Journal of Mathematical Inequalities
Volume 19, Number 3 (2025), 909–919 doi:10.7153/jmi-2025-19-58

NEW THREE PROOFS OF CÎRTOAJE INEQUALITY

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(*Communicated by L. Mihokovi´*)

Abstract. In this paper, we present three new proofs of the following Cîrtoaje inequality: If a and b are nonnegative real numbers such that $a+b=1$, then $a^{2b}+b^{2a}\leqslant 1$.

1. Introduction

The study of inequalities involving power-exponential functions is one of the most interesting areas of analytical research. Many mathematicians [1]–[17] studied inequalities with power-exponential functions and conjectured many open inequalities. Among them, especially, the following symmetric inequality is the one of the simplest shaped form:

THEOREM 1. *If a and b are nonnegative real numbers such that $a+b=1$, then $a^{2b}+b^{2a}\leqslant 1$.*

The inequality is posed by Cîrtoaje [3] as Conjecture 4.8 and proved by himself in [4] and Matejíčka [7]. Although the above symmetric inequality is very simple forms, its proof is not simple and is quite technical. In this paper, we will show three new proofs of Theorem 1. The first proof is a proof by contradiction. The second proof divides the interval of a and b into two and uses upper bounds of a^{2b} and b^{2a} in each interval. This is the method of proof attempted by Hisasue [5]. Hisasue's proof is incorrect because the inequality used in the proof is incorrect. The third proof uses upper bounds of a^{2b} and b^{2a}, but without dividing the range of a and b. A proof using such an approximation formula is already known by Cîrtoaje [4], but this is a new proof in that our approximation formula differs from Cîrtoaje's approximation formula.

Mathematics subject classification (2020): 26D20.
Keywords and phrases: Inequality, power-exponential function, Cîrtoaje inequality.
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2. Some Lemmas

We present some lemmas necessary for the proof of Theorem 1.

LEMMA 1. *For any* $0<a<\frac{1}{2}$, *we have* $(1-a)^{2a-1}<1+a-a^2-2a^3$.

Proof. We set

$$f_1(a)=\ln(1+a-a^2-2a^3)-(2a-1)\ln(1-a)$$

and the derivatives of $f_1(a)$ are

$$f_1'(a)=\frac{-a(2+a-6a^2+4a^3)}{(1-a)(1+a-a^2-2a^3)}-2\ln(1-a)$$

and

$$f_1''(a)=\frac{a^2f_2(a)}{(1-a)^2(1+a-a^2-2a^3)^2},$$

where $f_2(a)=8-30a+5a^2+34a^3-8a^4-8a^5$. Since the derivative of $f_2(a)$ is

$$f_2'(a)=-2(15-5a-51a^2+16a^3+20a^4)<0$$

for $0<a<\frac{1}{2}$, $f_2(a)$ is strictly decreasing for $0<a<\frac{1}{2}$. From $f_2(0)=8$ and $f_2\left(\frac{1}{2}\right)=-\frac{9}{4}$, there exists a unique real number a_0 with $0<a_0<\frac{1}{2}$ such that $f_2(a)>0$ for $0<a<a_0$ and $f_2(a)<0$ for $a_0<a<\frac{1}{2}$. Therefore, $f_1'(a)$ is strictly increasing for $0<a<a_0$ and strictly decreasing for $a_0<a<\frac{1}{2}$. By $f_1'(0)=0$ and $f_1'\left(\frac{1}{2}\right)=-\frac{3}{2}+2\ln 2\cong-0.113706$, there exists a unique real number a_1 with $0<a_1<\frac{1}{2}$ such that $f_1'(a)>0$ for $0<a<a_1$ and $f_1'(a)<0$ for $a_1<a<\frac{1}{2}$. Hence, $f_1(a)$ is strictly increasing for $0<a<a_1$ and strictly decreasing for $a_1<a<\frac{1}{2}$. From $f_1(0)=0$ and $f_1\left(\frac{1}{2}\right)=0$, we have $f_1(a)>0$ for $0<a<\frac{1}{2}$. $\square$

LEMMA 2. *For any* $0<a<\frac{1}{2}$, *we have*

$$(1-a)^{2a}\left(a^2+(1-a)^2-a(1-a)\ln(a(1-a))\right)<(1-a)^2-a(1-a)\ln a.$$

Proof. By Lemma 1, we have

$$\begin{aligned}&(1-a)^{2a}\left(a^2+(1-a)^2-a(1-a)\ln(a(1-a))\right)\\&\quad<(1-a)\left(1+a-a^2-2a^3\right)\left(a^2+(1-a)^2-a(1-a)\ln(a(1-a))\right)\end{aligned}$$

and we may show $g_1(a)=(1-a)-a\ln a-\left(1+a-a^2-2a^3\right)\left(a^2+(1-a)^2-a(1-a)\ln(a(1-a))\right)>0$ for $0<a<\frac{1}{2}$. The derivatives of $g_1(a)$ are

$$g_1'(a)=a-9a^2-8a^3+24a^4-\ln a+(1-6a^2-4a^3+10a^4)\ln(a(1-a))$$

and

$$g_1''(a) = \frac{-a(25-2a-138a^2+116a^3)}{1-a} + (-12a-12a^2+40a^3)\ln(a(1-a))$$
$$= (12a+12a^2-40a^3)g_2(a),$$

where

$$g_2(a) = \frac{-a(25-2a-138a^2+116a^3)}{(1-a)(12a+12a^2-40a^3)} - \ln(a(1-a)).$$

The derivative of $g_2(a)$ is

$$g_2'(a) = \frac{g_3(a)}{2a(1-a)^2(3+3a-10a^2)^2},$$

where $g_3(a) = -18+21a+263a^2-392a^3-376a^4+904a^5-400a^6$. The derivative of $g_3(a)$ is

$$g_3'(a) = 21+526a-1176a^2-1504a^3+4520a^4-2400a^5$$
$$> 21+526a-1176a^2-1504a^3+4520a^4-2400a^4\left(\frac{1}{2}\right)$$
$$= 21+526a-1176a^2-1504a^3+3320a^4 = g_4(a).$$

The derivatives of $g_4(a)$ are $g_4'(a) = 526-2352a-4512a^2+13280a^3$ and $g_4''(a) = -2352-9024a+39840a^2$. Since $g_4''(a)$ is convex upwards with $g_4''(0) = -2352$ and $g_4''\left(\frac{1}{2}\right) = 3096$, there exists a unique real number a_0 with $0 < a_0 < \frac{1}{2}$ such that $g_4''(a) < 0$ for $0 < a < a_0$ and $g_4''(a) > 0$ for $a_0 < a < \frac{1}{2}$. Hence, $g_4'(a)$ is strictly decreasing for $0 < a < a_0$ and strictly increasing for $a_0 < a < \frac{1}{2}$. From $g_4'(0) = 526$ and $g_4'\left(\frac{1}{2}\right) = -118$, there exists a unique real number a_1 with $0 < a_1 < \frac{1}{2}$ such that $g_4'(a) > 0$ for $0 < a < a_1$ and $g_4'(a) < 0$ for $a_1 < a < \frac{1}{2}$. Hence, $g_4(a)$ is strictly increasing for $0 < a < a_1$ and strictly decreasing for $a_1 < a < \frac{1}{2}$. From $g_4(0) = 21$ and $g_4\left(\frac{1}{2}\right) = \frac{19}{2}$, we have $g_4(a) > 0$ for $0 < a < \frac{1}{2}$. Since $g_3(a)$ is strictly increasing for $0 < a < \frac{1}{2}$ and $g_3(0) = -18$ and $g_3\left(\frac{1}{2}\right) = \frac{31}{4}$, there exists a unique real number a_2 with $0 < a_2 < \frac{1}{2}$ such that $g_3(a) < 0$ for $0 < a < a_2$ and $g_3(a) > 0$ for $a_2 < a < \frac{1}{2}$. Hence, $g_2(a)$ is strictly decreasing for $0 < a < a_2$ and strictly increasing for $a_2 < a < \frac{1}{2}$. By $g_2(0) = +\infty$ and $g_2\left(\frac{1}{2}\right) = -1+2\ln 2 \cong 0.386294$ and $g_2\left(\frac{1}{4}\right) = -\frac{283}{150} + \ln\frac{16}{3} \cong -0.21269$, there exists only two real numbers a_3 and a_4 with $0 < a_3 < \frac{1}{4} < a_4 < \frac{1}{2}$ such that $g_2(a) > 0$ for $0 < a < a_3, a_4 < a < \frac{1}{2}$ and $g_2(a) < 0$ for $a_3 < a < a_4$. Hence, $g_1'(a)$ is strictly increasing for $0 < a < a_3, a_4 < a < \frac{1}{2}$ and strictly decreasing for $a_3 < a < a_4$. From $g_1'(0) = 0$ and $g_1'\left(\frac{1}{2}\right) = -\frac{5}{4} + \frac{7\ln 2}{4} \cong -0.0369924$, there exists a unique real number a_5 with $0 < a_5 < \frac{1}{2}$ such that $g_1'(a) > 0$ for $0 < a < a_5$ and $g_1'(a) < 0$ for $a_5 < a < \frac{1}{2}$. Hence, $g_1(a)$ is strictly increasing for $0 < a < a_5$ and strictly decreasing for $a_5 < a < \frac{1}{2}$. By $g_1(0) = g_1\left(\frac{1}{2}\right) = 0$, we obtain $g_1(a) > 0$ for $0 < a < \frac{1}{2}$. □

LEMMA 3. *For any* $-1 \leqslant x \leqslant 1$, *we have*

$$(1+x)^{1-x} \leqslant \frac{1}{4}(1+x)^2(2-x^2)(2-2x+x^2).$$

Lemma 3 is proved by Miyagi and Nishizawa [11].

LEMMA 4. *For any* $0 < x < \frac{1}{2}$, *we have* $2^{\pm x} < 1 \pm x\ln 2 + \frac{271}{1000}x^2$.

Proof. First, we show that $2^x < 1 + x\ln 2 + \frac{271}{1000}x^2$. We set

$$h_1(x) = \ln\left(1 + x\ln 2 + \frac{271}{1000}x^2\right) - x\ln 2$$

and the derivative of $h_1(x)$ is

$$h_1'(x) = \frac{xh_2(x)}{1000 + 1000x\ln 2 + 271x^2},$$

where $h_2(x) = 542 - 1000(\ln 2)^2 - 271x\ln 2$. Since $h_2(x)$ is strictly decreasing for $0 < x < \frac{1}{2}$ and $h_2(0) = 542 - 1000(\ln 2)^2 \cong 61.547$ and $h_2\left(\frac{1}{2}\right) = \frac{1}{2}(1084 - 2000(\ln 2)^2 - 271\ln 2) \cong -32.3745$, there exists a unique real number x_1 with $0 < x_1 < \frac{1}{2}$ such that $h_2(x) > 0$ for $0 < x < x_1$ and $h_2(x) < 0$ for $x_1 < x < \frac{1}{2}$. Thus, $h_1(x)$ is strictly increasing for $0 < x < x_1$ and strictly decreasing for $x_1 < x < \frac{1}{2}$. From $h_1(0) = 0$ and $h_1\left(\frac{1}{2}\right) = \ln\left(\frac{4271}{4000} + \frac{\ln 2}{2}\right) - \frac{1}{2}\ln 2 = \ln\frac{4271 + 2000\ln 2}{4000\sqrt{2}} > \ln\frac{4271 + \frac{2000\cdot 6931}{10000}}{\frac{4000\cdot 141422}{100000}} = \ln\frac{70715}{70711} > 0$, we obtain $h_1(x) > 0$ for $0 < x < \frac{1}{2}$. Next, we show that $2^{-x} < 1 - x\ln 2 + \frac{271}{1000}x^2$. We set

$$h_3(x) = \ln\left(1 - x\ln 2 + \frac{271}{1000}x^2\right) + x\ln 2$$

and the derivative of $h_3(x)$ is

$$h_3'(x) = \frac{x(542 - 1000(\ln 2)^2 + 271x\ln 2)}{1000 - 1000x\ln 2 + 271x^2}.$$

From $542 - 1000(\ln 2)^2 + 271x\ln 2 > 542 - 1000(\ln 2)^2 \cong 61.547$ and $1000 - 1000x\ln 2 + 271x^2 > 1000 - 1000\frac{1}{2}\ln 2 \cong 653.426$, $h_3(x)$ is strictly increasing for $0 < x < \frac{1}{2}$. By $h_3(0) = 0$, we obtain $h_3(x) > 0$ for $0 < x < \frac{1}{2}$. $\square$

LEMMA 5. *For any* $0 < a < \frac{1}{2}$, *we have* $2\left(a - \frac{1}{2}\right) - 2\left(a - \frac{1}{2}\right)^2 - \ln 2 > \ln a$ *and* $-2\left(a - \frac{1}{2}\right) - 2\left(a - \frac{1}{2}\right)^2 - \ln 2 < \ln(1-a)$.

Proof. We set

$$j_1(a) = 2\left(a - \frac{1}{2}\right) - 2\left(a - \frac{1}{2}\right)^2 - \ln 2 - \ln a$$

and the derivative of $j_1(a)$ is $j_1'(a) = \frac{-(1-2a)^2}{a} < 0$ for $0 < a < \frac{1}{2}$. Since $j_1(a)$ is strictly decreasing for $0 < a < \frac{1}{2}$ and $j_1(\frac{1}{2}) = 0$, we obtain $j_1(a) > 0$ for $0 < a < \frac{1}{2}$. We set

$$j_2(a) = -2\left(a - \frac{1}{2}\right) - 2\left(a - \frac{1}{2}\right)^2 - \ln 2 - \ln(1-a)$$

and the derivative of $j_2(a)$ is $j_2'(a) = \frac{(1-2a)^2}{1-a} > 0$ for $0 < a < \frac{1}{2}$. Since $j_2(a)$ is strictly increasing for $0 < a < \frac{1}{2}$ and $j_2(\frac{1}{2}) = 0$, we obtain $j_2(a) < 0$ for $0 < a < \frac{1}{2}$. □

LEMMA 6. *For any* $0 < a < \frac{1}{2}$, *we have*

$$4 - 4a - 4(1-a)(1-2a)\ln 2 + (1-2a)^2(\ln 2)^2 > a^{-2a}.$$

Proof. We set

$$k_1(a) = \ln\left(4 - 4a - 4(1-a)(1-2a)\ln 2 + (1-2a)^2(\ln 2)^2\right) + 2a\ln a$$

and the derivative of $k_1(a)$ is

$$k_1'(a) = 2 + \frac{-4 + 4(3-4a)\ln 2 - 4(1-2a)(\ln 2)^2}{4 - 4a - 4(1-a)(1-2a)\ln 2 + (1-2a)^2(\ln 2)^2} + 2\ln a.$$

By Lemma 5, we have

$$\begin{aligned} k_1'(a) < 2 &+ \frac{-4 + 4(3-4a)\ln 2 - 4(1-2a)(\ln 2)^2}{4 - 4a - 4(1-a)(1-2a)\ln 2 + (1-2a)^2(\ln 2)^2} \\ &+ 2\left(2\left(a - \frac{1}{2}\right) - 2\left(a - \frac{1}{2}\right)^2 - \ln 2\right) \\ &= \frac{-(1-2a)k_2(a)}{k_3(a)}, \end{aligned}$$

where $k_2(a) = 8 - 4a - 4(2-a)(1-2a)\ln 2 - (3 + 8a - 4a^2)(\ln 2)^2 + 2(\ln 2)^3$ and $k_3(a) = 4 - 4a - 4(1-a)(1-2a)\ln 2 + (1-2a)^2(\ln 2)^2$. Since the derivatives of $k_2(a)$ are $k_2'(a) = -4 + 4(1-2a)\ln 2 + 8(2-a)\ln 2 - 8(1-a)(\ln 2)^2$ and $k_2''(a) = -16\ln 2 + 8(\ln 2)^2 \cong -7.24673$, $k_2'(a)$ is strictly decreasing for $0 < a < \frac{1}{2}$. From $k_2'(\frac{1}{2}) = -4 + 12\ln 2 - 4(\ln 2)^2 \cong 2.39595$, $k_2(a)$ is strictly increasing for $0 < a < \frac{1}{2}$. By $k_2(0) = 8 - 8\ln 2 - 3(\ln 2)^2 + 2(\ln 2)^3 \cong 1.67951$, we have $k_2(a) > 0$ for $0 < a < \frac{1}{2}$. Since we have $k_3(a) > 4 - 4a - 4(1-a)(1-2a)\ln 2 = 4(1-a)(1 - \ln 2 + 2a\ln 2) > 4(1-a)(1-\ln 2) \cong 4(1-a) \times 0.306853 > 0$, we obtain $k_1'(a) < 0$ for $0 < a < \frac{1}{2}$ and $k_1(a)$ is strictly decreasing for $0 < a < \frac{1}{2}$. From $k_1(\frac{1}{2}) = 0$, we have $k_1(a) > 0$ for $0 < a < \frac{1}{2}$. □

LEMMA 7. *For any* $0 < a < \frac{1}{2}$, *we have*

$$1 - a^2\left(4 - 4a - 4(1-a)(1-2a)\ln 2 + (1-2a)^2(\ln 2)^2\right) > (1-a)^{2a}.$$

Proof. We set

$$l_1(a) = 1 - a^2\left(4 - 4a - 4(1-a)(1-2a)\ln 2 + (1-2a)^2(\ln 2)^2\right) - (1-a)^{2a}.$$

By Lemma 1 and $\frac{693}{1000} < \ln 2 < \frac{694}{1000}$, we have

$$\begin{aligned} l_1(a) &> 1 - a^2\left(4 - 4a - 4(1-a)(1-2a)\frac{693}{1000} + (1-2a)^2\left(\frac{694}{1000}\right)^2\right) \\ &\quad - (1-a)(1+a-a^2-2a^3) \\ &= \frac{3a^2(1-2a)(24197-67394a)}{250000} \\ &> \frac{3a^2(1-2a)\left(24197-67394\left(\frac{35}{100}\right)\right)}{250000} \\ &= \frac{18273a^2(1-2a)}{250000} > 0 \end{aligned}$$

for $0 < a < \frac{35}{100}$. We set

$$l_2(a) = \ln\left(1 - a^2\left(4 - 4a - 4(1-a)(1-2a)\ln 2 + (1-2a)^2(\ln 2)^2\right)\right) - 2a\ln(1-a)$$

and by Lemma 5, the derivative of $l_2(a)$ is

$$\begin{aligned} l_2'(a) &= \frac{2a}{1-a} + \frac{-8a+12a^2+4a(2-9a+8a^2)\ln 2 - 2a(1-2a)(1-4a)(\ln 2)^2}{1-4a^2+4a^3+4(1-a)(1-2a)a^2\ln 2 - (1-2a)^2a^2(\ln 2)^2} \\ &\quad - 2\ln(1-a) \\ &< \frac{2a}{1-a} + \frac{-8a+12a^2+4a(2-9a+8a^2)\ln 2 - 2a(1-2a)(1-4a)(\ln 2)^2}{1-4a^2+4a^3+4(1-a)(1-2a)a^2\ln 2 - (1-2a)^2a^2(\ln 2)^2} \\ &\quad - 2\left(-2\left(a-\frac{1}{2}\right) - 2\left(a-\frac{1}{2}\right)^2 - \ln 2\right) \\ &= \frac{(1-2a)\,l_3(a)}{(1-a)\left(1-4a^2+4a^3+4(1-a)(1-2a)a^2\ln 2 - (1-2a)^2a^2(\ln 2)^2\right)}, \end{aligned}$$

where

$$\begin{aligned} l_3(a) = &-1 - 7a + 14a^2 - 4a^3 - 12a^4 + 8a^5 \\ &+ 2(1-a)(1+6a-12a^2+6a^3+8a^4-8a^5)\ln 2 \\ &- a(2-19a+29a^2-10a^3-12a^4+8a^5)(\ln 2)^2 \\ &- 2(1-a)a^2(1-2a)(\ln 2)^3. \end{aligned}$$

From $1+6a-12a^2+6a^3+8a^4-8a^5>0$ and $2-19a+29a^2-10a^3-12a^4+8a^5<0$ for $\frac{35}{100}<a<\frac{1}{2}$, we have

$$\begin{aligned}
l_3(a) &< -1-7a+14a^2-4a^3-12a^4+8a^5\\
&\quad +2(1-a)(1+6a-12a^2+6a^3+8a^4-8a^5)\left(\frac{694}{1000}\right)\\
&\quad -a(2-19a+29a^2-10a^3-12a^4+8a^5)\left(\frac{694}{1000}\right)^2\\
&\quad -2(1-a)a^2(1-2a)\left(\frac{693}{1000}\right)^3\\
&= \frac{l_4(a)}{500000000},
\end{aligned}$$

where

$$\begin{aligned}
l_4(a) &= 194000000-511636000a-1249270557a^2+4506715671a^3-2869445114a^4\\
&\quad -4214184000a^5+3625456000a^6.
\end{aligned}$$

$$\begin{aligned}
l_4(a) &< 1000000\left(194-511a-1249a^2+4510a^3-2869a^4-4214a^5+3626a^6\right)\\
&< 1000000\left(194-511a-1249a^2+4510a^3-2869a^4-4214a^5+3626a^5\left(\frac{1}{2}\right)\right)\\
&= 1000000\left(194-511a-1249a^2+4510a^3-2869a^4-2401a^5\right)\\
&= 1000000 l_5(a).
\end{aligned}$$

The derivative of $l_5(a)$ is

$$\begin{aligned}
l_5'(a) &= -511-2498a+13530a^2-11476a^3-12005a^4\\
&< -511-2498a+13530a^2-11476a^2\left(\frac{35}{100}\right)-12005a^4\\
&= -511-2498a+\frac{47567}{5}a^2-12005a^4 = l_6(a)
\end{aligned}$$

The derivative of $l_6(a)$ is

$$\begin{aligned}
l_6'(a) &= -2498+\frac{95134}{5}a-48020a^3\\
&> -2498+\frac{95134}{5}a-48020\left(\frac{1}{2}\right)a^2\\
&= -2498+\frac{95134}{5}a-24010a^2 = l_7(a).
\end{aligned}$$

Since $l_7(a)$ is upward convex and $l_7(a) > \min\left\{l_7(\frac{35}{100}) = \frac{244031}{200}, l_7(\frac{1}{2}) = \frac{10129}{10}\right\}$, we have $l_7(a) > 0$ for $\frac{35}{100} < a < \frac{1}{2}$ and $l_6(a)$ is strictly increasing for $\frac{35}{100} < a < \frac{1}{2}$. From $l_6(\frac{1}{2}) = -\frac{10557}{80}$, $l_6(a) < 0$ for $\frac{35}{100} < a < \frac{1}{2}$ and $l_5(a)$ is strictly decreasing for $\frac{35}{100} < a < \frac{1}{2}$. By $l_5(\frac{35}{100}) = -\frac{478987}{3200000} < 0$, we have $l_i(a) < 0\,(i = 3,4,5)$ for $\frac{35}{100} < a < \frac{1}{2}$. From $1 - 4a^2 + 4a^3 + 4(1-a)(1-2a)a^2\ln 2 - (1-2a)^2a^2(\ln 2)^2 > 0$ for $0 < a < \frac{1}{2}$, $l_2'(a) < 0$ for $\frac{35}{100} < a < \frac{1}{2}$. Since $l_2(a)$ is strictly decreasing for $\frac{35}{100} < a < \frac{1}{2}$ and $l_2(\frac{1}{2}) = 0$, we obtain $l_2(a) > 0$ for $\frac{35}{100} < a < \frac{1}{2}$. $\square$

3. Proofs of Theorem 1

We may assume $0 \leqslant a \leqslant \frac{1}{2} \leqslant b \leqslant 1$. If $(a,b) = (0,1), \left(\frac{1}{2}, \frac{1}{2}\right)$, then we have $a^{2b} + b^{2a} = 1$. Therefore, we consider the case of $0 < a < \frac{1}{2} < b < 1$ and we set $F(a) = a^{2(1-a)} + (1-a)^{2a} - 1$.

First proof of Theorem 1. The derivative of $F(a)$ is

$$F'(a) = a^{2(1-a)}\left(\frac{2(1-a)}{a} - 2\ln a\right) + (1-a)^{2a}\left(2\ln(1-a) - \frac{2a}{1-a}\right).$$

Since we have $F(0) = F\left(\frac{1}{2}\right) = 0$, if $F(a) > 0$ for some a with $0 < a < \frac{1}{2}$, then there exists at least one real number c with $0 < c < \frac{1}{2}$ such that $F'(c) = 0$ and $F(c) > 0$. Hence, we have

$$c^{2(1-c)}\left(\frac{1-c}{c} - \ln c\right) = (1-c)^{2c}\left(\frac{c}{1-c} - \ln(1-c)\right)$$

and

$$c^{2(1-c)} + (1-c)^{2c} - 1 > 0.$$

We multiply the above inequality by $\frac{1-c}{c} - \ln c > 0$ to get

$$c^{2(1-c)}\left(\frac{1-c}{c} - \ln c\right) + (1-c)^{2c}\left(\frac{1-c}{c} - \ln c\right) - \left(\frac{1-c}{c} - \ln c\right) > 0.$$

From

$$c^{2(1-c)}\left(\frac{1-c}{c} - \ln c\right) = (1-c)^{2c}\left(\frac{c}{1-c} - \ln(1-c)\right),$$

we have

$$(1-c)^{2c}\left(\frac{c}{1-c} - \ln(1-c)\right) + (1-c)^{2c}\left(\frac{1-c}{c} - \ln c\right) > \frac{1-c}{c} - \ln c,$$

$$(1-c)^{2c}\left(\frac{c^2 + (1-c)^2}{c(1-c)} - \ln(c(1-c))\right) > \frac{1-c}{c} - \ln c.$$

Thus, we obtain

$$(1-c)^{2c}\Big(c^2+(1-c)^2-c(1-c)\ln(c(1-c))\Big) > (1-c)^2-c(1-c)\ln c$$

for some c with $0<c<\frac{1}{2}$. Also, by Lemma 2, the above inequality is not holds for any $0<c<\frac{1}{2}$. From the contradiction, we can get $F(a)<0$ for $0<a<\frac{1}{2}$ and this completes the proof of Theorem 1. □

Second proof of Theorem 1. We consider the two case (i) $0<a<\frac{1}{4}$ and $\frac{3}{4}<b<1$, (ii) $\frac{1}{4}\leqslant a<\frac{1}{2}<b\leqslant\frac{3}{4}$. First, we consider the case (i). By Bernoulli inequality, we have $(1-a)^{2a}<1-2a^2$ for $0<a<\frac{1}{4}$. Hence, we can get $F(a)<a^{2(1-a)}-2a^2=a^2(a^{-2a}-2)$. Since a^{-2a} is strictly increasing for $0<a<\frac{1}{4}$ and $a^{-2a}<\left(\frac{1}{4}\right)^{-2(1/4)}=2$, we have $a^{-2a}-2\leqslant 0$ for $0<a\leqslant\frac{1}{4}$ and therefore, we obtain $F(a)<0$ for $0<a<\frac{1}{4}$. Next, we consider the case (ii). We set $a=\frac{1-x}{2}$ and $b=\frac{1+x}{2}$, then we can get $0<x\leqslant\frac{1}{2}$ and the following inequality by Lemmas 3 and 4.

$$\begin{aligned}
F(a) &= \left(\frac{1-x}{2}\right)^{1+x}+\left(\frac{1+x}{2}\right)^{1-x}-1\\
&=\frac{1}{2}\left((1-x)^{1+x}2^{-x}+(1+x)^{1-x}2^{x}\right)-1\\
&\leqslant\frac{1}{2}\left(\frac{1}{4}(1-x)^2(2-x^2)(2+2x+x^2)\left(1-x\ln 2+\frac{271}{1000}x^2\right)\right.\\
&\qquad\left.+\frac{1}{4}(1+x)^2(2-x^2)(2-2x+x^2)\left(1+x\ln 2+\frac{271}{1000}x^2\right)\right)-1\\
&=\frac{\left(2-x^2\right)\left(2000-458x^2+729x^4+271x^6+2000x^2\ln 2\right)-4000}{4000}\\
&=G(x)\,.
\end{aligned}$$

The derivative of $G(x)$ is $G'(x)=\frac{x}{2000}H(x)$, where $H(x)=-2916+3832x^2-561x^4-1084x^6+4000\ln 2-4000x^2\ln 2$. The derivative of $H(x)$ is

$$\begin{aligned}
H'(x) &= 4x(1916-561x^2-1626x^4-2000\ln 2)\\
&\geqslant 4x\left(1916-561\left(\frac{1}{2}\right)^2-1626\left(\frac{1}{2}\right)^4-2000\ln 2\right)\\
&=4x\left(\frac{13393}{8}-2000\ln 2\right)\\
&\cong 4x\times 287.831>0\,.
\end{aligned}$$

Since $H(x)$ is strictly increasing for $0<x<\frac{1}{2}$ and $H(0)=-2916+4000\ln 2\cong -143.411$ and $H\left(\frac{1}{2}\right)=-2010+3000\ln 2\cong 69.4415$, there exists a unique real number x_1 with $0<x_1<\frac{1}{2}$ such that $H(x)<0$ for $0<x<x_1$ and $H(x)>0$ for $x_1<x<\frac{1}{2}$.

Therefore, $G(x)$ is strictly decreasing for $0 < x < x_1$ and strictly increasing for $x_1 < x < \frac{1}{2}$. From $G(0) = 0$ and $G\left(\frac{1}{2}\right) = \frac{-156987+224000\ln 2}{1024000} < \frac{-156987+224000\frac{694}{1000}}{1024000} = \frac{-1531}{1024000}$, we obtain $G(x) < 0$ for $0 < x \leqslant \frac{1}{2}$ and $F(a) < 0$ for $\frac{1}{4} \leqslant a < \frac{1}{2}$. Thus, this completes the proof of Theorem 1. □

Third proof of Theorem 1. From Lemmas 6 and 7, we have $F(a) < 0$ for $0 < a < \frac{1}{2}$, immediately. □

Acknowledgements. The authors would like to thank the referees for his/her comments that helped us improve this article.

REFERENCES

[1] A. CORONEL AND F. HUANCAS, *The proof of three power-exponential inequalities*, J. Inequal. Appl. 2014, **509** (2014).

[2] A. CORONEL AND F. HUANCAS, *On the inequality* $a^{2a} + b^{2b} + c^{2c} \geqslant a^{2b} + b^{2c} + c^{2a}$, Aust. J. Math. Anal. Appl. **9**, no. 1, Art. 3 (2012).

[3] V. CÎRTOAJE, *On some inequalities with power-exponential functions*, J. Inequal. Pure Appl. Math. **10**, no. 1, Art. 21 (2009).

[4] V. CÎRTOAJE, *Proofs of three open inequalities with power-exponential functions*, J. Nonlinear Sci. Appl. **4**, no. 2 (2011), 130–137.

[5] M. HISASUE, *Solution of Inequalities with Power-Exponential Functions by Cîrtoaje*, Aust. J. Math. Anal. Appl. **9**, issue 2, Art. 4, (2012).

[6] L. MATEJÍČKA, *On an open problem posed in the paper "Inequalities of power-exponential functions"*, J. Inequal. Pure Appl. Math. **9**, no. 3, Art. 75 (2008).

[7] L. MATEJÍČKA, *Solution of one conjecture on inequalities with power-exponential functions*, J. Inequal. Pure Appl. Math. textbf10, no. 3, Art. 72 (2009).

[8] L. MATEJÍČKA, *Proof of one open inequality*, J. Nonlinear Sci. Appl. **7**, no. 1, (2014), 51–62.

[9] L. MATEJÍČKA, *On the Cîrtoaje's conjecture*, J. Inequal. Appl. 2016, **152** (2016).

[10] M. MIYAGI AND Y. NISHIZAWA, *Proof of an open inequality with double power-exponential functions*, J. Inequal. Appl. 2013, **468** (2013).

[11] M. MIYAGI AND Y. NISHIZAWA, *A short proof of an open inequality with power-exponential functions*, Aust. J. Math. Anal. Appl. **11**, no. 1, Art. 6 (2014).

[12] M. MIYAGI AND Y. NISHIZAWA, *Extension of an inequality with power exponential functions*, Tamkang J. Math. **46** (2015), 427–433.

[13] M. MIYAGI AND Y. NISHIZAWA, *A stronger inequality of Cirtoaje's one with power exponential functions*, J. Nonlinear Sci. Appl. **8** (2015), 224–230.

[14] Y. NISHIZAWA, *Symmetric inequalities with power-exponential functions*, Indian J. Pure Appl. Math. **48**, no. 3 (2017), 335–344.

[15] Y. NISHIZAWA, *A lower bound of the power exponential function*, J. Classical. Anal. **15**, no. 1 (2019), 1–6.

[16] Y. NISHIZAWA, *The best possible constants of the inequalities with power exponential functions*, Indian J. Pure Appl. Math. **51**, no. 4 (2020), 1761–1768.

[17] F. QI AND L. DEBNATH, *Inequalities for power-exponential functions*, J. Inequal. Pure Appl. Math. **1**, no. 2, Art. 15 (2000).

(Received March 27, 2025)

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