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COMPLETE CONVERGENCE FOR THE MAXIMUM OF WEIGHTED SUMS OF m–WIDELY ORTHANT DEPENDENT RANDOM VARIABLES SEQUENCES

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Abstract. In this paper, we investigate complete convergence for the maximum of weighted sums of m-widely orthant dependent (m-WOD) random variables sequences under general conditions. The m-WOD random variable sequences represent a broad class of dependency structures, so our results extend and improve the corresponding ones in the literature.

1. Introduction

Independent random variables are often impractical in many probabilistic and statistical models. To address this, scholars have introduced various types of dependent random variables. For example, negatively associated (NA, for short) random variables, negatively orthant dependent (NOD, for short) random variables, and extend negatively dependent (END, for short) random variables and so on. Among these, widely orthant dependent (WOD, for short) random variables represent one of the most general forms of dependence. They were first introduced by Wang et al. [12], defined as follows:

DEFINITION 1.1. The random variables $\{X_n, n \geqslant 1\}$ are called to be widely upper orthant dependent (WUOD, for short) random variables, if there exists a finite sequence of real numbers $\{g_U(n), n \geqslant 1\}$ satisfying for each $n \geqslant 1$, $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant g_U(n) \prod_{i=1}^{n} P(X_i > x_i). \tag{1.1}$$

The random variables $\{X_n, n \geqslant 1\}$ are called to be widely lower orthant dependent (WLOD, for short) random variables, if there exists a finite sequence of real numbers $\{g_L(n), n \geqslant 1\}$ satisfying for each $n \geqslant 1$, $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant g_L(n) \prod_{i=1}^{n} P(X_i \leqslant x_i). \tag{1.2}$$

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The $\{X_n, n \geqslant 1\}$ are called to be WOD random variables, if $\{X_n, n \geqslant 1\}$ are both WUOD and WLOD random variables. Let $g(n) = \max\{g_U(n), g_L(n)\}$, $n \geqslant 1$ be called dominated coefficients.

When $g_U(n) = g_L(n) = 1$, $\{X_n, n \geqslant 1\}$ are NOD random variables, which were introduced by Ebrahimi and Ghosh [3]; when $g_U(n) = g_L(n) = M \geqslant 1$, $\{X_n, n \geqslant 1\}$ are END random variables which were introduced by Liu [7]. END random variables include NA and positive random variables. Therefore, WOD random variables represent a broad structure of dependent random variables.

Since the concept of WOD random variables was introduced, many scholars have devoted efforts to studying their limit properties and applications, achieving significant results. For example, Wang *et al.* [13] obtained the precise large deviations; Qiu *et al.* [11] established the complete convergence and moment complete convergence of the weighted sums; Liu *et al.* [8] derived the moment complete convergence; Wang *et al.* [15] and Chen *et al.* [1] studied the asymptotics of ruin probabilities in renewal risk models based on WOD sequences; Shen [9] proved the Bernstein-type probability inequality; Wang *et al.* [14] investigated complete convergence and its applications in nonparametric regression models; Ding *et al.* [2] provided results on the complete convergence of weighted sums, Song *et al.* [10] analyzed the convergence of moving average processes generated by WOD random variables, and so on.

Inspired by m-END and WOD dependence structures, Fang *et al.* [4] introduced the concept of m-WOD random variables, defined as follows:

DEFINITION 1.2. For fix integer $m \geqslant 1$, the random variables $\{X_n, n \geqslant 1\}$ is called to be m-WOD if for any $n \geqslant 2$, $i_1, i_2, \dots, i_n \in \mathbb{N}^+$, such that $|i_k - i_j| \geqslant m$, for all $1 \leqslant k \neq j \leqslant n$,we get the $X_{i_1}, X_{i_2}, \dots, X_{i_n}$ are WOD random variables.

From the definition, we see that m-WOD random variables represent a broader class of dependence than WOD random variables. Therefore, investigating the complete convergence of m-WOD random variables is very interesting.

Complete convergence plays a fundamental role in probability theory and mathematical statistics. This concept was first introduced by Hsu and Robbins [5].

A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to converge completely to a constant θ, if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty, \quad \forall \varepsilon > 0.$$

In view of the Borel-Cantelli lemma, the complete convergence implies that $X_n \to \theta$ almost surely. Therefore, complete convergence is a very important tool in establishing almost sure convergence for sequences of random variables, as well as for weighted sums of random variables.

Recently, Wu [17] obtained the complete convergence for the maximum of weighted sums of END random variables.

THEOREM A. *Let $\{X_n, n \geqslant 1\}$ be a sequence of END and identically distributed random variables with $EX_1 = 0$, and let $\{a_{nj}, 1 \leqslant j \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying $\sum_{j=1}^{n} a_{nj}^2 = O(n^{-\alpha})$ and $\max_{1 \leqslant j \leqslant n} |a_{nj}| = O(n^{-\alpha})$. For some $p \geqslant 2$,*

$1/p \leqslant \alpha < 1$, *if* $E|X_1|^p < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}X_j| > \varepsilon) < \infty, \quad \forall \varepsilon > 0.$$

In this paper, we aims to investigate the complete convergence for the maximum of weighted sums of m-WOD random variables, which extend and improve Theorem A.

DEFINITION 1.3. The real valued function l, positive and measurable on $(0,\infty)$ is said to be slowly varying at infinity if for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1.$$

DEFINITION 1.4. The random variables $\{X_n, n \geqslant 1\}$ are called be stochastically dominated by a random variable X, if for any $x > 0$,

$$P(|X_n| > x) \leqslant CP(|X| > x), \ \ n \geqslant 1,$$

where the constant $C > 0$.

In this paper, $I(A)$ denotes the indicator function of an event A, the symbol C represents a positive constant, which can take different values in different places, even in the same formula. Let $\log n = \ln\max\{x,e\}$, $X^+ = XI\ (X > 0)$, $g(n) = \max\{g_U(n), g_L(n)\}$.

2. Some Lemmas and main results

LEMMA 2.1. (Fang [4]) *The sequence* $\{X_n, n \geqslant 1\}$ *are* m*-WOD random variables, if the function sequences* $\{f_n, n \geqslant 1\}$ *are non-decreasing (non-increasing), then random variables* $\{f_n(X_n), n \geqslant 1\}$ *are also* m*-WOD random variables with the same dominating coefficients.*

LEMMA 2.2. (Fang [4]) *The sequence* $\{X_n, n \geqslant 1\}$ *are* m*-WOD random variables with dominating coefficients* $g(n)$*. For every* $j \geqslant 1$*, the* $EX_j = 0$ *and* $E|X_j|^p < \infty$*. Then, there exist positive constants* $C_1 = C(p,m)$*,* $C_2 = C(p,m)$ *depending only on* p *and* m*, such that*

$$E(|\sum_{j=1}^{n} X_j|^p) \leqslant [C_1(p,m) + C_2(p,m)g(n)] \sum_{j=1}^{n} E|X_j|^p, \quad 1 < p \leqslant 2,$$

$$E(|\sum_{j=1}^{n} X_j|^p) \leqslant C_1(p,m) \sum_{j=1}^{n} E|X_j|^p + C_2(p,m)g(n)(\sum_{j=1}^{n} E|X_j|^2)^{p/2}, \quad p > 2.$$

LEMMA 2.3. (Fang [4]) *The sequence* $\{X_n, n \geqslant 1\}$ *are* m*-WOD random variables with dominating coefficients* $g(n)$*. For every* $j \geqslant 1$*, the* $EX_j = 0$ *and* $E|X_j|^p < \infty$*.*

Then, there exist positive constants $C_1 = C(p,m)$, $C_2 = C(p,m)$ depending only on p and m, such that

$$E(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}X_j|^p) \leqslant [C_1(p,m)+C_2(p,m)g(n)]\log^p n\sum_{j=1}^{n}E|X_j|^p, \quad 1<p\leqslant 2,$$

$$E(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}X_i|^p) \leqslant C_1(p,m)\log^p n\sum_{j=1}^{n}E|X_j|^p + C_2(p,m)g(n)\log^p n(\sum_{j=1}^{n}E|X_j|^2)^{p/2},$$
$$p>2.$$

LEMMA 2.4. (Wu [16]) *Let $\{X_n, n\geqslant 1\}$ be stochastically dominated by X, $a>0$, $b>0$ are constant, then there exist positive constant C_1, C_2 such that following inequalities are established:*

$$E|X_n|^a I(|X_n|\leqslant b) \leqslant C_1[E|X|^a I(|X|\leqslant b)+b^a P(|X|>b)],$$
$$E|X_n|^a I(|X_n|>b) \leqslant C_2 E|X|^a I(|X|>b).$$

LEMMA 2.5. (Zhou [18]) *If l is slowly varying at infinity, then for positive integer n, we have*

(1) $\sum\limits_{k=1}^{n} k^s l(k) \leqslant Cn^{s+1}l(n)$, *for* $s>-1$;

(2) $\sum\limits_{k=n}^{\infty} k^s l(k) \leqslant Cn^{s+1}l(n)$, *for* $s<-1$.

Now, we present the main results, the proofs for them will be postponed in the next section.

THEOREM 2.1. *Let $\{X_n, n\geqslant 1\}$ be a sequence of m-WOD random variables stochastically dominated by a random variable X with dominating coefficients $g(n) = O(n^\delta)$, $\delta\geqslant 0$, $n\geqslant 1$. $l(n)$ is a slowly varying function. Let $\{a_{nj}, 1\leqslant j\leqslant n, n\geqslant 1\}$ be an array of real numbers satisfying $\sum\limits_{j=1}^{n} a_{nj}^2 = O(n^{-\alpha})$ and $\max\limits_{1\leqslant j\leqslant n}|a_{nj}| = O(n^{-\alpha})$. For some $p>1$, $1/p\leqslant\alpha<1$, if $E|X|^p l(X^{\frac{1}{\alpha}})<\infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)P(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}a_{nj}(X_j-EX_j)|>\varepsilon)<\infty, \quad \forall\varepsilon>0. \tag{2.1}$$

THEOREM 2.2. *Let $\{X_n, n\geqslant 1\}$ be a sequence of m-WOD random variables stochastically dominated by a random variable X with dominating coefficients $g(n) = O(n^\delta)$, $\delta\geqslant 0$, $n\geqslant 1$. $l(n)$ is a slowly varying function. Let $\{a_{nj}, 1\leqslant j\leqslant n, n\geqslant 1\}$ be an array of real numbers satisfying $\sum\limits_{j=1}^{n} a_{nj}^2 = O(n^{-\alpha})$ and $E|X|^p l(X^{\frac{1}{\alpha}})<\infty$. Assume that one of the following conditions holds:*

(A_1) *let* $p>2$ *and* $\frac{1}{p}\leqslant\alpha<\frac{2}{p}$,

(A_2) *let* $1\leqslant p\leqslant 2$ *and* $0<\alpha<1$, $0<\delta<1-\alpha$.

Then (2.1) *holds.*

THEOREM 2.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-WOD random variables stochastically dominated by a random variable X with dominating coefficients $g(n) = O(n^\delta)$, $\delta \geqslant 0$, $n \geqslant 1$. $l(n)$ is a slowly varying function. Let $\{a_{nj}, 1 \leqslant j \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying $\max_{1\leqslant j\leqslant n} |a_{nj}| = O(n^{-\alpha})$. For some $p > 1$, $\alpha \geqslant 1$, if $E|X|^p l(X^{\frac{1}{\alpha}}) < \infty$, then (2.1) holds.*

Taking $\alpha p = 1$, $l(n) = 1$ in Theorem 2.1, we have

COROLLARY 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-WOD random variables stochastically dominated by a random variable X with dominating coefficients $g(n) = O(n^\delta)$, $\delta \geqslant 0$, $n \geqslant 1$. Let $\{a_{nj}, 1 \leqslant j \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying $\sum_{j=1}^n a_{nj}^2 = O(n^{-\alpha})$ and $\max_{1\leqslant j\leqslant n} |a_{nj}| = O(n^{-\alpha})$, $0 < \alpha < 1$. If $E|X|^{\frac{1}{\alpha}} < \infty$, then*

$$\sum_{j=1}^n a_{nj}(X_j - EX_j) \to 0, \quad a.s., \text{ as } n \to \infty.$$

Taking $\alpha p = 2$, $l(n) = 1$ in Theorem 2.1 and combining with the Borel-Cantelli lemma, we have.

COROLLARY 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-WOD random variables stochastically dominated by a random variable X with dominating coefficients $g(n) = O(n^\delta)$, $\delta \geqslant 0$, $n \geqslant 1$. $l(n)$ is a slowly varying function. Let $\{a_{nj}, 1 \leqslant j \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying $\sum_{j=1}^n a_{nj}^2 = O(n^{-\alpha})$ and $\max_{1\leqslant j\leqslant n} |a_{nj}| = O(n^{-\alpha})$, $0 < \alpha < 1$. If $E|X|^{\frac{2}{\alpha}} < \infty$, then*

$$\sum_{n=1}^\infty P\Big(\max_{1\leqslant k\leqslant n} \Big|\sum_{j=1}^k a_{nj}(X_j - EX_j)\Big| > \varepsilon\Big) < \infty, \quad \forall \varepsilon > 0.$$

and

$$\sum_{j=1}^n a_{nj}(X_j - EX_j) \to 0, \quad a.s., \text{ as } n \to \infty.$$

REMARK 2.1. The class of m-WOD random variables encompass WOD, m-NA, m-NOD, m-END, among others. Thus, the results presented in this paper extend and improve upon existing results.

REMARK 2.2. Since stochastic domination is a weaker condition than identical distribution, the results in this paper also hold under the condition of identical distribution.

REMARK 2.3. Taking $g(n) = M$, $l(n) = 1$, $p > 2$ in Theorems 2.1, we obtain the result of Theorem A in Wu [17]. Compared with Theorem 2.1 of the paper, we remove the condition $\max_{1\leqslant j\leqslant n} |a_{nj}| = O(n^{-\alpha})$ in Theorems 2.2, and eliminate the condition $\sum_{j=1}^n a_{nj}^2 = O(n^{-\alpha})$ in Theorems 2.3. Therefore, our results improve Theorem A.

REMARK 2.4. By replacing the real numbers $\{a_{nj}, 1 \leqslant j \leqslant n, n \geqslant 1\}$ with a random sequence, this paper's conclusions remain hold.

3. Proof of Theorems

Proof of Theorem 2.1. For a fixed $n \geqslant 1$, for $1 \leqslant j \leqslant n$, denote

$$Y_{nj} = -n^{\alpha} I(X_j < -n^{\alpha}) + X_j I(|X_j| \leqslant n^{\alpha}) + n^{\alpha} I(X_j > n^{\alpha}),$$
$$Z_{nj} = X_j - Y_{nj} = (X_j + n^{\alpha}) I(X_j < -n^{\alpha}) + (X_j - n^{\alpha}) I(X_j > n^{\alpha}).$$

Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(X_j - EX_j)| > \varepsilon) \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(Y_{nj} - EY_{nj})| > \varepsilon) + \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\bigcup_{j=1}^{n} \{|X_j| > n^{\alpha}\}) \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(Y_{nj} - EY_{nj})| > \varepsilon) + \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \sum_{j=1}^{n} P(|X_j| > n^{\alpha}) \\
&=: I_1 + I_2.
\end{aligned}$$

Thus, to prove (2.1), we only need to show that $I_1 < \infty$ and $I_2 < \infty$.

By Lemmas 2.4–2.5 and the condition $\alpha p - 1 > -1$, we have

$$\begin{aligned}
I_2 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|X| > n^{\alpha}) \\
&= C \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) \sum_{k=n}^{\infty} P(k^{\alpha} < |X| \leqslant (k+1)^{\alpha}) \\
&= C \sum_{k=1}^{\infty} P(k^{\alpha} < |X| \leqslant (k+1)^{\alpha}) \sum_{n=1}^{k} n^{\alpha p-1} l(n) \\
&= C \sum_{k=1}^{\infty} k^{\alpha p} l(k) P(k^{\alpha} < |X| \leqslant (k+1)^{\alpha}) \leqslant CE|X|^p l(|X|^{\frac{1}{\alpha}}) < \infty. \qquad (3.1)
\end{aligned}$$

By Lemma 2.1, for each $n \geqslant 1, \{a_{nj}(Y_{nj} - EY_{nj})\}$ are also m-WOD random variables with the same dominating coefficients.

For I_1, by Lemma 2.3 and Markov's inequality, we have that for any $v > \max\{2, p\}$,

$$\begin{aligned}
I_1 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) E\{\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(Y_{nj} - EY_{nj})|^{v}\} \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) (\log n)^{v} \{\sum_{j=1}^{n} E|a_{nj} Y_{nj}|^{v} + g(n)(\sum_{j=1}^{n} E|a_{nj} Y_{nj}|^2)^{\frac{v}{2}}\} \\
&=: I_{11} + I_{12}. \qquad (3.2)
\end{aligned}$$

For I_{11}, combining Lemma 2.4 with the conditions $\sum_{j=1}^{n} a_{nj}^2 = O(n^{-\alpha})$ and $\max_{1\leqslant j\leqslant n} |a_{nj}| = O(n^{-\alpha})$, where $\alpha < 1$, we obtain

$$
\begin{aligned}
I_{11} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)(\log n)^{v} \sum_{j=1}^{n} |a_{nj}|^{v}[E|X_j|^{v} I(|X_j| \leqslant n^{\alpha}) + n^{\alpha v} P(|X_j| > n^{\alpha})] \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)(\log n)^{v} [E|X|^{v} I(|X| \leqslant n^{\alpha}) + n^{\alpha v} P(|X| > n^{\alpha})] \sum_{j=1}^{n} |a_{nj}|^{v} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)(\log n)^{v} [E|X|^{v} I(|X| \leqslant n^{\alpha}) + n^{\alpha v} P(|X| > n^{\alpha})] \\
&\quad \times (\max_{1\leqslant j\leqslant n} |a_{nj}|)^{v-2} \sum_{j=1}^{n} |a_{nj}|^{2} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-av+a} l(n)(\log n)^{v} [E|X|^{p} n^{\alpha(v-p)} I(X \leqslant n^{\alpha}) \\
&\quad + E|X|^{p} n^{\alpha(v-p)} I(|X| > n^{\alpha})] \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-(1-a)} l(n)(\log n)^{v} < \infty. \qquad (3.3)
\end{aligned}
$$

For I_{12}, we have

$$
\begin{aligned}
I_{12} &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n)(\log n)^{v} g(n) \{\sum_{j=1}^{n} |a_{nj}|^{2} [E|X_j|^{2} I(|X_j| \leqslant n^{\alpha}) + n^{2\alpha} P(|X_j| > n^{\alpha})]\}^{\frac{v}{2}} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\frac{av}{2}+\delta} l(n)(\log n)^{v} [EX^{2} I(|X| \leqslant n^{\alpha}) + n^{2\alpha} P(|X| > n^{\alpha})]^{\frac{v}{2}}. \qquad (3.4)
\end{aligned}
$$

To prove $I_{12} < \infty$, we consider the following two cases:

Case 1: When $p \geqslant 2$. From the condition $E|X|^{p} l(X^{\frac{1}{\alpha}}) < \infty$, we obtain $E|X|^{2} < \infty$, taking $v > \max\{p, \frac{2(\alpha p-1+\delta)}{\alpha}\}$, then

$$
I_{12} \leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\frac{av}{2}+\delta} l(n)(\log n)^{v} (E|X|^{2})^{\frac{v}{2}} < \infty. \qquad (3.5)
$$

Case 2: When $1 < p < 2$. From the condition $E|X|^{p} l(X^{\frac{1}{\alpha}}) < \infty$, taking $v > \max\{2, \frac{2(\alpha p-1+\delta)}{\alpha(p-1)}\}$, we have

$$
\begin{aligned}
I_{12} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\frac{av}{2}+\delta} l(n)(\log n)^{v} n^{\frac{va(2-p)}{2}} (E|X|^{p})^{\frac{v}{2}} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\delta+\frac{av}{2}-\frac{apv}{2}} l(n)(\log n)^{v} (E|X|^{p})^{\frac{v}{2}} < \infty. \qquad (3.6)
\end{aligned}
$$

From (3.1)–(3.6), the proof Theorem 2.1 is completed. □

Proof of Theorem 2.2. *Case* 1: When condition (A_1) holds, following the proof of Theorem 2.1, it suffices to prove $I_{11} < \infty$.

Since $p > 2$ and $\frac{1}{p} \leqslant \alpha < \frac{2}{p}$, we take $p < v < \frac{2}{a}$. By the Jensen's inequality for any $v > 2$, we get $\sum_{j=1}^{n} |a_{nj}|^v \leqslant (\sum_{j=1}^{n} a_{nj}^2)^{\frac{v}{2}} \leqslant n^{\frac{-\alpha v}{2}}$.

$$\begin{aligned} I_{11} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)(\log n)^v [E|X|^v I(|X| \leqslant n^\alpha) + n^{\alpha v} P(|X| > n^\alpha)] \sum_{j=1}^{n} |a_{nj}|^v \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)(\log n)^v [E|X|^v I(|X| \leqslant n^\alpha) + n^{\alpha v} P(|X| > n^\alpha)] (\sum_{j=1}^{n} |a_{nj}|^2)^{\frac{v}{2}} \\ &\leqslant C\sum_{n=1}^{\infty} n^{-1-\frac{2-av}{2}} l(n)(\log n)^v < \infty. \end{aligned} \tag{3.7}$$

Case 2: When condition (A_2) holds, following the proof of Theorem 2.1, it suffices to prove $I_1 < \infty$. By Lemma 2.3 and Markov's inequalities, taking $\max\{\frac{2\delta}{1-\alpha}, p\} < v < 2$, when $1 \leqslant p < 2$ and $v = 2$, when $p = 2$.

$$\begin{aligned} I_1 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) E\{\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(Y_{nj} - EY_{nj})|^v\} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)(\log n)^v (1+g(n)) \sum_{j=1}^{n} E|a_{nj}Y_{nj}|^v \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\delta} l(n)(\log n)^v [E|X|^v I(|X| \leqslant n^\alpha) + n^{\alpha v} P(|X| > n^\alpha)] \sum_{j=1}^{n} |a_{nj}|^v. \end{aligned} \tag{3.8}$$

By Hölder's inequality, we have

$$\sum_{j=1}^{n} |a_{nj}|^v \leqslant (\sum_{j=1}^{n} (|a_{nj}|^v)^{\frac{2}{v}})^{\frac{v}{2}} (\sum_{j=1}^{n} 1)^{1-\frac{v}{2}} \leqslant n^{\frac{-av-v}{2}+1}.$$

For parameters satisfying $1 \leqslant p \leqslant 2$ and $0 < \alpha < 1-\delta$, we have

$$\begin{aligned} I_1 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1+\delta+\frac{-av-v}{2}} l(n)(\log n)^v [E|X|^v I(|X| \leqslant n^\alpha) + n^{\alpha v} P(|X| > n^\alpha)] \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1+\delta+\frac{-av-v}{2}+av-ap} l(n)(\log n)^v \\ &\leqslant C\sum_{n=1}^{\infty} n^{-1+\delta+\frac{av-v}{2}} l(n)(\log n)^v < \infty. \end{aligned} \tag{3.9}$$

From (3.7)–(3.9), the proof Theorem 2.2 is completed. □

Proof of Theorem 2.3. Since $E|X|^p l(X^{\frac{1}{\alpha}}) < \infty, p > 1$, it follows that $E|X| < \infty$. Consequently, there exists a positive integer N such that $E|X|I(|X| > N) < \frac{\varepsilon}{20}$. For $n^\alpha > N, j \geqslant 1$, let

$$\begin{aligned}
X_j^{(n,1)} &= -NI(X_j < -N) + X_j I(|X_j| \leqslant N) + NI(X_j > N),\\
X_j^{(n,2)} &= (X_j - N)I(N < X_j \leqslant n^\alpha) + (n^\alpha - N)I(X_j > n^\alpha),\\
X_j^{(n,3)} &= (X_j + N)I(-n^\alpha < X_j \leqslant -N) + (-n^\alpha + N)I(X_j < -n^\alpha),\\
X_j^{(n,4)} &= (X_j - n^\alpha)I(X_j > n^\alpha),\\
X_j^{(n,5)} &= (X_j + n^\alpha)I(X_j < -n^\alpha).
\end{aligned}$$

then $X_j = \sum_{l=1}^{5} X_j^{(n,l)}$. By Lemma 2.4 and $\max_{1\leqslant j\leqslant n} |a_{nj}| = O(n^{-\alpha})$, $\alpha \geqslant 1$, we have

$$\begin{aligned}
&\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} E a_{nj} X_j^{(n,2)}| \leqslant \sum_{j=1}^{n} |a_{nj}| E|X_j^{(n,2)}|\\
&\leqslant \sum_{j=1}^{n} |a_{nj}| E|X_j| I(|X_j| > N)\\
&\leqslant \max_{1\leqslant j\leqslant n} |a_{nj}| \sum_{j=1}^{n} E|X|I(|X| > N)\\
&\leqslant n^{-\alpha+1} E|X|I(|X| > N) < \frac{\varepsilon}{20}.
\end{aligned} \tag{3.10}$$

Similar to (3.10), we also obtain

$$\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} E a_{nj} X_j^{(n,l)}| \leqslant \sum_{j=1}^{n} |a_{nj}| E|X_j^{(n,l)}| < \frac{\varepsilon}{20}, \;\; l = 3,4,5.$$

For (2.1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(X_j - EX_j)| > \varepsilon)\\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \sum_{l=1}^{5} P(\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(X_j^{(n,l)} - EX_j^{(n,l)})| > \frac{\varepsilon}{5})\\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(X_j^{(n,1)} - EX_j^{(n,1)})| > \frac{\varepsilon}{5})\\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \sum_{l=2}^{5} P(\sum_{j=1}^{n} |a_{nj} X_j^{(n,l)}| > \frac{3\varepsilon}{20})
\end{aligned}$$

$$\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(X_j^{(n,1)} - EX_j^{(n,1)})| > \frac{\varepsilon}{5})$$
$$+ \sum_{l=2}^{3}\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\sum_{j=1}^{n} |a_{nj}(X_j^{(n,l)} - EX_j^{(n,l)})| > \frac{\varepsilon}{10})$$
$$+ \sum_{l=4}^{5}\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\sum_{j=1}^{n} |a_{nj} X_j^{(n,l)}| > \frac{3\varepsilon}{20})$$
$$=: J_1 + J_2 + J_3 + J_4 + J_5.$$

Thus, to prove (2.1), it suffices to show that $J_i < \infty$, $i = 1,2,3,4,5$.

For J_1, noting that $\{a_{nj}X_j^{(n,l)}, j \geqslant 1\}$ is a sequence of m-WOD with dominating coefficients $g(n)$ for $n \geqslant 1$. Applying Markov's inequality and Lemmas 2.3–2.4 with $v > \max\{p, 2, \frac{\alpha p-1+\delta}{\alpha-1/2}\}$, we have

$$J_1 \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) E\{\max_{1\leqslant k\leqslant n} |\sum_{j=1}^{k} a_{nj}(X_j^{(n,1)} - EX_j^{(n,1)})|^v\}$$
$$\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) (\log n)^v \{\sum_{j=1}^{n} E|a_{nj}X_j^{(n,1)}|^v + g(n)(\sum_{j=1}^{n} E|a_{nj}X_j^{(n,1)}|^2)^{\frac{v}{2}}\}$$
$$=: J_{11} + J_{12}. \tag{3.11}$$

For J_{11}, by Lemma 2.4 and $\max_{1\leqslant j\leqslant n} |a_{nj}| = O(n^{-\alpha})$, we have

$$J_{11} \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) (\log n)^v \sum_{j=1}^{n} |a_{nj}|^v [E|X_j|^v I(|X_j| \leqslant N) + N^v P(|X_j| > N)]$$
$$\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) (\log n)^v \sum_{j=1}^{n} |a_{nj}|^v$$
$$\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) (\log n)^v (\max_{1\leqslant j\leqslant n} |a_{nj}|)^v \sum_{j=1}^{n} 1$$
$$\leqslant C \sum_{n=1}^{\infty} n^{-\alpha(v-p)-1} l(n) (\log n)^v < \infty. \tag{3.12}$$

For J_{12}, by Lemma 2.4, and $\max_{1\leqslant j\leqslant n} |a_{nj}| = O(n^{-\alpha})$, we get

$$J_{12} \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) (\log n)^v g(n) \{\sum_{j=1}^{n} |a_{nj}|^2 [E|X_j|^2 I(|X_j| \leqslant N) + N^2 P(|X_j| > N)]\}^{\frac{v}{2}}$$
$$\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2+\delta} l(n) (\log n)^v (\sum_{j=1}^{n} |a_{nj}|^2)^{\frac{v}{2}}$$

$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2+\delta} l(n)(\log n)^{v}[(\max_{1\leqslant j\leqslant n}|a_{nj}|)^{2}\sum_{j=1}^{n} 1]^{\frac{v}{2}}$$

$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-av+\delta+\frac{v}{2}} l(n)(\log n)^{v} < \infty. \tag{3.13}$$

For J_2, by Markov's inequality and Lemma 2.3–2.4, for $v > \max\{p,2\}$, we have

$$\begin{aligned} J_2 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)E|\sum_{j=1}^{n} a_{nj}(X_j^{(n,2)}-EX_j^{(n,2)})|^{v} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\{\sum_{j=1}^{n} E|a_{nj}X_j^{(n,2)}|^{v}+g(n)(\sum_{j=1}^{n} E|a_{nj}X_j^{(n,2)}|^{2})^{\frac{v}{2}}\} \\ &=: J_{21}+J_{22}. \end{aligned} \tag{3.14}$$

For J_{21}, by Lemma 2.4, we have

$$\begin{aligned} J_{21} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\sum_{j=1}^{n}|a_{nj}|^{v}[E|X_j|^{v}I(|X_j|\leqslant n^{\alpha})+n^{\alpha v}P(|X_j|>n^{\alpha})] \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)[E|X|^{v}I(|X|\leqslant n^{\alpha})+n^{\alpha v}P(|X|>n^{\alpha})]\sum_{j=1}^{n}|a_{nj}|^{v} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-av-1} l(n)E|X|^{v}I(|X|\leqslant n^{\alpha})+C\sum_{n=1}^{\infty} n^{\alpha p-1} l(n)P(|X|>n^{\alpha}) \\ &=: J_{211}+J_{212}. \end{aligned} \tag{3.15}$$

For J_{211}, by Lemma 2.5, we have

$$\begin{aligned} J_{211} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-av-1} l(n)\sum_{k=1}^{n} E|X|^{v}I((k-1)^{\alpha}<|X|\leqslant k^{\alpha}) \\ &\leqslant C\sum_{k=1}^{\infty} E|X|^{v}I((k-1)^{\alpha}<|X|\leqslant k^{\alpha})\sum_{n=k}^{\infty} n^{\alpha p-av-1} l(n) \\ &\leqslant C\sum_{k=1}^{\infty} k^{\alpha p-av} l(k)E|X|^{v}I((k-1)^{\alpha}<|X|\leqslant k^{\alpha}) \\ &\leqslant CE|X|^{p} l(X^{\frac{1}{\alpha}}) < \infty. \end{aligned} \tag{3.16}$$

For J_{212}, by Lemma 2.5, we have

$$\begin{aligned} J_{212} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1} l(n)\sum_{k=n}^{\infty} P(k^{\alpha}<|X|\leqslant (k+1)^{\alpha}) \\ &\leqslant C\sum_{k=1}^{\infty} P(k^{\alpha}<|X|\leqslant (k+1)^{\alpha})\sum_{n=1}^{k} n^{\alpha p-1} l(n) \\ &\leqslant C\sum_{k=1}^{\infty} k^{\alpha p} l(k)P(k^{\alpha}<|X|\leqslant (k+1)^{\alpha}) \\ &\leqslant CE|X|^{p} l(X^{\frac{1}{\alpha}}) < \infty. \end{aligned} \tag{3.17}$$

The proof of $J_{22} < \infty$ will be divided into two cases:

Case 1: $p \geqslant 2$. Since $E|X|^p l(X^{\frac{1}{\alpha}}) < \infty$, it follows that $E|X|^2 < \infty$. Taking $v > \max\{p, \frac{2(\alpha p-1+\delta)}{2\alpha-1}\}$, we obtain

$$J_{22} \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2+\delta-av+\frac{v}{2}} l(n)(E|X|^2)^{\frac{v}{2}} < \infty. \tag{3.18}$$

Case 2: $1 < p < 2$. By $E|X|^p l(X^{\frac{1}{\alpha}}) < \infty$, taking $v > \max\{2, \frac{2(\alpha p-1+\delta)}{\alpha p-1}\}$, we have

$$J_{22} \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2+\delta-av+\frac{v}{2}+(2-p)\frac{av}{2}} l(n)(E|X|^p)^{\frac{v}{2}} < \infty. \tag{3.19}$$

Analogous to the proof for J_2, we also obtain $J_3 < \infty$.
For J_4, by (3.17), we get

$$\begin{aligned} J_4 &= \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\sum_{j=1}^{n} |a_{nj} X_j^{(n,4)}| > \frac{\varepsilon}{10}) \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\exists j: 1 \leqslant j \leqslant n, |X_j| > n^{\alpha}) \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \sum_{j=1}^{n} P(|X_j| > n^{\alpha}) \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|X| > n^{\alpha}) \\ &\leqslant CE|X|^p l(|X|^{\frac{1}{\alpha}}) < \infty. \end{aligned} \tag{3.20}$$

Similar to J_4, we also can get $J_5 < \infty$.
From (3.10)–(3.20), the proof of Theorem 2.3 is completed. □

Proof of Corollary 3.1. The proof is similar to that of Corollary 3.1 in Lang [6] and is therefore omitted. □

Proof of Corollary 3.2. The proof is similar to that of Corollary 3.2 in Lang [6] and is therefore omitted. □

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