y}|)\right)y,y\right\rangle\right).$$

The desired result is achieved by taking the supremum over all vectors $x,y \in \mathscr{H}$, with $\|x\| = \|y\| = 1$. □

The triangle inequality for the operator norm is improved by Theorem 3.6, as shown in the following remark.

REMARK 3.1. Theorem 3.6, is a significant improvement of the triangle inequality for the operator norm. In fact,

$$\begin{aligned}\|\mathscr{X}+\mathscr{Y}\| &\leqslant \frac{1-\tau}{2}\left(\left\||X|^{\frac{1}{2}}U|\mathscr{X}^*|^{\frac{1}{2}}\right\| + \left\||\mathscr{Y}|^{\frac{1}{2}}V|\mathscr{Y}^*|^{\frac{1}{2}}\right\|\right)\\ &\quad+\frac{1+\tau}{4}\left(\||\mathscr{X}^*|+|\mathscr{Y}^*|\| + \||\mathscr{X}|+|\mathscr{Y}|\|\right)\\ &\leqslant \frac{1-\tau}{2}(\|\mathscr{X}\|+\|\mathscr{Y}\|) + \frac{1+\tau}{4}(2\|\mathscr{X}\|+2\|\mathscr{Y}\|)\\ &= \|\mathscr{X}\|+\|\mathscr{Y}\|.\end{aligned}$$

Conclusion

In this paper, we have presented new refinements of Kato's inequality, offering significant contributions to the study of numerical radius and norm inequalities for bounded linear operators on Hilbert spaces. By introducing improved versions of Young's inequality and employing a generalized form of Buzano's inequality, we derived a series of sharper bounds that enhance classical results in operator theory.

Theoretical developments were complemented by various applications, including refined forms of the triangle inequality and extensions of Furuta-type inequalities. These results not only generalize several known inequalities but also introduce new perspectives on the relationships between the numerical radius, operator norms, and functional calculus.

The integration of convexity particularly through the notion of doubly convex functions played a central role in expanding the analytical framework, allowing for a broader class of operator inequalities. Our methods are robust and adaptable, suggesting further avenues for investigation in related fields such as matrix analysis, spectral theory, and quantum computing.

Overall, the results obtained underline the effectiveness of combining refined functional inequalities with operator theoretic techniques. Future research may focus on extending these results to unbounded operators, non Hilbertian settings, or applications in concrete problems where norm estimates are essential.

Acknowledgements. The authors would like to express their sincere gratitude to the anonymous reviewers for their careful reading of the manuscript and their valuable comments. Their constructive suggestions have significantly improved the clarity and quality of this paper. We deeply appreciate their insightful feedback and attention to detail.

Declarations

Ethical approval. This statement is not applicable here.

Competing interest. The authors declare no competing interests.

Authors' contribution. All authors have contributed equally to this work.

Funding. This work is supported by the Natural Science Foundation of Henan (252300421797).

Availability of data and materials. This statement does not apply.

REFERENCES

[1] M. AL-DOLAT, F. KITTANEH, *Upper bounds for the numerical radii of powers of Hilbert space operators*, Quaestiones Mathematicae, **47** (2024), 341–352.
[2] A. ABU OMAR AND F. KITTANEH, *Numerical radius inequalities for $n \times n$ operator matrices*, Linear Algebra Appl., **468** (2015), 18–26.
[3] M. AKKOUCHI AND M. A. IGHACHANE, *Some refinements to Hölder's inequality and applications*, Proyecciones (Antofagasta), vol. 39, no. 1, Antofagasta feb. 2020.
[4] R. BHATIA, *Positive definite matrices*, Princeton Univ. Press, Princeton (2007).
[5] M. L. BUZANO, *Generalizzazione della diseguaglianza di Cauchy-Schwarz*, (Italian), Rend. Sem. Mat. Univ. e Politech. Torino, **31** (1971/73), (1974), 405–409.
[6] S. S. DRAGOMIR AND M. T. GARAYEV, *Inequalities for power series of products of operators in Hilbert spaces with applications to numerical radius*, J. Math. Inequal., **19** (1) (2025), 99–118.
[7] M. EL-HADDAD AND F. KITTANEH, *Numerical radius inequalities for Hilbert space operators II*, Studia Math., **182** (2007), no. 2, 133–140.
[8] T. FURUTA, *An extension of the Heinz-Kato theorem*, Proc. Am. Math. Soc., **120**, (1994), 785–787.
[9] I. GUMUS, O. HIRZALLAH, AND F. KITTANEH, *Norm inequalities involving accretive-dissipative 2×2 block matrices*, Linear Algebra Appl., **538** (2017), 76–93.
[10] T. HIROSHIMA, *Majorization criterion for distillability of a bipartite quantum state*, Phys. Rev. Lett., **91** (5) (2003), 057902.
[11] T. KATO, *Notes on some inequalities for linear operators*, Math. Ann., **125**, (1952), 208–212.
[12] F. KITTANEH, *Notes on some inequalities for Hilbert Space operators*, Publ. Res. Inst. Math. Sci., **24** (2), (1988), 283–293.
[13] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158** (2003), 11–17.
[14] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Ibid., **168** (2005), 73–80.

[15] F. KITTANEH, H. R. MORADI, AND M. SABABHEH, *Refined Kato inequality and applications to norm and numerical radius inequalities*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2024) **118**: 97, https://doi.org/10.1007/s13398-024-01600-4.
[16] M. S. MOSLEHIAN, M. KHOSRAVI, AND R. DRNOVŠEK, *A commutator approach to Buzano inequality*, Filomat, **26** (2012), no. 4, 827–832.
[17] S. PANG AND Y. X. LIANG, *Numerical radius inequalities for the weighted sums of Hilbert space operators*, J. Math. Inequal., **19** (1) (2025), 307–331.
[18] C. YANG AND D. LI, *Some improvements about numerical radius inequalities for Hilbert space operators*, J. Math. Inequal., **18** (1) (2024), 219–234.
[19] C. YANG, *Some generalizations of numerical radius inequalities for* 2×2 *operator matrices*, J. Math. Inequal., **19** (1) (2025), 247–259.

(Received February 22, 2025)

Yonghui Ren
School of Mathematics and Statistics
Zhoukou Normal University
Zhoukou 466001, China
e-mail: yonghuiren1992@163.com

Mohamed Amine Ighachane
Sciences and Technologies Team (ESTE)
Higher School of Education and Training of El Jadida
Chouaib Doukkali University
El Jadida, Morocco
e-mail: mohamedamineighachane@gmail.com