Journal of Mathematical Inequalities
Volume 19, Number 3 (2025), 935–942 doi:10.7153/jmi-2025-19-60

NEW PROOFS ON TWO RECENT INEQUALITIES RELATED TO THE SPECTRAL NORM

JIQIN CHEN, XIAOHUI FU* AND QI SONG

(*Communicated by M. Krnić*)

Abstract. Afraz et al. [1] recently obtained some norm inequalities involving a special class of functions for sector matrices. In this short note, we give alternative proofs of Afraz et al.'s two results [1] under the spectral norm.

1. Introduction

Let $\mathbb{M}_n$ be the set of all $n\times n$ complex matrices. The identity matrix of $\mathbb{M}_n$ is denoted by I. For any $X\in\mathbb{M}_n$, X^* stands for the conjugate transpose of X. For two Hermitian matrices X,Y of the same size, $X\geqslant Y$ $(X>Y)$ means that $X-Y\geqslant 0$ $(X-Y>0)$. For $X\in\mathbb{M}_n$, if the eigenvalues of X are real, then they are arranged nonincreasingly $\lambda_1(X)\geqslant\cdots\geqslant\lambda_n(X)$; the singular values of X are denoted by $s_j(X)=\lambda_j(|X|)=\lambda_j((X^*X)^{\frac{1}{2}})$, which are similarly arranged. Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is unitarily invariant if $\|UXV\|=\|X\|$ for any $X\in\mathbb{M}_n$ and unitary matrices $U,V\in\mathbb{M}_n$. The spectral norm, written as $\|\cdot\|_\infty$, is defined by $\|X\|_\infty=\max\limits_{\|x\|=1}\|Xx\|$ for $X\in\mathbb{M}_n$ and $x\in\mathbb{C}^n$. Note that $\|X\|_\infty=s_1(X)$. The spectral norm is a special class of unitarily invariant norms. Some papers [2, 3, 4, 17] are devoted to the study of unitarily invariant norm inequalities.

A matrix $T\in\mathbb{M}_{2n}$ can be partitioned as a 2×2 block matrix

$$T=\begin{pmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}, \tag{1}$$

where $T_{ij}\in\mathbb{M}_n$, $i,j=1,2$. For any $T\in\mathbb{M}_{2n}$, recall that the Cartesian decomposition of T (see [5, p. 6] and [11, p. 7]) is

$$T=A+iB,$$

where $A=\Re T=\frac{T+T^*}{2}$ and $B=\Im T=\frac{T-T^*}{2i}$.

Mathematics subject classification (2020): 15A42, 47A63, 15A45, 47B65.
Keywords and phrases: Submultiplicative function, sector matrix, spectral norm.
* Corresponding author.

In this paper, the decomposition is represented as

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} + i \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}.$$

The matrix is called accretive-dissipative if in its Cartesian decomposition, $T = A + iB$, the matrices A and B are positive semidefinite. Such matrices have found many applications [8, 10, 12, 16]. Recent works devoted to studying this kind of matrix are [7, 9, 13, 14, 15].

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax | x \in \mathbb{C}_n, x^*x = 1\}.$$

For $\alpha \in [0, \pi/2)$, let

$$S_\alpha = \{z \in \mathbb{C} | \Re z \geqslant 0, |\Im z| \leqslant (\Re z) \tan(\alpha)\}$$

be a sector region on the complex plane. A matrix whose numerical range is contained in a sector region S_α is called a sector matrix.

Afraz et al. [1] presented several unitarily norm inequalities related to sector matrices involving convex and concave functions. They proved that if a sector matrix T is partitioned as in (1) and f is a submultiplicative convex function, then

$$\begin{aligned} \|f(|T_{12}|^2) + f(|T_{21}^*|^2)\| &\leqslant \|f^r(\sqrt{2}\sec(\alpha)\Re(T_{11}))\|^{\frac{1}{r}} \|f^s(\sqrt{2}\sec(\alpha)\Re(T_{22}))\|^{\frac{1}{s}} \\ &\leqslant \|f^r(\sqrt{2}\sec(\alpha)|T_{11}|)\|^{\frac{1}{r}} \|f^s(\sqrt{2}\sec(\alpha)|T_{22}|)\|^{\frac{1}{s}}, \end{aligned} \tag{2}$$

where r and s are positive real numbers with $\frac{1}{r} + \frac{1}{s} =$ and $\alpha \in [0, \frac{\pi}{2})$.

However, if f is a submultiplicative concave function, then inequality (2) becomes

$$\begin{aligned} \|f(|T_{12}|^2) + f(|T_{21}^*|^2)\| &\leqslant 2\sec^2(\alpha) \|f^r(\Re(T_{11}))\|^{\frac{1}{r}} \|f^s(\Re(T_{22}))\|^{\frac{1}{s}} \\ &\leqslant 2\sec^2(\alpha) \|f^r(|T_{11}|)\|^{\frac{1}{r}} \|f^s(|T_{22}|)\|^{\frac{1}{s}}. \end{aligned} \tag{3}$$

In this paper, we will give alternative proofs of inequalities (2) and (3) in the case of the spectral norm by using sectoral decomposition.

2. Main results

For presenting the new proofs, we give the following several lemmas.

LEMMA 2.1. ([5, p. 5]) *Let $X \in \mathbb{M}_n$ and let f be a nonnegative increasing function on $[0, \infty)$. Then*

$$f(s_j(X)) = s_j(f(|X|)).$$

LEMMA 2.2. ([5, p. 75] *Let $A, B \in \mathbb{M}_n$, $1 \leqslant i, j \leqslant n$, $i + j - 1 \leqslant n$. Then*

$$s_{i+j-1}(A + B) \leqslant s_i(A) + s_j(B),$$

$$s_{i+j-1}(AB) \leqslant s_i(A) s_j(B).$$

Part (a) of the following lemma has been given in [6], while a stronger version of part (b) can be obtained by invoking an argument similar to that used in the proof of Proposition 4.1 in [18].

LEMMA 2.3. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Then for every unitarily invariant norm,*

(a) $\|f(A)+f(B)\| \leqslant \|f(A+B)\|$ *for every nonnegative convex function on* $[0,\infty)$.

(b) $\|\frac{f(A)+f(B)}{2}\| \leqslant \|f\left(\frac{A+B}{2}\right)\|$ *for every nonnegative concave function on* $[0,\infty)$.

LEMMA 2.4. ([19, Theorem 2.1]) *Let A be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then there exist an invertible matrix X and a unitary and diagonal matrix $Z = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with all $|\theta_j| \leqslant \alpha$ such that $A = XZX^*$. Moreover, such a matrix Z is unique up to permutation.*

LEMMA 2.5. ([19, Corollary 2.3]) *Let A be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$ and let $A = XZX^*$ be a sectoral decomposition of A, where X is invertible and Z is unitary and diagonal. Then for any matrix R and all $j = 1, \ldots, n$, the following inequalities hold.*

$$RR^* \leqslant \sec(\alpha)(R(\Re Z)R^*) = \sec(\alpha)(\Re(RZR^*)), \tag{4}$$

and

$$s_j(RR^*) \leqslant \sec(\alpha)\lambda_j(R(\Re Z)R^*) \leqslant \sec(\alpha)s_j(RZR^*). \tag{5}$$

Next, we will present another proofs of inequalities (2) and (3) under the spectral norm.

THEOREM 2.6. *Let $T \in \mathbb{M}_{2n}$ partitioned as in (1) be a sector matrix and let f be increasing submultiplicative convex on $[0,\infty)$ and $\alpha \in [0, \frac{\pi}{2})$. If r and s are positive numbers with $\frac{1}{r} + \frac{1}{s} = 1$, then*

$$\begin{aligned} \left\|f\left(|T_{12}|^2\right) + f(|T_{21}^*|^2)\right\|_\infty &\leqslant \|f^r(\sqrt{2}\sec(\alpha)\Re(T_{11}))\|_\infty^{\frac{1}{r}} \|f^s(\sqrt{2}\sec(\alpha)\Re(T_{22}))\|_\infty^{\frac{1}{s}} \\ &\leqslant \|f^r(\sqrt{2}\sec(\alpha)|T_{11}|)\|_\infty^{\frac{1}{r}} \|f^s(\sqrt{2}\sec(\alpha)|T_{22}|)\|_\infty^{\frac{1}{s}}. \end{aligned} \tag{6}$$

Proof. We write $T = CZC^*$ with $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, $C_1 \in \mathbb{M}_{n\times 2n}$. Then $T_{12} = C_1 Z C_2^*$ and $T_{21} = C_2 Z C_1^*$.

Compute

$$\begin{aligned} &\|f(|T_{12}|^2) + f(|T_{21}^*|^2)\|_\infty \\ \leqslant &\left\|f\left(|T_{12}|^2 + |T_{21}^*|^2\right)\right\|_\infty \quad \text{(by Lemma 2.3 (a))} \\ = &\, s_1\left(f\left(C_2 Z^* C_1^* C_1 Z C_2^* + C_2 Z C_1^* C_1 Z^* C_2^*\right)\right) \\ = &\, f\left(s_1\left(C_2 Z^* C_1^* C_1 Z C_2^* + C_2 Z C_1^* C_1 Z^* C_2^*\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\leqslant f\left(s_1\left(C_2Z^*C_1^*C_1ZC_2^*\right)+s_1\left(C_2ZC_1^*C_1Z^*C_2^*\right)\right)\\
&\leqslant f\left(s_1(C_2)s_1(Z^*)s_1(C_1C_1^*)s_1(Z)s_1(C_2^*)+s_1(C_2)s_1(Z)s_1(C_1C_1^*)s_1(Z^*)s_1(C_2^*)\right)\\
&= f\left(2s_1(C_1C_1^*)s_1(C_2C_2^*)\right)\\
&\leqslant f\left(2\sec^2(\alpha)s_1(C_1\Re(Z)C_1^*)s_1(C_2\Re(Z)C_2^*)\right) \quad \text{(by (4))}\\
&= f\left(2\sec^2(\alpha)s_1(\Re(C_1ZC_1^*))s_1(\Re(C_2ZC_2^*))\right)\\
&= f\left(s_1\left(\sqrt{2}\sec(\alpha)\Re(C_1ZC_1^*)\right)s_1\left(\sqrt{2}\sec(\alpha)\Re(C_2ZC_2^*)\right)\right)\\
&\leqslant f\left(s_1\left(\sqrt{2}\sec(\alpha)\Re(C_1ZC_1^*)\right)\right)f\left(s_1\left(\sqrt{2}\sec(\alpha)\Re(C_2ZC_2^*)\right)\right)\\
&= s_1\left(f\left(\sqrt{2}\sec(\alpha)\Re(T_{11})\right)\right)s_1\left(f\left(\sqrt{2}\sec(\alpha)\Re(T_{22})\right)\right)\\
&= s_1^{\frac{1}{r}}\left(f^r\left(\sqrt{2}\sec(\alpha)\Re(T_{11})\right)\right)s_1^{\frac{1}{s}}\left(f^s\left(\sqrt{2}\sec(\alpha)\Re(T_{22})\right)\right)\\
&= \left\|f^r\left(\sqrt{2}\sec(\alpha)\Re(T_{11})\right)\right\|_\infty^{\frac{1}{r}}\left\|f^s\left(\sqrt{2}\sec(\alpha)\Re(T_{22})\right)\right\|_\infty^{\frac{1}{s}}\\
&\leqslant \left\|f^r\left(\sqrt{2}\sec(\alpha)|T_{11}|\right)\right\|_\infty^{\frac{1}{r}}\left\|f^s\left(\sqrt{2}\sec(\alpha)|T_{22}|\right)\right\|_\infty^{\frac{1}{s}}. \quad \text{(by (5))}
\end{aligned}
$$

This completes the proof. □

In fact, by the same technique used in the above theorem, we provide a new proof of inequality (3) under the spectral norm.

THEOREM 2.7. *Let $T \in \mathbb{M}_{2n}$ partitioned as in (1) be a sector matrix and let f be increasing submultiplicative concave on $[0,\infty)$ with $f(0)=0$ and $\alpha \in [0,\frac{\pi}{2})$. If r and s are positive numbers with $\frac{1}{r}+\frac{1}{s}=1$, then*

$$
\begin{aligned}
\|f(|T_{12}|^2)+f(|T_{21}^*|^2)\|_\infty &\leqslant 2\sec^2(\alpha)\|f^r(\Re(T_{11}))\|_\infty^{\frac{1}{r}}\|f^s(\Re(T_{22}))\|_\infty^{\frac{1}{s}}\\
&\leqslant 2\sec^2(\alpha)\|f^r(|T_{11}|)\|_\infty^{\frac{1}{r}}\|f^s(|T_{22}|)\|_\infty^{\frac{1}{s}}.
\end{aligned} \tag{7}
$$

Proof. Compute

$$
\begin{aligned}
&\|f(|T_{12}|^2)+f(|T_{21}^*|^2)\|_\infty\\
&\leqslant 2\left\|f\left(\frac{|T_{12}|^2+|T_{21}^*|^2}{2}\right)\right\|_\infty \quad \text{(by Lemma 2.3 (b))}\\
&= 2s_1\left(f\left(\frac{C_2Z^*C_1^*C_1ZC_2^*}{2}+\frac{C_2ZC_1^*C_1Z^*C_2^*}{2}\right)\right)\\
&= 2f\left(s_1\left(\frac{1}{2}C_2Z^*C_1^*C_1ZC_2^*+\frac{1}{2}C_2ZC_1^*C_1Z^*C_2^*\right)\right)\\
&\leqslant 2f\left(\frac{1}{2}s_1(C_2Z^*C_1^*C_1ZC_2^*)+\frac{1}{2}s_1(C_2ZC_1^*C_1Z^*C_2^*)\right)\\
&\leqslant 2f\left(s_1(C_1C_1^*)s_1(C_2C_2^*)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 2f\big(\sec^2(\alpha)s_1(C_1\Re(Z)C_1^*)s_1(C_2\Re(Z)C_2^*)\big)\\
&= 2f\big(\sec^2(\alpha)s_1(\Re(C_1ZC_1^*))s_1(\Re(C_2ZC_2^*))\big)\\
&= 2f\Big(s_1\big(\sec(\alpha)\Re(C_1ZC_1^*)\big)s_1\big(\sec(\alpha)\Re(C_2ZC_2^*)\big)\Big)\\
&\leqslant 2f\Big(s_1\big(\sec(\alpha)\Re(C_1ZC_1^*)\big)\Big)f\Big(s_1\big(\sec(\alpha)\Re(C_2ZC_2^*)\big)\Big)\\
&= 2s_1\Big(f\big(\sec(\alpha)\Re(T_{11})\big)\Big)s_1\Big(f\big(\sec(\alpha)\Re(T_{22})\big)\Big)\\
&= 2s_1^{\frac{1}{r}}\Big(f^r\big(\sec(\alpha)\Re(T_{11})\big)\Big)s_1^{\frac{1}{s}}\Big(f^s\big(\sec(\alpha)\Re(T_{22})\big)\Big)\\
&= 2\,\|f^r(\sec(\alpha)\Re(T_{11}))\|_\infty^{\frac{1}{r}}\,\|f^s(\sec(\alpha)\Re(T_{22}))\|_\infty^{\frac{1}{s}}.
\end{aligned}
$$

Since f is concave, it follows that $f(aT)\leqslant af(T)$ for $T\in\mathbb{M}_n$ and $a>1$. Therefore,

$$
\begin{aligned}
\|f(|T_{12}|^2)+f(|T_{21}^*|^2)\|_\infty &\leqslant 2\sec^2(\alpha)\|f^r(\Re(T_{11}))\|_\infty^{\frac{1}{r}}\|f^s(\Re(T_{22}))\|_\infty^{\frac{1}{s}}\\
&\leqslant 2\sec^2(\alpha)\|f^r(|T_{11}|)\|_\infty^{\frac{1}{r}}\|f^s(|T_{22}|)\|_\infty^{\frac{1}{s}}. \quad \square
\end{aligned}
$$

By direct computations, we can get the following lemma.

LEMMA 2.8. *Let $A,B\in\mathbb{M}_n$ and $\alpha,\beta\in[0,1]$ such that $\alpha+\beta=1$. Then*

$$|A+(2\alpha-1)B|^2+4\alpha\beta|B|^2=|A|^2+|B|^2+(2\alpha-1)(A^*B+B^*A).$$

Next, we try to insert parameters α,β into inequality (6) and (7), but we just obtain two weaker results as follows.

THEOREM 2.9. *Let $T\in\mathbb{M}_{2n}$ partitioned as in (1) be a sector matrix and let f be increasing submultiplicative convex on $[0,\infty)$ and $\theta\in[0,\frac{\pi}{2})$. If r and s are positive numbers with $\frac{1}{r}+\frac{1}{s}=1$, then*

$$
\begin{aligned}
&\|f(|T_{12}+(2\alpha-1)T_{21}^*|^2)+f(4\alpha\beta|T_{21}^*|^2)\|_\infty\\
&\leqslant \|f^r(2\sec(\theta)\Re(T_{11}))\|_\infty^{\frac{1}{r}}\,\|f^s(2\sec(\theta)\Re(T_{22}))\|_\infty^{\frac{1}{s}}\\
&\leqslant \|f^r(2\sec(\theta)|T_{11}|)\|_\infty^{\frac{1}{r}}\|f^s(2\sec(\theta)|T_{22}|)\|_\infty^{\frac{1}{s}},
\end{aligned}
$$

where $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$.

Proof.

$$
\begin{aligned}
&\|f(|T_{12}+(2\alpha-1)T_{21}^*|^2)+f(4\alpha\beta|T_{21}^*|^2)\|_\infty\\
&\leqslant \|f(|T_{12}+(2\alpha-1)T_{21}^*|^2+4\alpha\beta|T_{21}^*|^2)\|_\infty\\
&= s_1\left(f(|C_1ZC_2^*|^2+|C_1Z^*C_2^*|^2+(2\alpha-1)(C_2Z^*C_1^*C_1Z^*C_2^*+C_2ZC_1^*C_1ZC_2^*)\right)\\
&= f\big(s_1(C_2Z^*C_1^*C_1(Z+(2\alpha-1)Z^*)C_2^*+C_2ZC_1^*C_1(Z^*+(2\alpha-1)Z)C_2^*)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant f\big(s_1(C_2)s_1(Z^*)s_1(C_1^*C_1)s_1(Z+(2\alpha-1)Z^*)s_1(C_2^*)\\
&\quad +s_1(C_2)s_1(Z)s_1(C_1^*C_1)s_1(Z^*+(2\alpha-1)Z)s_1(C_2^*)\big)\\
&= f\big(s_1(C_2C_2^*)s_1(C_1C_1^*)s_1(Z+(2\alpha-1)Z^*)\\
&\quad +s_1(C_2C_2^*)s_1(C_1C_1^*)s_1(Z^*+(2\alpha-1)Z)\big)\\
&\leqslant f\big(4s_1(C_2C_2^*)s_1(C_1C_1^*)\big)\\
&\leqslant f\big(4\sec^2(\theta)s_1(C_1\Re(Z)C_1^*)s_1(C_2\Re(Z)C_2^*)\big)\\
&= f\big(4\sec^2(\theta)s_1(\Re(C_1ZC_1^*))s_1(\Re(C_2ZC_2^*))\big)\\
&= f\Big(s_1\big(2\sec(\theta)\Re(C_1ZC_1^*)\big)s_1\big(2\sec(\theta)\Re(C_2ZC_2^*)\big)\Big)\\
&\leqslant f\Big(s_1\big(2\sec(\theta)\Re(C_1ZC_1^*)\big)\Big)f\Big(s_1\big(2\sec(\theta)\Re(C_2ZC_2^*)\big)\Big)\\
&= s_1\Big(f\big(2\sec(\theta)\Re(T_{11})\big)\Big)s_1\Big(f\big(2\sec(\theta)\Re(T_{22})\big)\Big)\\
&= s_1^{\frac{1}{r}}\Big(f^r\big(2\sec(\theta)\Re(T_{11})\big)\Big)s_1^{\frac{1}{s}}\Big(f^s\big(2\sec(\theta)\Re(T_{22})\big)\Big)\\
&= \|f^r(2\sec(\theta)\Re(T_{11}))\|_\infty^{\frac{1}{r}}\|f^s(2\sec(\theta)\Re(T_{22}))\|_\infty^{\frac{1}{s}}\\
&\leqslant \|f^r(2\sec(\theta)|T_{11}|)\|_\infty^{\frac{1}{r}}\|f^s(2\sec(\theta)|T_{22}|)\|_\infty^{\frac{1}{s}}. \quad \square
\end{aligned}
$$

THEOREM 2.10. *Let $T \in \mathbb{M}_{2n}$ partitioned as in (1) be a sector matrix and let f be increasing submultiplicative concave on $[0,\infty)$ and $\theta \in [0,\frac{\pi}{2})$. If r and s are positive numbers with $\frac{1}{r}+\frac{1}{s}=1$, then*

$$
\begin{aligned}
&\|f(|T_{12}+(2\alpha-1)T_{21}^*|^2)+f(4\alpha\beta|T_{21}^*|^2)\|_\infty\\
\leqslant & 2\left\|f^r\left(\sqrt{2}\sec(\theta)\Re(T_{11})\right)\right\|_\infty^{\frac{1}{r}}\left\|f^s\left(\sqrt{2}\sec(\theta)\Re(T_{22})\right)\right\|_\infty^{\frac{1}{s}}\\
\leqslant & 2\left\|f^r\left(\sqrt{2}\sec(\theta)|T_{11}|\right)\right\|_\infty^{\frac{1}{r}}\left\|f^s\left(\sqrt{2}\sec(\theta)|T_{22}|\right)\right\|_\infty^{\frac{1}{s}},
\end{aligned}
$$

where $\alpha,\beta \in [0,1]$ with $\alpha+\beta=1$.

Proof.

$$
\begin{aligned}
&\|f(|T_{12}+(2\alpha-1)T_{21}^*|^2)+f(4\alpha\beta|T_{21}^*|^2)\|_\infty\\
\leqslant\ & 2\left\|f\left(\frac{|T_{12}+(2\alpha-1)T_{21}^*|^2+4\alpha\beta|T_{21}^*|^2}{2}\right)\right\|_\infty\\
=\ & 2s_1\left(f\left(\frac{1}{2}C_2Z^*C_1^*C_1(Z+(2\alpha-1)Z^*)C_2^*+\frac{1}{2}C_2ZC_1^*C_1(Z^*+(2\alpha-1)Z)C_2^*\right)\right)\\
=\ & 2f\left(s_1\left(\frac{1}{2}C_2Z^*C_1^*C_1(Z+(2\alpha-1)Z^*)C_2^*+\frac{1}{2}C_2ZC_1^*C_1(Z^*+(2\alpha-1)Z)C_2^*\right)\right)
\end{aligned}
$$

$$\leqslant 2f\left(\frac{1}{2}s_1(C_2C_2^*)s_1(C_1^*C_1)s_1(Z+(2\alpha-1)Z^*)\right.$$
$$\left.+\frac{1}{2}s_1(C_2C_2^*)s_1(C_1C_1^*)s_1(Z^*+(2\alpha-1)Z)\right)$$
$$\leqslant 2f\big(2s_1(C_2C_2^*)s_1(C_1C_1^*)\big)$$
$$\leqslant 2\left\|f^r\left(\sqrt{2}\sec(\theta)\Re(T_{11})\right)\right\|_\infty^{\frac{1}{r}}\left\|f^s\left(\sqrt{2}\sec(\theta)\Re(T_{22})\right)\right\|_\infty^{\frac{1}{s}}$$
$$\leqslant 2\left\|f^r\left(\sqrt{2}\sec(\theta)|T_{11}|\right)\right\|_\infty^{\frac{1}{r}}\left\|f^s\left(\sqrt{2}\sec(\theta)|T_{22}|\right)\right\|_\infty^{\frac{1}{s}}. \quad \square$$

Use of AI tools declaration. The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions. Xiaohui Fu wrote the main manuscript text and Jiqin Chen and Qi Song checked the proofs. All authors contributed equally to the manuscript. Jiqin Chen, Xiaohui Fu and Qi Song are co-first authors.

Acknowledgement. The work is supported by Hainan Provincial Natural Science Foundation for High-level Talents (grant no. 123RC474, 124RC503), National Natural Science Foundation of China (grant no. 12261030), Hainan Provincial Natural Science Foundation (grant no. 125MS075), the National Natural Science Foundation of China (grant no. 12261029), the Hainan Province Academician Workstation (Changbin Yu), the Key Laboratory of Computational Science and Application of Hainan Province and Hainan Provincial Graduate Innovation Research Program (grant no. Qhys2024-397, Qhys2024-398).

REFERENCES

[1] D. AFRAZ, R. LASHKARIPOUR, M. BAKHERAD, *Norm inequalities involving a special class of functions for sector matrices*, J. Inequal. Appl. (2020) **2020**: 122.

[2] D. AFRAZ, R. LASHKARIPOUR, M. BAKHERAD, *Further norm and numerical radius inequalities for sum of Hilbert space operators*, Filomat. **38** (2024) 3235–3242.

[3] M. BAKHERAD, R. LASHKARIPOUR, M. HAJMOHAMADI, *Extensions of interpolation between the arithmetic-geometric mean inequality for matrices*, J. Inequal. Appl. (2017) **2017**: 209.

[4] M. BAKHERAD, *Unitarily invariant norm inequalities involving G_1 operators*, Commun. Korean Math. Soc. **33** (2018) 889–899.

[5] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[6] J.-C. BOURIN, M. UCHIYAMA, *A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$*, Linear Algebra Appl. **423** (2007) 512–518.

[7] X. FU, C. HE, *On some Fischer type determinantal inequalities for accretive-dissipative matrices*, J. Inequal. Appl. (2013) **2013**: 316.

[8] A. GEORGE, KH. D. IKRAMOV, A. B. KUCHEROV, *On the growth factor in Gaussian elimination for generalized Higham matrices*, Numer. Linear Algebra Appl. **9** (2002) 107–114.

[9] A. GEORGE, KH. D. IKRAMOV, *On the properties of accretive-dissipative matrices*, Math. Notes **77** (2005) 767–776.

[10] M. D. GUNZBURGER, R. J. PLEMMONS, *Energy conserving norms for the solution of hyperbolic systems of partial differential equations*, Math. Comp. **33** (1979) 1–10.
[11] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, 2nd ed, Cambridge University Press, Cambridge, 2013.
[12] N. J. HIGHAM, *Factorizing complex symmetric matrices with positive real and imaginary parts*, Math. Comp. **67** (1998) 1591–1599.
[13] KH. D. IKRAMOV, *Determinantal inequalities for accretive-dissipative matrices*, J. Math. Sci. (New York) **121** (2004) 2458–2464.
[14] M. LIN, *Reversed determinantal inequalities for accretive-dissipative matrices*, Math. Inequal. Appl. **12** (2012) 955–958.
[15] M. LIN, *Fisher type determinantal inequalities for accretive-dissipative matrices*, Linear Algebra Appl. **438** (2013) 2808–2812.
[16] M. LIN, *A note on the growth factor in Gaussian elimination for accretive-dissipative matrices*, Calcolo. **51** (2014) 363–366.
[17] K. SHEBRAWI, M. BAKHERAD, *Generalizations of the Aluthge transform of operators*, Filomat. **32** (2018) 6465–6474.
[18] M. UCHIYAMA, *Subadditivity of eigenvalue sums*, Proc. Amer. Math. Soc. **134** (2006) 1405–1412.
[19] F. ZHANG, *A matrix decomposition and its applications*, Linear Multilinear Algebra. **10** (2015) 2033–2042.

(Received April 29, 2025)

Jiqin Chen
School of Mathematics and Statistics
Hainan Normal University
Haikou, China

Xiaohui Fu
School of Mathematics and Statistics
Hainan Normal University
Haikou, China
and
School of Mathematics and Statistics
Hainan Normal University
Haikou, China
e-mail: fxh6662@sina.com

Qi Song
School of Mathematics and Statistics
Hainan Normal University
Haikou, China