Journal of Mathematical Inequalities
Volume 19, Number 3 (2025), 893–907 doi:10.7153/jmi-2025-19-57

BEREZIN RADIUS INEQUALITIES VIA CLASSICAL INEQUALITIES

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(*Communicated by M. Krnić*)

Abstract. We use classical inequalities, including the Cauchy-Schwarz inequality and its extension, the Buzano inequality, as well as their generalizations, to prove new Berezin radius inequalities for operators on reproducing kernel Hilbert spaces.

1. Introduction

Let $\mathscr{H} = \mathscr{H}(F)$ be a Hilbert space of complex-valued functions φ on some set F such that the evaluation functional $\varphi \to \varphi(\omega)$ is continuous for every $\omega \in F$. Then by the Riesz representation theorem there exists a unique function $k_\omega \in \mathscr{H}$ satisfying the reproducing property

$$\varphi(\omega) = \langle \varphi, k_\omega \rangle \quad \text{for all } \varphi \in \mathscr{H} \text{ and } \omega \in F.$$

We denote the normalized reproducing kernel by $\hat{k}_\omega := \frac{k_\omega}{\|k_\omega\|_{\mathscr{H}}}$ and define the Berezin symbol $\widetilde{\mathrm{A}}$ of the bounded linear operator A on $\mathscr{H}$ (i.e., $\mathrm{A} \in \mathscr{B}(\mathscr{H})$) by the following formula:

$$\widetilde{\mathrm{A}}(\omega) := \left\langle \mathrm{A}\hat{k}_\omega(z), \hat{k}_\omega(z) \right\rangle, \ \omega \in F.$$

It is obvious that $|\widetilde{\mathrm{A}}(\omega)| \leqslant \|\mathrm{A}\|$ for all $\omega \in F$, and hence $\widetilde{\mathrm{A}}$ is a bounded function on F. The Berezin radius (or Berezin number [23, 24, 26]) $\mathrm{ber}(\mathrm{A})$ is defined as

$$\mathrm{ber}(\mathrm{A}) := \sup_{\omega \in F} |\widetilde{\mathrm{A}}(\omega)|.$$

Clearly, $\mathrm{ber}(\mathrm{A}) \leqslant w(\mathrm{A})$ (numerical radius) for any $\mathrm{A} \in \mathscr{B}(\mathscr{H})$. The Berezin set is defined as the range of the function $\widetilde{\mathrm{A}}$ (see Karaev [26]):

$$\mathrm{Ber}(\mathrm{A}) := \mathrm{Range}(\widetilde{\mathrm{A}}) = \{\widetilde{\mathrm{A}}(\omega) : \omega \in F\}.$$

Mathematics subject classification (2020): 47A30, 47A63.
Keywords and phrases: Cauchy-Schwarz inequality, reproducing kernel, Berezin radius, Berezin norm.

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Obviously, $\mathrm{Ber}(A) \subseteq W(A)$ (numerical range) for any $A \in \mathscr{B}(\mathscr{H})$. The Berezin norms of the operator A are defined by the formulas:

$$\|A\|_{b,1} := \sup\left\{\left\|A\hat{k}_\omega\right\|_{\mathscr{H}} : \omega \in \digamma\right\}$$
$$\|A\|_{b,2} := \sup\left\{\left|\left\langle A\hat{k}_\omega, \hat{k}_\rho\right\rangle\right| : \omega, \rho \in \digamma\right\}.$$

It is clear that $\|A\|_{b,2} \leqslant \|A\|_{b,1} \leqslant \|A\|$ and $\mathrm{ber}(A) \leqslant \|A\|_{b,2} \leqslant \|A\|_{b,1}$.

Note that the correspondence between an operator and its Berezin symbol is one-to-one in many cases, i.e., $A = 0$ if and only if $\widetilde{A} = 0$ (see, Zhu [45], Engliš [13], and Karaev [25, 27]). Thus, all information about an operator is contained in its Berezin symbol.

The classical Cauchy-Schwarz inequality says that for any two vectors $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in $\mathscr{H}$,

$$|\langle \mathfrak{a}_1, \mathfrak{a}_2\rangle_{\mathscr{H}}| \leqslant \|\mathfrak{a}_1\|_{\mathscr{H}} \|\mathfrak{a}_2\|_{\mathscr{H}}, \tag{1}$$

where $\langle .,.\rangle := \langle .,.\rangle_{\mathscr{H}}$ is the inner product, and $\|\mathfrak{a}_1\|_{\mathscr{H}} = \langle \mathfrak{a}_1, \mathfrak{a}_1\rangle^{\frac{1}{2}}$. Equality in (1) holds if and only if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are linearly dependent. This inequality is important in mathematics (see, e.g., [11, 31, 38]).

Kittaneh et al. [31, Lemma 3] enhanced inequality (1) after being inspired by it. Inspired on the same inequality (1), the authors created the following improved inequality in [16] :

$$\begin{aligned} |\langle \mathfrak{a}_1, \mathfrak{a}_2\rangle| &\leqslant \left[\rho\left(\|\mathfrak{a}_1\|^2\|\mathfrak{a}_2\|^2 - |\langle \mathfrak{a}_1, \mathfrak{a}_2\rangle|^2\right) + |\langle \mathfrak{a}_1, \mathfrak{a}_2\rangle|^{2\rho}\|\mathfrak{a}_1\|^{2(1-\rho)}\|\mathfrak{a}_2\|^{2(1-\rho)}\right]^{\frac{1}{2}} \\ &\leqslant \|\mathfrak{a}_1\|\|\mathfrak{a}_2\|. \end{aligned} \tag{2}$$

Recall that the Cauchy-Schwarz inequality for positive operators says that if A is a positive operator in $\mathscr{B}(\mathscr{H})$, then for any vectors $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathscr{H}$

$$|\langle A\mathfrak{a}_1, \mathfrak{a}_2\rangle|^2 \leqslant \langle A\mathfrak{a}_1, \mathfrak{a}_1\rangle\langle A\mathfrak{a}_2, \mathfrak{a}_2\rangle. \tag{3}$$

Kato [30] introduced the following mixed Schwarz inequality

$$|\langle A\mathfrak{a}_1, \mathfrak{a}_2\rangle|^2 \leqslant \langle |A|^{2\nu}\mathfrak{a}_1, \mathfrak{a}_1\rangle\langle |A^*|^{2(1-\nu)}\mathfrak{a}_2, \mathfrak{a}_2\rangle, \quad 0 \leqslant \nu \leqslant 1, \tag{4}$$

for every operator $A \in \mathscr{B}(\mathscr{H})$ and any vectors $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathscr{H}$, where $|A| := (A^*A)^{1/2}$. In particular, it can be obtained

$$|\langle A\mathfrak{a}_1, \mathfrak{a}_2\rangle| \leqslant \sqrt{\langle |A|\mathfrak{a}_1, \mathfrak{a}_1\rangle\langle |A^*|\mathfrak{a}_2, \mathfrak{a}_2\rangle} \tag{5}$$

(see Halmos [15, pp. 75–76]).

A generalization of the mixed Schwarz inequality (4) is the following inequality due to Kittaneh [32, Theorem 1].

LEMMA 1. *Let $A \in \mathscr{B}(\mathscr{H})$ be an operator. If $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are non-negative continuous functions on $[0,\infty)$ such that $\mathfrak{f}_1(\xi)\mathfrak{f}_2(\xi) = \xi$ for all $\xi \in [0,\infty)$, then*

$$|\langle A\mathfrak{a}_1, \mathfrak{a}_2\rangle| \leqslant \|\mathfrak{f}_1(|A|\mathfrak{a}_1)\|\|\mathfrak{f}_2(|A^*|\mathfrak{a}_2)\| \tag{6}$$

for every $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathscr{H}$.

In this paper, we establish certain new inequalities related with Berezin number and Berezin norms of operators.

2. Berezin radius inequalities via Cauchy-Schwarz type inequalities

Let us start with the following lemma.

LEMMA 2. *Let* $\mathrm{A} \in \mathscr{B}(\mathscr{H})$ *be an operator,* $\mathfrak{f}_1$ *and* $\mathfrak{f}_2$ *be non-negative continuous functions on* $[0,\infty)$ *satisfying the relation* $\mathfrak{f}_1(\xi)\mathfrak{f}_2(\xi) = \xi$ *for every* $0 \leqslant \xi < \infty$*. Then*

$$\begin{aligned}\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \right\rangle\right|^2 &\leqslant \varkappa \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle \\ &\quad + (1-\varkappa)\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \right\rangle\right| \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle^{\frac{1}{2}} \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle^{\frac{1}{2}} \\ &\leqslant \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle \end{aligned} \tag{7}$$

for every $\omega, \rho \in \digamma$ *and* $0 \leqslant \varkappa \leqslant 1$.

Proof. Indeed, by Lemma 1, we have for all $\varkappa \in [0,1]$ that

$$\begin{aligned}\left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \rangle\right|^2 &= \varkappa \left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \rangle\right|^2 + (n-\varkappa)\left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \rangle\right| \left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \rangle\right| \\ &\leqslant \varkappa \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle \\ &\quad + (1-\varkappa)\left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \rangle\right| \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle^{\frac{1}{2}} \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle^{\frac{1}{2}}. \end{aligned} \tag{8}$$

Also, we have

$$\begin{aligned}&\varkappa \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle \\ &\quad + (1-\varkappa)\left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \rangle\right| \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle^{\frac{1}{2}} \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle^{\frac{1}{2}} \\ &\leqslant \varkappa \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle \\ &\quad + (1-\varkappa)\left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle^{\frac{1}{2}} \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle^{\frac{1}{2}} \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle^{\frac{1}{2}} \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle^{\frac{1}{2}} \\ &= \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle \end{aligned} \tag{9}$$

Combining (8) and (9), we have

$$\begin{aligned}\left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \rangle\right|^2 &= \varkappa \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle \\ &\quad + (1-\varkappa)\left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\rho \rangle\right| \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle^{\frac{1}{2}} \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle^{\frac{1}{2}} \\ &\leqslant \left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega, \hat{k}_\omega \right\rangle \left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\rho, \hat{k}_\rho \right\rangle, \end{aligned} \tag{10}$$

as desired. This completes the proof. □

COROLLARY 1. *For* $\mathfrak{f}_1(\xi)=\xi^{\alpha}$ *and* $\mathfrak{f}_2(\xi)=\xi^{1-\alpha}$, $0\leqslant\alpha\leqslant 1$, *in the inequality* (7), *we have:*

(i)
$$\begin{aligned}\left|\langle \mathrm{A}\hat{k}_\omega,\hat{k}_\rho\rangle\right|^2 &\leqslant \varkappa\left\langle |\mathrm{A}|^{2\alpha}\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |\mathrm{A}^*|^{2(1-\alpha)}\hat{k}_\rho,\hat{k}_\rho\right\rangle\\ &\quad+(1-\varkappa)\left|\langle \mathrm{A}\hat{k}_\omega,\hat{k}_\rho\rangle\right|\left(\left\langle |\mathrm{A}|^{2\alpha}\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |\mathrm{A}^*|^{2(1-\alpha)}\hat{k}_\rho,\hat{k}_\rho\right\rangle\right)^{\frac{1}{2}}\\ &\leqslant \left\langle |\mathrm{A}|^{2\alpha}\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |\mathrm{A}^*|^{2(1-\alpha)}\hat{k}_\rho,\hat{k}_\rho\right\rangle,\ 0\leqslant\varkappa\leqslant 1. \end{aligned}\tag{11}$$

(ii)
$$\begin{aligned}\|\mathrm{A}\|_{b,2}^2 &\leqslant \varkappa\,\mathrm{ber}\left(|\mathrm{A}|^{2\alpha}\right)\mathrm{ber}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right)\\ &\quad+(1-\varkappa)\|\mathrm{A}\|_{b,2}\left(\mathrm{ber}\left(|\mathrm{A}|^{2\alpha}\right)\mathrm{ber}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right)\right)^{\frac{1}{2}}\\ &\leqslant \mathrm{ber}\left(|\mathrm{A}|^{2\alpha}\right)\mathrm{ber}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right),\ 0\leqslant\varkappa\leqslant 1. \end{aligned}\tag{12}$$

(iii)
$$\begin{aligned}\mathrm{ber}(\mathrm{A})^2 &\leqslant \varkappa\,\mathrm{ber}\left(|\mathrm{A}|^{2\alpha}\right)\mathrm{ber}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right)\\ &\quad+(1-\varkappa)\,\mathrm{ber}(\mathrm{A})\left(\mathrm{ber}\left(|\mathrm{A}|^{2\alpha}\right)\mathrm{ber}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right)\right)^{\frac{1}{2}}\\ &\leqslant \mathrm{ber}\left(|\mathrm{A}|^{2\alpha}\right)\mathrm{ber}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right),\ 0\leqslant\varkappa\leqslant 1. \end{aligned}\tag{13}$$

We have the following result as an application of Lemma 2, which generalizes all the results in this direction.

THEOREM 1. *Let* $\mathrm{A}\in\mathscr{B}(\mathscr{H})$ *be an operator and let* $\mathfrak{f}_1,\mathfrak{f}_2$ *be non-negative continuous functions on* $[0,\infty)$ *satisfying the relation* $\mathfrak{f}_1(\xi)\mathfrak{f}_2(\xi)=\xi$ *for every* $\xi\in[0,\infty)$. *Then*

$$\mathrm{ber}^2(\mathrm{A})\leqslant\frac{\varkappa}{2}\left\|\mathfrak{f}_1^4(|\mathrm{A}|)+\mathfrak{f}_2^4(|\mathrm{A}^*|)\right\|_{b,1}+\frac{1-\varkappa}{2}\mathrm{ber}(\mathrm{A})\left\|\mathfrak{f}_1^2(|\mathrm{A}|)+\mathfrak{f}_2^2(|\mathrm{A}^*|)\right\|_{b,1}\tag{14}$$

for $0\leqslant\varkappa\leqslant 1$.

Proof. Applying the AM-GM inequality and the well-known McCarthy inequality (see [36, 38]) to the first half of Lemma 2, where $\rho=\omega$, we obtain:

$$\begin{aligned}\left|\left\langle \mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2 &\leqslant \varkappa\left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\omega,\hat{k}_\omega\right\rangle\\ &\quad+(1-\varkappa)\left|\left\langle \mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega,\hat{k}_\omega\right\rangle^{\frac{1}{2}}\left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\omega,\hat{k}_\omega\right\rangle^{\frac{1}{2}}\\ &\leqslant \frac{\varkappa}{2}\left(\left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega,\hat{k}_\omega\right\rangle^2+\left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\omega,\hat{k}_\omega\right\rangle^2\right)\\ &\quad+\frac{1-\varkappa}{2}\left|\left\langle \mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left(\left\langle \mathfrak{f}_1^2(|\mathrm{A}|)\hat{k}_\omega,\hat{k}_\omega\right\rangle+\left\langle \mathfrak{f}_2^2(|\mathrm{A}^*|)\hat{k}_\omega,\hat{k}_\omega\right\rangle\right)\\ &\leqslant \frac{\varkappa}{2}\left\langle\left(\mathfrak{f}_1^4(|\mathrm{A}|)+\mathfrak{f}_2^4(|\mathrm{A}^*|)\right)\hat{k}_\omega,\hat{k}_\omega\right\rangle\\ &\quad+\frac{1-\varkappa}{2}\left|\left\langle \mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle\left(\mathfrak{f}_1^2(|\mathrm{A}|)+\mathfrak{f}_2^2(|\mathrm{A}^*|)\right)\hat{k}_\omega,\hat{k}_\omega\right\rangle. \end{aligned}$$

Now, in order to get the desired inequality, it remains only to take the supremum over $\omega \in F$. This proves the theorem. $\square$

COROLLARY 2. *For* $\mathfrak{f}_1(\xi) = \xi^{\alpha}$ *and* $\mathfrak{f}_2(\xi) = \xi^{1-\alpha}$, $0 \leqslant \alpha \leqslant 1$, *in the inequality* (14), *we have the inequality*

$$\mathrm{ber}^2(\mathrm{A}) \leqslant \frac{\varkappa}{2} \left\| |\mathrm{A}|^{4\alpha} + |\mathrm{A}^*|^{4(1-\alpha)} \right\|_{b,1} + \frac{1-\varkappa}{2} \mathrm{ber}(\mathrm{A}) \left\| |\mathrm{A}|^{2\alpha} + |\mathrm{A}^*|^{2(1-\alpha)} \right\|_{b,1} \tag{15}$$

for $0 \leqslant \varkappa \leqslant 1$.

The following significant and intriguing expansion of the well-known Cauchy-Schwarz inequality was found by Buzano [8] :

$$|\langle \mathfrak{a}_1, e\rangle \langle e, \mathfrak{a}_2\rangle| \leqslant \frac{1}{2} \left[\|\mathfrak{a}_1\|_{\mathscr{H}} \|\mathfrak{a}_2\|_{\mathscr{H}} + |\langle \mathfrak{a}_1, \mathfrak{a}_2\rangle| \right], \tag{16}$$

where $\mathfrak{a}_1, \mathfrak{a}_2, e$ are vectors in $\mathscr{H}$ with $\|e\| = 1$. When $\mathfrak{a}_1, \mathfrak{a}_2, e$ are vectors in $\mathscr{H}$ with $\|e\| = 1$ and $\beta \in \mathbb{C}$, Moslehian et al. were able to derive the following extension of Buzano's inequality (see [35, Corollary 2.5]):

$$|\beta \langle \mathfrak{a}_1, e\rangle \langle e, \mathfrak{a}_2\rangle - \langle \mathfrak{a}_1, \mathfrak{a}_2\rangle| \leqslant \max\{1, |\beta - 1|\} \|\mathfrak{a}_1\|_{\mathscr{H}} \|\mathfrak{a}_2\|_{\mathscr{H}} \tag{17}$$

The inequality (17) was also recently established by Bottazi and Conde [9, Proposition 3.1] using a rank-one operator. If $\beta \in \mathbb{C} \setminus \{0\}$, then the inequality is equivalent to

$$\left| \langle \mathfrak{a}_1, e\rangle \langle e, \mathfrak{a}_2\rangle - \frac{1}{\beta} \langle \mathfrak{a}_1, \mathfrak{a}_2\rangle \right| \leqslant |\beta|^{-1} \max\{1, |\beta - 1|\} \|\mathfrak{a}_1\|_{\mathscr{H}} \|\mathfrak{a}_2\|_{\mathscr{H}}.$$

Whence,

$$\left| |\langle \mathfrak{a}_1, e\rangle \langle e, \mathfrak{a}_2\rangle| - |\beta|^{-1} \right| |\langle \mathfrak{a}_1, \mathfrak{a}_2\rangle| \leqslant |\beta|^{-1} \max\{1, |\beta - 1|\} \|\mathfrak{a}_1\|_{\mathscr{H}} \|\mathfrak{a}_2\|_{\mathscr{H}},$$

which implies that

$$|\langle \mathfrak{a}_1, e\rangle \langle e, \mathfrak{a}_2\rangle| \leqslant |\beta|^{-1} \left[|\langle \mathfrak{a}_1, \mathfrak{a}_2\rangle| + \max\{1, |\beta - 1|\} \|\mathfrak{a}_1\|_{\mathscr{H}} \|\mathfrak{a}_2\|_{\mathscr{H}} \right]. \tag{18}$$

We have the Buzano inequality (16), for $\beta = 2$ in inequality (18). We obtain

$$\begin{aligned}
|\langle \mathfrak{f}_1, e\rangle \langle e, \mathfrak{f}_2\rangle|^2 &\leqslant |\langle \mathfrak{f}_1, e\rangle \langle e, \mathfrak{f}_2\rangle| |\beta|^{-1} \left[|\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle| + \max\{1, |\beta - 1|\} \|\mathfrak{f}_1\|_{\mathscr{H}} \|\mathfrak{f}_2\|_{\mathscr{H}} \right] \\
&= |\langle \mathfrak{f}_1, e\rangle \langle e, \mathfrak{f}_2\rangle| \frac{|\rho|}{|\beta|} \left[|\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle| + \max\{1, |\beta - 1|\} \|\mathfrak{f}_1\|_{\mathscr{H}} \|\mathfrak{f}_2\|_{\mathscr{H}} \right] \\
&\quad + |\langle \mathfrak{f}_1, e\rangle \langle e, \mathfrak{f}_2\rangle| \frac{1-\rho}{|\beta|} \left[|\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle| + \max\{1, |\beta - 1|\} \|\mathfrak{f}_1\|_{\mathscr{H}} \|\mathfrak{f}_2\|_{\mathscr{H}} \right] \\
&\leqslant \frac{\rho}{|\beta|^2} \left[|\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle| + \max\{1, |\beta - 1|\} \|\mathfrak{f}_1\|_{\mathscr{H}} \|\mathfrak{f}_2\|_{\mathscr{H}} \right]^2 \\
&\quad + |\langle \mathfrak{f}_1, e\rangle \langle e, \mathfrak{f}_2\rangle| \frac{1-\rho}{|\beta|} \left[|\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle| + \max\{1, |\beta - 1|\} \|\mathfrak{f}_1\|_{\mathscr{H}} \|\mathfrak{f}_2\|_{\mathscr{H}} \right]
\end{aligned}$$

$$= \frac{\rho}{|\beta|^2}\left[|\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle|^2 + \max\{1, |\beta-1|\}\|\mathfrak{f}_1\|_{\mathscr{H}}^2\|\mathfrak{f}_2\|_{\mathscr{H}}^2\right] + \frac{2\rho}{|\beta|^2}|\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle|\max\{1, |\beta-1|\}\|\mathfrak{f}_1\|_{\mathscr{H}}\|\mathfrak{f}_2\|_{\mathscr{H}} + |\langle \mathfrak{f}_1, e\rangle\langle e, \mathfrak{f}_2\rangle|\frac{1-\rho}{|\beta|}\left[|\langle \mathfrak{f}_1, \mathfrak{f}_2\rangle| + \max\{1, |\beta-1|\}\|\mathfrak{f}_1\|_{\mathscr{H}}\|\mathfrak{f}_2\|_{\mathscr{H}}\right]. \tag{19}$$

by using inequality (18) twice and assuming $0 \leqslant \rho \leqslant 1$.

Applying the inequality (19) now yields the following conclusion.

THEOREM 2. *Let* $A \in \mathscr{B}(\mathscr{H})$, $0 \leqslant \rho \leqslant 1$ *and* $\beta \in \mathbb{C}\setminus\{0\}$. *Then we have*

$$\operatorname{ber}^4(A) \leqslant \frac{\rho}{|\beta|^2}\operatorname{ber}^2\left(A^2\right) + \frac{\rho}{2|\beta|^2}\max\{1, |\beta-1|\}\left\||A|^4 + |A^*|^4\right\| + \frac{\rho}{|\beta|^2}\max\{1, |\beta-1|\}\operatorname{ber}\left(A^2\right)\left\||A|^2 + |A^*|^2\right\| + \frac{1-\rho}{|\beta|}\operatorname{ber}^2(A)\left[\operatorname{ber}\left(A^2\right) + \frac{\max\{1, |\beta-1|\}}{2}\left\||A^2| + |A^*|^2\right\|\right]. \tag{20}$$

Proof. In fact, let $\mathfrak{f}_1, \mathfrak{f}_2, e \in \mathscr{H}$ and $0 \leqslant \rho \leqslant 1$. We have from inequality (19) that

$$\left|\left\langle A\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|^4 \leqslant \frac{\rho}{|\beta|^2}\left[\left|\left\langle A^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|^2 + \max\{1, |\beta-1|\}\|A\hat{k}_\omega\|_{\mathscr{H}}^2\|A^*\hat{k}_\omega\|^2\right] + \frac{2\rho}{|\beta|^2}\left|\left\langle A^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|\max\{1, |\beta-1|\}\|A\hat{k}_\omega\|_{\mathscr{H}}\|A^*\hat{k}_\omega\|_{\mathscr{H}} + \frac{1-\rho}{|\beta|}\left|\left\langle A\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|^2\left[\left|\left\langle A^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\right| + \max\{1, |\beta-1|\}\|A\hat{k}_\omega\|_{\mathscr{H}}\|A^*\hat{k}_\omega\|_{\mathscr{H}}\right].$$

Now, using the AM-GM inequality in the second line below and the McCarthy inequalities known as

$$\langle Ax, x\rangle^r \leqslant \langle A^r x, x\rangle \quad \text{for } r \geqslant 1; \tag{21}$$

$$\langle A^r x, x\rangle \leqslant \langle Ax, x\rangle^r \quad \text{for } 0 < r \leqslant 1, \tag{22}$$

in the third line, we obtain the inequality

$$\left|\left\langle A\hat{k}_\omega, \hat{k}_\omega\right\rangle\right| \leqslant \frac{\rho}{|\beta|^2}\left\langle A^2\hat{k}_\omega, \hat{k}_\omega\right\rangle^2 + \max\{1, |\beta-1|\}\frac{\rho}{|\beta|^2}\left\langle |A|^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\left\langle |A^*|^2\hat{k}_\omega, \hat{k}_\omega\right\rangle + \max\{1, |\beta-1|\}\frac{2\rho}{|\beta|^2}\left\langle A^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\left(\left\langle |A|^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\left\langle |A^*|^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)^{\frac{1}{2}} + \frac{1-\rho}{|\beta|}\left|\left\langle A\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|^2 \times\left[\left|\left\langle A^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\right| + \max\{1, |\beta-1|\}\left(\left\langle |A|^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\left\langle |A^*|^2\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)^{\frac{1}{2}}\right]$$

$$\begin{aligned}
&\leqslant \frac{\rho}{|\beta|^2}\left|\left\langle A^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2+\max\{1,|\beta-1|\}\frac{\rho}{2|\beta|^2}\left[\left\langle A^2\hat{k}_\omega,\hat{k}_\omega\right\rangle^2+\left\langle |A^*|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle^2\right]\\
&\quad+\max\{1,|\beta-1|\}\frac{\rho}{|\beta|^2}\left|\left\langle A^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left[\left\langle A^2\hat{k}_\omega,\hat{k}_\omega\right\rangle^2+\left\langle |A^*|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle^2\right]\\
&\quad+\frac{1-\rho}{|\beta|}\left|\left\langle A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2\left[\left|\left\langle A^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|+\max\{1,|\beta-1|\}\frac{\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |A^*|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle}{2}\right]\\
&\leqslant \frac{\rho}{|\beta|^2}\left|\left\langle A^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2+\max\{1,|\beta-1|\}\frac{\rho}{2|\beta|^2}\left\langle\left(|A|^4+|A^*|^4\right)\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+\max\{1,|\beta-1|\}\frac{\rho}{|\beta|^2}\left|\left\langle A^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle\left(|A|^2+|A^*|^2\right)\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+\frac{1-\rho}{|\beta|}\left|\left\langle A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2\left[\left|\left\langle A^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|+\frac{\max\{1,|\beta-1|\}}{2}\left\langle\left(|A|^2+|A^*|^2\right)\hat{k}_\omega,\hat{k}_\omega\right\rangle\right].
\end{aligned}$$

So, we get our desired result by taking the supremum over all $\omega\in\mathcal{F}$. The theorem is proved. □

In [16], the authors proved the following generalized improvement of the Cauchy-Schwarz inequality.

LEMMA 3. *Let* $x,y\in\mathscr{H}$ *and* $0\leqslant\rho\leqslant 1$. *Then*

$$\begin{aligned}
|\langle x,y\rangle| &\leqslant \left[\rho\left(\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right)+|\langle x,y\rangle|^{2\rho}\|x\|^{2(1-\rho)}\|y\|^{2(1-\rho)}\right]^{1/2}\\
&\leqslant \|x\|\,\|y\|.
\end{aligned}\tag{23}$$

THEOREM 3. *Let* $A,B\in\mathscr{B}(\mathscr{H})$ *and* $0\leqslant\rho\leqslant 1$. *Then*

$$\mathrm{ber}^2(B^*A)\leqslant\frac{\rho(1-\rho)}{1+\rho-\rho^2}\,\mathrm{ber}(B^*A)\left\||A|^2+|B|^2\right\|+\frac{1-\rho+\rho^2}{2(1+\rho-\rho^2)}\left\||A|^4+|B|^4\right\|.\tag{24}$$

Proof. We have from the lower inequality in (23) that

$$|\langle\mathfrak{a}_1,\mathfrak{a}_2\rangle|^2\leqslant\rho\left(\|\mathfrak{a}_1\|^2\|\mathfrak{a}_2\|^2-|\langle\mathfrak{a}_1,\mathfrak{a}_2\rangle|^2\right)+|\langle\mathfrak{a}_1,\mathfrak{a}_2\rangle|^{2\rho}\|\mathfrak{a}_1\|^{2(1-\rho)}\|\mathfrak{a}_2\|^{2(1-\rho)},$$

so

$$|\langle\mathfrak{a}_1,\mathfrak{a}_2\rangle|^2\leqslant\frac{\rho}{1+\rho}\|\mathfrak{a}_1\|^2\|\mathfrak{a}_2\|^2+\frac{1}{1+\rho}|\langle\mathfrak{a}_1,\mathfrak{a}_2\rangle|^{2\rho}\|\mathfrak{a}_1\|^{2(1-\rho)}\|\mathfrak{a}_2\|^{2(1-\rho)}.$$

Replacing $\mathfrak{a}_1$ by $A\hat{k}_\omega$ and $\mathfrak{a}_2$ by $B\hat{k}_\omega$, we have

$$\begin{aligned}
&\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2\\
&\leqslant\frac{\rho}{1+\rho}\left\|A\hat{k}_\omega\right\|^2\left\|B\hat{k}_\omega\right\|^2+\frac{1}{1+\rho}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^{2\rho}\|A\hat{k}_\omega\|^{2(1-\rho)}\|B\hat{k}_\omega\|^{2(1-\rho)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2\\
&\leqslant \frac{\rho}{1+\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+\frac{1}{1+\rho}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^{2\rho}\left(\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right)^{1-\rho}\\
&=\frac{\rho}{1+\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+\frac{1}{1+\rho}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^{\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle^{1-\rho}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^{\rho}\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle^{1-\rho}. \quad (25)
\end{aligned}$$

Now, using the classical Young inequality, we get

$$\begin{aligned}
&\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2\\
&\leqslant \frac{\rho}{1+\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+\frac{1}{1+\rho}\left(\rho\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|+(1-\rho)\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right)\\
&\quad\times\left(\rho\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|+(1-\rho)\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right)\\
&=\frac{\rho}{1+\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+\frac{\rho^2}{1+\rho}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2+\frac{\rho(1-\rho)}{1+\rho}\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\\
&\quad+\frac{\rho(1-\rho)}{1+\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|+\frac{(1-\rho)^2}{1+\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&=\frac{\rho}{1+\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+\frac{\rho^2}{1+\rho}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2+\frac{\rho(1-\rho)}{1+\rho}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle\left(|A|^2+|B|^2\right)\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+\frac{(1-\rho)^2}{1+\rho}\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle. \quad (26)
\end{aligned}$$

After an elementary calculation, we have that

$$\begin{aligned}
(1+\rho-\rho^2)\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2 &\leqslant \rho(1-\rho)\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle(|A|^2+|B|^2)\hat{k}_\omega,\hat{k}_\omega\right\rangle\\
&\quad+(1-\rho+\rho^2)\left\langle |A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle |B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle. \quad (27)
\end{aligned}$$

Also, using again the AM-GM inequality and the McCarthy inequalities, we have

$$\begin{aligned}(1+\rho-\rho^2)\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2 &\leqslant \rho(1-\rho)\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle(|A|^2+|B|^2)\hat{k}_\omega,\hat{k}_\omega\right\rangle \\ &\quad+\frac{1-\rho+\rho^2}{2}\left(\left\langle|A|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle^2+\left\langle|B|^2\hat{k}_\omega,\hat{k}_\omega\right\rangle^2\right) \\ &\leqslant \rho(1-\rho)\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle(|A|^2+|B|^2)\hat{k}_\omega,\hat{k}_\omega\right\rangle \\ &\quad+\frac{1-\rho+\rho^2}{2}\left[\left\langle|A|^4\hat{k}_\omega,\hat{k}_\omega\right\rangle+\left\langle|B|^4\hat{k}_\omega,\hat{k}_\omega\right\rangle\right]. \end{aligned}\tag{28}$$

Finally, we have:

$$\begin{aligned}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2 \leqslant \frac{\rho(1-\rho)}{(1+\rho-\rho^2)}\left|\left\langle B^*A\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle(|A|^2+|B|^2)\hat{k}_\omega,\hat{k}_\omega\right\rangle \\ +\frac{1-\rho+\rho^2}{2(1+\rho-\rho^2)}\left[\left\langle|A|^4\hat{k}_\omega,\hat{k}_\omega\right\rangle+\left\langle|B|^4\hat{k}_\omega,\hat{k}_\omega\right\rangle\right],\end{aligned}\tag{29}$$

from which by taking the supremum over all $\omega \in \digamma$, we obtain the desired result. This proves the theorem. □

COROLLARY 3. *For $\rho = 0$ or $\rho = 1$ in inequality (24), we have the inequality*

$$\mathrm{ber}^2\left(B^*A\right) \leqslant \frac{1}{2}\left\||A|^4+|B|^4\right\|_{B,1}. \tag{30}$$

3. Berezin radius inequalities via geometric convexity

Some applications of geometrically convex functions to Berezin radius inequalities are shown in this section. If I is a sub-interval of $(0,\infty)$ and $\mathfrak{f}: I \to (0,\infty)$, then $\mathfrak{f}$ is called geometrically convex if (see [36])

$$\mathfrak{f}\left(a^{1-\rho}b^{\rho}\right) \leqslant \mathfrak{f}^{1-\rho}(a)\mathfrak{f}^{\rho}(b),\ 0 \leqslant \rho \leqslant 1. \tag{31}$$

THEOREM 4. *Let $A \in \mathscr{B}(\mathscr{H})$ and let $\mathfrak{f}$ be an increasing geometrically convex function. If, in addition, $\mathfrak{f}$ is convex, then*

$$\begin{aligned}\mathfrak{f}\left(\mathrm{ber}^2(A)\right) &\leqslant \rho\left\|\alpha\mathfrak{f}\left(|A|^2\right)+(1-\alpha)\mathfrak{f}\left(|A^*|^2\right)\right\| \\ &\quad+\frac{1-\rho}{2}\mathfrak{f}(\mathrm{ber}(A))\left\|\mathfrak{f}\left(|A|^{2\alpha}\right)+\mathfrak{f}\left(|A^*|^{2(1-\alpha)}\right)\right\|_{b,1}\end{aligned}\tag{32}$$

for every $0 \leqslant \rho \leqslant 1$ and $0 \leqslant \alpha \leqslant 1$.

Proof. Indeed, using monotonicity of $\mathfrak{f}$ and some known inequalities, we have for any $\omega \in \mathcal{F}$ that

$$
\begin{aligned}
&\mathfrak{f}\left(\left|\widetilde{\mathrm{A}}(\omega)\right|^2\right)\\
&= \mathfrak{f}\left(\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|^2\right) = \rho\mathfrak{f}\left(\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|^2\right) + (1-\rho)\mathfrak{f}\left(\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|^2\right)\\
&\leqslant \rho\mathfrak{f}\left(\left\langle |\mathrm{A}|^{2\alpha}\hat{k}_\omega, \hat{k}_\omega\right\rangle \left\langle |\mathrm{A}^*|^{2(1-\alpha)}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)\\
&\quad + (1-\rho)\mathfrak{f}\left(\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|\right)\mathfrak{f}\left(\sqrt{\left\langle |\mathrm{A}|^{2\alpha}\hat{k}_\omega, \hat{k}_\omega\right\rangle \left\langle |\mathrm{A}^*|^{2(1-\alpha)}\hat{k}_\omega, \hat{k}_\omega\right\rangle}\right)\\
&\leqslant \rho\mathfrak{f}\left(\left\langle |\mathrm{A}|^{2}\hat{k}_\omega, \hat{k}_\omega\right\rangle^{\alpha} \left\langle |\mathrm{A}^*|^{2}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)^{(1-\alpha)}\\
&\quad + (1-\rho)\mathfrak{f}\left(\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|\right)\mathfrak{f}\left(\sqrt{\left\langle |\mathrm{A}|^{2\alpha}\hat{k}_\omega, \hat{k}_\omega\right\rangle \left\langle |\mathrm{A}^*|^{2(1-\alpha)}\hat{k}_\omega, \hat{k}_\omega\right\rangle}\right)\\
&\leqslant \rho\mathfrak{f}^{\alpha}\left(\left\langle |\mathrm{A}|^{2}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)\mathfrak{f}^{1-\alpha}\left(\left\langle |\mathrm{A}^*|^{2}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)\\
&\quad + (1-\rho)\mathfrak{f}\left(\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|\right)\left(\sqrt{\mathfrak{f}\left(\left\langle |\mathrm{A}|^{2\alpha}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)\mathfrak{f}\left(\left\langle |\mathrm{A}^*|^{2(1-\alpha)}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)}\right)\\
&\leqslant \rho\left(\alpha\mathfrak{f}\left(\left\langle |\mathrm{A}|^{2}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right) + (1-\alpha)\mathfrak{f}\left(\left\langle |\mathrm{A}^*|^{2}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right)\right)\\
&\quad + (1-\rho)\mathfrak{f}\left(\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|\right)\left(\sqrt{\left\langle \mathfrak{f}|\left(\mathrm{A}|^{2\alpha}\right)\hat{k}_\omega, \hat{k}_\omega\right\rangle \left\langle \mathfrak{f}_1\left(|\mathrm{A}^*|^{2(1-\alpha)}\right)\hat{k}_\omega, \hat{k}_\omega\right\rangle}\right)\\
&\leqslant \rho\left\langle\left(\alpha\mathfrak{f}\left(|\mathrm{A}|^2\right) + (1-\alpha)\mathfrak{f}\left(|\mathrm{A}^*|^2\right)\right)\hat{k}_\omega, \hat{k}_\omega\right\rangle\\
&\quad + \frac{1-\rho}{2}\mathfrak{f}\left(\left|\left\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\right\rangle\right|\right)\left\langle\left(\mathfrak{f}\left(|\mathrm{A}|^{2\alpha}\right) + \mathfrak{f}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right)\right)\hat{k}_\omega, \hat{k}_\omega\right\rangle\\
&= \rho\left(\alpha\mathfrak{f}\left(|\mathrm{A}|^2\right) + (1-\alpha)\mathfrak{f}\left(|\mathrm{A}^*|^2\right)\right)^{\sim}(\omega)\\
&\quad + \frac{1-\rho}{2}\mathfrak{f}\left(|\widetilde{\mathrm{A}}(\omega)|\right)\mathfrak{f}\left(\left(|\mathrm{A}|^{2\alpha}\right) + \mathfrak{f}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right)\right)^{\sim}(\omega).
\end{aligned}
$$

By taking the supremum over $\omega \in \mathcal{F}$, we deduce

$$
\begin{aligned}
\mathfrak{f}\left(\mathrm{ber}^2(\mathrm{A})\right) \leqslant \rho\left\|\alpha\mathfrak{f}\left(|\mathrm{A}|^2\right) + (1-\alpha)\mathfrak{f}\left(|\mathrm{A}^*|^2\right)\right\|_{b,1}\\
+ \frac{1-\rho}{2}\mathfrak{f}\left(\mathrm{ber}(\mathrm{A})\right)\left\|\mathfrak{f}\left(|\mathrm{A}|^{2\alpha}\right) + \mathfrak{f}\left(|\mathrm{A}^*|^{2(1-\alpha)}\right)\right\|_{b,1},
\end{aligned}
$$

this proves the theorem. $\square$

COROLLARY 4. *Let* $\mathrm{A} \in \mathscr{B}(\mathscr{H})$ *and* $\mathfrak{f}(\xi) = \xi^r$, $r \geqslant 1$ *in Theorem 4, we get*

$$
\mathrm{ber}^{2r}(\mathrm{A}) \leqslant \rho\left\|\alpha|\mathrm{A}|^{2r} + (1-\alpha)|\mathrm{A}^*|^{2r}\right\|_{b,1} + \frac{1-\rho}{2}\mathrm{ber}^r(\mathrm{A})\left\||\mathrm{A}|^{2r\alpha} + |\mathrm{A}^*|^{2r(1-\alpha)}\right\|_{b,1}
$$

for $0 \leqslant \alpha \leqslant 1$ *and* $0 \leqslant \rho \leqslant 1$.

In conclusion, we establish the following two results.

THEOREM 5. *Let* $\mathrm{A} \in \mathscr{B}(\mathscr{H})$ *be an operator. Then we have*

$$\operatorname{ber}^2(\mathrm{A}) \leqslant \frac{1}{2}\left[\|\mathrm{A}\|_{b,1}^2 \left(\min_{0\leqslant\xi\leqslant 1} \|\xi \mathrm{A}^*\mathrm{A} + (1-\xi)\mathrm{A}\mathrm{A}^*\|_{b,1}\right) + \operatorname{ber}^2\left(\mathrm{A}^2\right) + \operatorname{ber}\left(\mathrm{A}^2\right)\|\mathrm{A}^*\mathrm{A}+\mathrm{A}\mathrm{A}^*\|_{b,1}\right]^{\frac{1}{2}}.$$

Proof. Let $\omega \in \digamma$. Then we have:

$$\begin{aligned}
&\left|\widetilde{\mathrm{A}}(\omega)\right|^2 \\
&= \left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\rangle\right|^2 = \left|\langle \mathrm{A}\hat{k}_\omega, \hat{k}_\omega\rangle \langle \hat{k}_\omega, \mathrm{A}^*\hat{k}_\omega\rangle\right| && (33)\\
&\leqslant \frac{1}{2}\left(\left|\langle \mathrm{A}\hat{k}_\omega, \mathrm{A}^*\hat{k}_\omega\rangle\right| + \|\mathrm{A}\hat{k}_\omega\|\,\|\mathrm{A}^*\hat{k}_\omega\|\right) && \text{(by Buzano's inequality)}\\
&= \frac{1}{2}\left\{\left|\langle \mathrm{A}\hat{k}_\omega, \mathrm{A}^*\hat{k}_\omega\rangle\right|^2 + \|\mathrm{A}\hat{k}_\omega\|^2\|\mathrm{A}^*\hat{k}_\omega\|^2 + 2\left|\langle \mathrm{A}\hat{k}_\omega, \mathrm{A}^*\hat{k}_\omega\rangle\right|\|\mathrm{A}\hat{k}_\omega\|\,\|\mathrm{A}^*\hat{k}_\omega\|\right\}^{\frac{1}{2}}\\
&\leqslant \frac{1}{2}\left\{\begin{array}{c}\left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|^2 + \langle \mathrm{A}^*\mathrm{A}\hat{k}_\omega, \hat{k}_\omega\rangle\langle \mathrm{A}\mathrm{A}^*\hat{k}_\omega, \hat{k}_\omega\rangle\\ +\left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|\langle(\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A})\hat{k}_\omega, \hat{k}_\omega\rangle\end{array}\right\}^{\frac{1}{2}}\\
&= \frac{1}{2}\left\{\begin{array}{c}\left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|^2 + \langle \mathrm{A}^*\mathrm{A}\hat{k}_\omega, \hat{k}_\omega\rangle^{\xi}\langle \mathrm{A}\mathrm{A}^*\hat{k}_\omega, \hat{k}_\omega\rangle^{1-\xi}\langle \mathrm{A}^*\mathrm{A}\hat{k}_\omega, \hat{k}_\omega\rangle^{1-\xi}\\ \times\langle \mathrm{A}\mathrm{A}^*\hat{k}_\omega, \hat{k}_\omega\rangle^{\xi} + \left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|\langle(\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A})\hat{k}_\omega, \hat{k}_\omega\rangle\end{array}\right\}^{\frac{1}{2}}\\
&\leqslant \frac{1}{2}\left\{\begin{array}{c}\left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|^2 + \langle(\xi\mathrm{A}^*\mathrm{A}+(1-\xi)\mathrm{A}\mathrm{A}^*)\hat{k}_\omega, \hat{k}_\omega\rangle\\ \times\langle((1-\xi)\mathrm{A}^*\mathrm{A}+\xi\mathrm{A}\mathrm{A}^*)\hat{k}_\omega, \hat{k}_\omega\rangle + \left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|\langle(\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A})\hat{k}_\omega, \hat{k}_\omega\rangle\end{array}\right\}^{\frac{1}{2}}\\
&\qquad \text{(by the inequality } a^{\alpha}b^{1}-\alpha \leqslant \alpha a+(1-\alpha)b \text{ for } a,b\geqslant 0 \text{ and } 0\leqslant\alpha\leqslant 1)\\
&\leqslant \frac{1}{2}\left\{\begin{array}{c}\left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|^2 + \|\xi\mathrm{A}^*\mathrm{A}+(1-\xi)\mathrm{A}\mathrm{A}^*\|_{b,1}\|(1-\xi)\mathrm{A}^*\mathrm{A}+\xi\mathrm{A}\mathrm{A}^*\|\\ +\left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|\langle(\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A})\hat{k}_\omega, \hat{k}_\omega\rangle\end{array}\right\}^{1/2}\\
&\leqslant \frac{1}{2}\left\{\begin{array}{c}\left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|^2 + \|\xi\mathrm{A}^*\mathrm{A}+(1-\xi)\mathrm{A}\mathrm{A}^*\|_{b,1}\|\mathrm{A}\|_{b,1}^2\\ +\left|\langle \mathrm{A}^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|\langle(\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A})\hat{k}_\omega, \hat{k}_\omega\rangle\end{array}\right\}^{1/2}\\
&\leqslant \frac{1}{2}\left\{\operatorname{ber}^2\left(\mathrm{A}^2\right) + \|\xi\mathrm{A}^*\mathrm{A}+(1-\xi)\mathrm{A}\mathrm{A}^*\|_{b,1}\|\mathrm{A}\|_{b,1}^2 + \operatorname{ber}\left(\mathrm{A}^2\right)\|\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A}\|_{b,1}\right\}^{1/2}.
\end{aligned}$$

So, taking the supremum over $\omega \in \digamma$, we have

$$\operatorname{ber}^2(\mathrm{A}) \leqslant \frac{1}{2}\left[\|\mathrm{A}\|_{b,1}^2\left(\|\xi\mathrm{A}^*\mathrm{A} + (1-\xi)\mathrm{A}\mathrm{A}^*\|_{b,1}\right) + \operatorname{ber}^2\left(\mathrm{A}^2\right) + \operatorname{ber}\left(\mathrm{A}^2\right)\|\mathrm{A}^*\mathrm{A}+\mathrm{A}\mathrm{A}^*\|_{b,1}\right]^{\frac{1}{2}},$$

for all $\xi \in [0,1]$. Taking the minimum over $\xi \in [0,1]$, we obtain the desired result:

$$\begin{aligned}\operatorname{ber}^2(\mathrm{A}) \leqslant \frac{1}{2}\Big[\|\mathrm{A}\|_{b,1}^2\left(\min_{0\leqslant\xi\leqslant1}\|\xi\mathrm{A}^*\mathrm{A}+(1-\xi)\mathrm{A}\mathrm{A}^*\|_{b,1}\right)+\operatorname{ber}^2\left(\mathrm{A}^2\right)\\ +\operatorname{ber}\left(\mathrm{A}^2\right)\|\mathrm{A}^*\mathrm{A}+\mathrm{A}\mathrm{A}^*\|_{b,1}\Big]^{\frac{1}{2}}.\end{aligned}$$

the theorem is proven. □

THEOREM 6. *Let* $\mathrm{A} \in \mathscr{B}(\mathscr{H})$. *Then we have:*

$$\begin{aligned}\operatorname{ber}^4(\mathrm{A}) \leqslant \frac{1}{4}\Big[\operatorname{ber}^2\left(\mathrm{A}^2\right)+\frac{1}{4}\left\|(\mathrm{A}^*\mathrm{A})^2+(\mathrm{A}\mathrm{A}^*)^2\right\|_{b,1}+\frac{1}{2}\operatorname{ber}\left(\mathrm{A}^*\mathrm{A}^2\mathrm{A}^*\right)\\ +\operatorname{ber}\left(\mathrm{A}^2\right)\|\mathrm{A}^*\mathrm{A}+\mathrm{A}\mathrm{A}^*\|_{b,1}\Big].\end{aligned}$$

Proof. For any $\omega \in F$, we have:

$$\begin{aligned}\widetilde{\mathrm{A}^*\mathrm{A}}(\omega)\widetilde{\mathrm{A}\mathrm{A}^*}(\omega) &= \left\langle \mathrm{A}^*\mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle \mathrm{A}\mathrm{A}^*\hat{k}_\omega,\hat{k}_\omega\right\rangle = \left\langle \mathrm{A}^*\mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle \hat{k}_\omega,\mathrm{A}\mathrm{A}^*\hat{k}_\omega\right\rangle\\ &\leqslant \frac{\left\|\mathrm{A}^*\mathrm{A}\hat{k}_\omega\right\|\left\|\mathrm{A}\mathrm{A}^*\hat{k}_\omega\right\|+\left|\left\langle \mathrm{A}\mathrm{A}^*\hat{k}_\omega,\mathrm{A}^*\mathrm{A}\hat{k}_\omega\right\rangle\right|}{2}\\ &\leqslant \frac{1}{4}\left(\left\|\mathrm{A}^*\mathrm{A}\hat{k}_\omega\right\|^2+\left\|\mathrm{A}\mathrm{A}^*\hat{k}_\omega\right\|^2+\frac{1}{2}\left|\left\langle \mathrm{A}^*\mathrm{A}^2\mathrm{A}^*\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\right)\\ &= \frac{1}{4}\left\langle\left((\mathrm{A}^*\mathrm{A})^2+(\mathrm{A}\mathrm{A}^*)^2\right)\hat{k}_\omega,\hat{k}_\omega\right\rangle+\frac{1}{2}\left|\left\langle \mathrm{A}^*\mathrm{A}^2\mathrm{A}^*\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\\ &\leqslant \frac{1}{4}\left\|(\mathrm{A}^*\mathrm{A})^2+(\mathrm{A}\mathrm{A}^*)^2\right\|_{b,1}+\frac{1}{2}\operatorname{ber}\left(\mathrm{A}^*\mathrm{A}^2\mathrm{A}^*\right).\end{aligned}$$

Then we obtain that

$$\begin{aligned}|\widetilde{\mathrm{A}}(\omega)|^4 &= \left|\left\langle \mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^4\\ &\leqslant \frac{1}{4}\Big\{\left|\left\langle \mathrm{A}^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|^2+\left\langle \mathrm{A}^*\mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle \mathrm{A}\mathrm{A}^*\hat{k}_\omega,\hat{k}_\omega\right\rangle\\ &\quad+\left|\left\langle \mathrm{A}^2\hat{k}_\omega,\hat{k}_\omega\right\rangle\right|\left\langle(\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A})\hat{k}_\omega,\hat{k}_\omega\right\rangle\Big\}\\ &\leqslant \frac{1}{4}\Big\{\operatorname{ber}^2\left(\mathrm{A}^2\right)+\left\langle \mathrm{A}^*\mathrm{A}\hat{k}_\omega,\hat{k}_\omega\right\rangle\left\langle \mathrm{A}\mathrm{A}^*\hat{k}_\omega,\hat{k}_\omega\right\rangle+\operatorname{ber}\left(\mathrm{A}^2\right)\|\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A}\|_{b,1}\Big\}\\ &\leqslant \frac{1}{4}\Big\{\operatorname{ber}^2\left(\mathrm{A}^2\right)+\frac{1}{4}\left\|(\mathrm{A}^*\mathrm{A})^2+(\mathrm{A}\mathrm{A}^*)^2\right\|_{b,1}+\frac{1}{2}\operatorname{ber}\left(\mathrm{A}^*\mathrm{A}^2\mathrm{A}^*\right)\\ &\quad+\operatorname{ber}\left(\mathrm{A}^2\right)\|\mathrm{A}\mathrm{A}^*+\mathrm{A}^*\mathrm{A}\|_{b,1}\Big\}.\end{aligned}$$

Taking the supremum over all $\omega \in F$, we get the required inequality. The theorem is proven. □

For the related results for numerical radius and Berezin radius of operators, see, for instance, [1, 3, 4, 5, 6, 7, 10, 12, 14, 16, 17, 18, 19, 20, 21, 28, 29, 31, 33, 39, 40, 41, 42, 43, 44].

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(Received March 4, 2025)

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