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TIGHTENING BOUNDS ON THE NUMERICAL RADIUS FOR HILBERT SPACE OPERATORS

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Abstract. Let S be a bounded linear operator on a Hilbert space. We show that if S is accretive (resp. dissipative in the sense that $\frac{S-S^*}{2i}$ is positive) in the sense that $\frac{S+S^*}{2}$ is positive, then

$$\frac{1}{2}\sqrt{\|S\|^2+\|\Re S\|\,\|\Im S\|}\leqslant \omega(S),$$

where $\|\cdot\|$ and $\omega(\cdot)$ denote the operator norm and the numerical radius, respectively.

1. Introduction

Let $\mathbb{H}$ be an arbitrary Hilbert space, endowed with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. The notation $\mathbb{B}(\mathbb{H})$ will be used to denote the C^*-algebra of all bounded linear operators on $\mathbb{H}$. Upper case letters will be used to denote the element of $\mathbb{B}(\mathbb{H})$. For $S\in\mathbb{B}(\mathbb{H})$, the adjoint operator S^* is the operator defined by $\langle Sx,y\rangle=\langle x,S^*y\rangle$ for $x,y\in\mathbb{H}$, and the operator norm of S is defined by $\|S\|=\sup_{\|x\|=1}\|Sx\|$. If an operator $S\in\mathbb{B}(\mathbb{H})$ satisfies $\langle Sx,x\rangle\geqslant 0$ for all $x\in\mathbb{H}$, it will be a positive operator. In this case, we write $S\geqslant O$. Here $|S|$ stands for the positive operator $(S^*S)^{\frac{1}{2}}$. Related to the operator norm, the numerical radius of $S\in\mathbb{B}(\mathbb{H})$ is defined by $\omega(S)=\sup_{\|x\|=1}|\langle Sx,x\rangle|$. This latter quantity defines a norm on $\mathbb{B}(\mathbb{H})$ that is equivalent to the operator norm, where we have the equivalence

$$\frac{1}{2}\|S\|\leqslant\omega(S)\leqslant\|S\|;\ \ S\in\mathbb{B}(\mathbb{H}). \tag{1.1}$$

Several numerical radius inequalities that improve upon those in (1.1) have been recently established in [3, 4, 5, 6, 11, 15, 16, 17, 18, 19].

Sharpening the first inequality in (1.1), Kittaneh showed that [10]

$$\frac{1}{4}\|S^*S+SS^*\|\leqslant\omega^2(S). \tag{1.2}$$

Let $m(S)$ be the nonnegative number defined by $m(S)=\inf_{\|x\|=1}|\langle Sx,x\rangle|$. It has been shown in [2] that

$$\frac{1}{4}\|S^*S+SS^*\|+\frac{1}{2}m\left(S^2\right)\leqslant\omega^2(S). \tag{1.3}$$

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Of course (1.3) improves (1.2).

Let $S\in\mathbb{B}(\mathbb{H})$. S is accretive (resp. dissipative) if in its Cartesian decomposition $S=\Re S+i\Im S$, $\Re S=\frac{S+S^*}{2}$ (resp. $\Im S=\frac{S-S^*}{2i}$) is positive and S is accretive-dissipative if both $\Re S$ and $\Im S$ are positive.

The first inequality in (1.1) has been improved considerably in [13]. Indeed, it has been shown in [13, Theorem 2.6] that if S is an accretive-dissipative operator, then

$$\frac{\sqrt{2}}{2}\|S\|\leqslant\omega(S).$$

As we proceed in this paper, the following basic fact will be used (see, e.g., [14, Ineq. (2.3) and Ineq. (2.4)]):

$$\|\Re S\|,\|\Im S\|\leqslant\omega(S), \tag{1.4}$$

for any $S\in\mathbb{B}(\mathbb{H})$.

Let $|||\cdot|||$ denote any unitarily invariant norm, i.e., a norm with the property that $|||USV|||=|||S|||$ for all S, and all unitary U, V. As an extension of the definition of the numerical radius, it has been defined in [1] that

$$\omega_{|||\cdot|||}(S)=\sup_{\theta\in\mathbb{R}}\left|\left|\left|\Re\left(e^{i\theta}S\right)\right|\right|\right|. \tag{1.5}$$

The essential properties and the inequalities associated with this concept have been examined in the same reference.

The paper includes two main results, namely Theorems 2.2 and 2.3. The first main result concerning accretive operators reads as follows: If $S\in\mathbb{B}(\mathbb{H})$ is an accretive operator, then

$$\frac{1}{2}\sqrt{\|S\|^2+\|\Re S\|\,\|\Im S\|}\leqslant\omega(S).$$

The above inequality is also true when S is a dissipative operator. As our second main result of this paper, for an accretive-dissipative operator S, we will show that

$$\frac{1}{2}\sqrt{\|S\|^2+2\|\Re S\|\,\|\Im S\|}\leqslant\omega(S).$$

To prove our main results, we need some lemmas.

LEMMA 1.1. [8, Theorem 1] *Let $S,T\in\mathbb{B}(\mathbb{H})$ be such that S or T is positive. Then*

$$\|ST-TS\|\leqslant\|S\|\,\|T\|.$$

LEMMA 1.2. [9, Corollary 1] *Let $S,T\in\mathbb{B}(\mathbb{H})$ be positive. Then*

$$\max\{\|S\|,\|T\|\}-\left\|S^{\frac{1}{2}}T^{\frac{1}{2}}\right\|\leqslant\|S-T\|.$$

LEMMA 1.3. [7] *Let $S,T\in\mathbb{B}(\mathbb{H})$ and let S be normal with the Cartesian decomposition $S=\Re S+i\Im S$. Then, for every unitarily invariant norm,*

$$|||ST-TS|||\leqslant\sqrt{\|\Re S\|^2+\|\Im S\|^2}\,|||T|||,$$

when $\Re S$, $\Im S\geqslant O$.

2. Results

We open this section with the following useful lemma concerning accretive operators.

LEMMA 2.1. *Let $S \in \mathbb{B}(\mathbb{H})$ be an accretive operator. Then*

$$\frac{\sqrt{3}}{3}\|S\| \leqslant \omega(S).$$

The above inequality is also true when S is a dissipative operator.

Proof. According to the assumption, $\Re S \geqslant O$. If we use the triangle inequality for the operator norm and Lemma 1.1, we can write

$$\begin{aligned}
\|S\|^2 &= \|SS^*\| \\
&= \left\|(\Re S + i\Im S)(\Re S + i\Im S)^*\right\| \\
&= \|(\Re S + i\Im S)(\Re S - i\Im S)\| \\
&= \left\|(\Re S)^2 + (\Im S)^2 - i((\Re S)(\Im S) - (\Im S)(\Re S))\right\| \\
&\leqslant \left\|(\Re S)^2 + (\Im S)^2\right\| + \|(\Re S)(\Im S) - (\Im S)(\Re S)\| \\
&\leqslant \left\|(\Re S)^2\right\| + \left\|(\Im S)^2\right\| + \|(\Re S)(\Im S) - (\Im S)(\Re S)\| \\
&\leqslant \|\Re S\|^2 + \|\Im S\|^2 + \|\Re S\|\,\|\Im S\|
\end{aligned}$$

i.e.,

$$\|S\|^2 \leqslant \|\Re S\|^2 + \|\Im S\|^2 + \|\Re S\|\,\|\Im S\|. \tag{2.1}$$

On the other hand, if we apply (1.4), we infer that

$$\|\Re S\|^2 + \|\Im S\|^2 + \|\Re S\|\,\|\Im S\| \leqslant 3\omega^2(S).$$

Indeed, we have shown that

$$\|S\|^2 \leqslant 3\omega^2(S),$$

which is equivalent to the desired result.

The other case (when $\Im S \geqslant O$) can be proven similarly; we omit the details. □

It is interesting to note that Lemma 2.1 can be improved in the following form.

THEOREM 2.1. *Let $S \in \mathbb{B}(\mathbb{H})$ be an accretive operator. Then*

$$\frac{\sqrt{3}}{3}\sqrt{\|S\|^2 + m(S^2)} \leqslant \omega(S).$$

The above inequality is also true when S is a dissipative operator.

Proof. Since

$$\begin{aligned}
\|S\|^2 &= \|SS^*\| \\
&= \|(\Re S + i\Im S)(\Re S - i\Im S)\| \\
&= \left\|(\Re S)^2 + (\Im S)^2 - i((\Re S)(\Im S) - (\Im S)(\Re S))\right\| \\
&\leqslant \left\|(\Re S)^2 + (\Im S)^2\right\| + \|(\Re S)(\Im S) - (\Im S)(\Re S)\| \\
&= \left\|\left(\frac{S+S^*}{2}\right)^2 + \left(\frac{S-S^*}{2i}\right)^2\right\| + \|(\Re S)(\Im S) - (IS)(\Re S)\| \\
&= \frac{1}{2}\|SS^* + S^*S\| + \|(\Re S)(\Im S) - (\Im S)(\Re S)\| \\
&\leqslant \frac{1}{2}\|SS^* + S^*S\| + \|\Re S\|\,\|\Im S\|,
\end{aligned}$$

we have

$$\|S\|^2 \leqslant \frac{1}{2}\|SS^* + S^*S\| + \|\Re S\|\,\|\Im S\|. \tag{2.2}$$

Now, from (1.3) and (1.4), we obtain

$$\begin{aligned}
\frac{1}{2}\|S\|^2 &\leqslant \frac{1}{4}\|SS^* + S^*S\| + \frac{1}{2}\|\Re S\|\,\|\Im S\| \\
&\leqslant \omega^2(S) + \frac{1}{2}\|\Re S\|\,\|\Im S\| - \frac{1}{2}m\left(S^2\right) \\
&\leqslant \frac{3}{2}\omega^2(S) - \frac{1}{2}m\left(S^2\right),
\end{aligned}$$

as required. □

We obtain the well-known inequality

$$\omega(ST) \leqslant 4\omega(S)\,\omega(T),\ S,T \in \mathbb{B}(\mathbb{H})$$

from (1.1) and the fact that the operator norm is sub-multiplicative. We can enhance this approximation under specific conditions, as demonstrated in the next result.

COROLLARY 2.1. *Let $S,T \in \mathbb{B}(\mathbb{H})$. If*

(i) S and T are accretive,
or

(ii) S and T are dissipative,
or

(iii) S is accretive (resp. dissipative) and T is dissipative (resp. accretive),

then

$$\omega(ST) \leqslant 3\omega(S)\,\omega(T) - \sqrt{m(S^2)\,m(T^2)}.$$

Proof. It follows from Theorem 2.1 that

$$\begin{aligned}\omega^2(ST) &\leqslant \|ST\|^2\\ &\leqslant \|S\|^2\|T\|^2\\ &\leqslant \left(3\omega^2(S)-m\left(S^2\right)\right)\left(3\omega^2(T)-m\left(T^2\right)\right)\\ &= \left(\left(\sqrt{3}\omega(S)\right)^2-\left(\sqrt{m(S^2)}\right)^2\right)\left(\left(\sqrt{3}\omega(T)\right)^2-\left(\sqrt{m(T^2)}\right)^2\right)\\ &\leqslant \left(3\omega(S)\,\omega(T)-\sqrt{m(S^2)\,m(T^2)}\right)^2,\end{aligned}$$

where the last inequality is obtained from the simple fact that $\left(a^2-b^2\right)\left(c^2-d^2\right) \leqslant (ac-bd)^2$, which holds for $a,b,c,d \in \mathbb{R}$. This completes the proof. □

Now, we are ready to prove the first result.

THEOREM 2.2. *Let $S \in \mathbb{B}(\mathbb{H})$ be an accretive operator. Then*

$$\frac{1}{2}\sqrt{\|S\|^2+\|\Re S\|\,\|\Im S\|} \leqslant \omega(S).$$

The above inequality is also true when S is a dissipative operator.

Proof. It follows from (2.1) that

$$\|S\|^2 \leqslant (\|\Re S\|+\|\Im S\|)^2-\|\Re S\|\,\|\Im S\|,$$

which is equivalent to

$$\|S\|^2+\|\Re S\|\,\|\Im S\| \leqslant (\|\Re S\|+\|\Im S\|)^2.$$

From this, we infer that

$$\|S\|^2+\|\Re S\|\,\|\Im S\| \leqslant 4\omega^2(S),$$

as required. □

Similar to Corollary 2.1, we obtain the following estimate for the numerical radii of the product of two operators.

COROLLARY 2.2. *Let $S,T \in \mathbb{B}(\mathbb{H})$. If*

(i) S and T are accretive,
or

(ii) S and T are dissipative,
or

(iii) S is accretive (resp. dissipative) and T is dissipative (resp. accretive),

then

$$\omega(ST) \leqslant 4\omega(S)\,\omega(T) - \sqrt{\|\Re S\|\,\|\Im S\|\,\|\Re T\|\,\|\Im T\|}.$$

It has been shown in [12] that if S is accretive-dissipative operator, then

$$\|S\|^2 \leqslant \|\Re S\|^2 + \|\Im S\|^2,$$

which is equivalent to

$$\|S\|^2 + 2\|\Re S\|\,\|\Im S\| \leqslant (\|\Re S\| + \|\Im S\|)^2.$$

Employing this inequality and using (1.4), we can infer the following result for accretive-dissipative operators.

THEOREM 2.3. *Let $S \in \mathbb{B}(\mathbb{H})$ be an accretive-dissipative operator. Then*

$$\frac{1}{2}\sqrt{\|S\|^2 + 2\|\Re S\|\,\|\Im S\|} \leqslant \omega(S).$$

As a direct consequence of the previous theorem, we have the following result.

COROLLARY 2.3. *Let $S,T \in \mathbb{B}(\mathbb{H})$ be two accretive-dissipative operators. Then*

$$\omega(ST) \leqslant 4\omega(S)\,\omega(T) - 2\sqrt{\|\Re S\|\,\|\Im S\|\,\|\Re T\|\,\|\Im T\|}.$$

The inequality $\left\|S^2\right\| \leqslant \|S\|^2$ for any bounded linear operator $S \in \mathbb{B}(\mathbb{H})$ is a fundamental result in operator theory. As a consequence, it is important to establish upper bounds for the nonnegative difference $\|S\|^2 - \left\|S^2\right\|$ under various assumptions. In the following result, we aim to derive such an inequality.

THEOREM 2.4. *Let $S \in \mathbb{B}(\mathbb{H})$ be an accretive operator. Then*

$$\|S\|^2 - \left\|S^2\right\| \leqslant 2\|\Re S\|\,\|\Im S\|. \tag{2.3}$$

The above inequality is also true when S is a dissipative operator.

Proof. Notice that

$$\begin{aligned}\|(\Re S)(\Im S) - (\Im S)(\Re S)\| &= \left\|\left(\frac{S+S^*}{2}\right)\left(\frac{S-S^*}{2i}\right) - \left(\frac{S-S^*}{2i}\right)\left(\frac{S+S^*}{2}\right)\right\| \\ &= \frac{1}{2}\|S^*S - SS^*\|.\end{aligned}$$

Therefore, from Lemma 1.1, we have

$$\begin{aligned}\|S^*S - SS^*\| &= 2\|(\Re S)(\Im S) - (\Im S)(\Re S)\| \\ &\leqslant 2\|\Re S\|\,\|\Im S\|.\end{aligned}$$

On the other hand, Lemma 1.2 ensures that

$$\begin{aligned}\|S\|^2-\left\|S^2\right\| &= \|S\|^2-\left\|(S^*S)^{\frac{1}{2}}(SS^*)^{\frac{1}{2}}\right\| \\ &= \max\{\|S^*S\|,\|SS^*\|\}-\left\|(S^*S)^{\frac{1}{2}}(SS^*)^{\frac{1}{2}}\right\| \\ &\leqslant \|S^*S-SS^*\|,\end{aligned}$$

due to the fact that $\|\,|A|\,|B|\,\|=\|AB^*\|$ for any $A,B\in\mathbb{B}(\mathbb{H})$ (see, e.g., [9, (8)]). $\square$

REMARK 2.1. It follows from (2.2) and (2.3) that

$$\|S\|^2\leqslant\|\Re S\|\,\|\Im S\|+\min\left\{\|\Re S\|\,\|\Im S\|+\left\|S^2\right\|,\frac{1}{2}\|SS^*+S^*S\|\right\}.$$

The following result concerning the unitarily invariant norms may be stated.

THEOREM 2.5. *Let $S\in\mathbb{B}(\mathbb{H})$.*

- *If S is an accretive operator, then*

$$|||SS^*|||\leqslant\left|\left|\left|(\Re S)^2+(\Im S)^2\right|\right|\right|+\|\Re S\|\ |||\Im S|||. \tag{2.4}$$

- *If S is a dissipative operator, then*

$$|||SS^*|||\leqslant\left|\left|\left|(\Re S)^2+(\Im S)^2\right|\right|\right|+\|\Im S\|\ |||\Re S|||.$$

Proof. Assume that S is accretive. It follows from Lemma 1.3 that

$$\begin{aligned}|||SS^*||| &= |||(\Re S+i\Im S)(\Re S-i\Im S)||| \\ &= \left|\left|\left|(\Re S)^2+(\Im S)^2-i\left((\Re S)(\Im S)-(\Im S)(\Re S)\right)\right|\right|\right| \\ &= \left|\left|\left|(\Re S)^2+(\Im S)^2\right|\right|\right|+|||(\Re S)(\Im S)-(\Im S)(\Re S)||| \\ &\leqslant \left|\left|\left|(\Re S)^2+(\Im S)^2\right|\right|\right|+\|\Re S\|\ |||\Im S|||\end{aligned}$$

i.e.,

$$|||SS^*|||\leqslant\left|\left|\left|(\Re S)^2+(\Im S)^2\right|\right|\right|+\|\Re S\|\ |||\Im S|||,$$

which completes the inequality in the first case.

Now, assume that S is dissipative. If we apply the same method as in the above, we can write

$$|||SS^*|||\leqslant\left|\left|\left|(\Re S)^2+(\Im S)^2\right|\right|\right|+\|\Im S\|\ |||\Re S|||,$$

thanks to Lemma 1.3. $\square$

REMARK 2.2. Inequalities in Theorem 2.5 can be written in the following form.

- If S is an accretive operator, then

$$|||SS^*||| \leqslant \frac{1}{2}|||S^*S+SS^*||| + \|\Re S\|\,|||\Im S|||.$$

- If S is a dissipative operator, then

$$|||SS^*||| \leqslant \frac{1}{2}|||S^*S+SS^*||| + \|\Im S\|\,|||\Re S|||.$$

We can conclude the following result as a direct consequence of Theorem 2.5.

COROLLARY 2.4. *Let $S \in \mathbb{B}(\mathbb{H})$ be an accretive dissipative operator. Then*

$$|||SS^*||| \leqslant \left|\left|\left|(\Re S)^2 + (\Im S)^2\right|\right|\right| + \min\{\|\Re S\| \;\; |||\Im S|||, \|\Im S\| \;\; |||\Re S|||\}.$$

Or in the equivalent form,

$$|||SS^*||| \leqslant \frac{1}{2}|||S^*S+SS^*||| + \min\{\|\Re S\|\,|||\Im S|||, \|\Im S\|\,|||\Re S|||\}.$$

Theorem 2.5 also enables us to prove the following result.

THEOREM 2.6. *Let $S \in \mathbb{B}(\mathbb{H})$ be an accretive operator. Then*

$$|||SS^*||| \leqslant \omega_{|||\cdot|||}(S)\left(2\omega_{|||\cdot|||}(S) + \omega(S)\right).$$

The above inequality is also true when S is a dissipative operator.

Proof. Assume that S is accretive. In this case, (2.4) holds. It observes from the triangle inequality for the unitarily invariant norm, (1.5), and (1.4) that

$$\begin{aligned}
|||SS^*||| &\leqslant \left|\left|\left|(\Re S)^2 + (\Im S)^2\right|\right|\right| + \|\Re S\| \;\; |||\Im S||| \\
&\leqslant \left|\left|\left|(\Re S)^2\right|\right|\right| + \left|\left|\left|(\Im S)^2\right|\right|\right| + \|\Re S\| \;\; |||\Im S||| \\
&\leqslant 2\omega^2_{|||\cdot|||}(S) + \omega_{|||\cdot|||}(S) \;\; \|\Re S\| \\
&\leqslant 2\omega^2_{|||\cdot|||}(S) + \omega_{|||\cdot|||}(S) \;\; \omega(S) \\
&= \omega_{|||\cdot|||}(S)\left(2\omega_{|||\cdot|||}(S) + \omega(S)\right)
\end{aligned}$$

i.e.,

$$|||SS^*||| \leqslant \omega_{|||\cdot|||}(S)\left(2\omega_{|||\cdot|||}(S) + \omega(S)\right).$$

The same result also holds when S is dissipative. □

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