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# BILINEAR FRACTIONAL INTEGRAL OPERATORS ON GRAND MORREY SPACES AND GRAND HARDY–MORREY SPACES

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*Abstract.* We establish the boundedness of bilinear fractional integral operators on grand Morrey spaces and grand Hardy-Morrey spaces. Our approach combines a refined extrapolation method with sparse domination techniques, extending recent results on linear fractional integrals to the multilinear setting. The key innovation is the adaptation of the extrapolation machinery to handle the nonlinear nature of bilinear operators while preserving the delicate balance between the fractional parameter and the integrability indices. As applications, we obtain bilinear Sobolev embeddings and fractional Leibniz rules in the grand Morrey space framework.

## 1. Introduction

The study of fractional integral operators has profound connections to potential theory, partial differential equations, and mathematical physics. In the classical setting, the fractional integral operator $I_\alpha$ of order $0 < \alpha < n$ acts on suitable functions $f$ on $\mathbb{R}^n$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

This operator arises naturally in the analysis of fractional Laplacians, the study of Sobolev spaces of fractional order, and in modeling physical phenomena involving long-range interactions [17, 1].

The multilinear theory of fractional integrals emerged from the pioneering work of Grafakos [4] and Kenig-Stein [11], who introduced the $m$-linear fractional integral operator

$$I_\alpha(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^{mn}} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}} dy_1\cdots dy_m.$$

These operators appear in the study of multilinear PDEs, particularly in the analysis of products of solutions to elliptic equations and in the theory of compensated compactness [2, 3].

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The grand Lebesgue spaces $L^{p)}(\mathbb{R}^n)$ were introduced by Iwaniec and Sbordone [10] in their study of the integrability of the Jacobian determinant. These spaces have since found applications in the regularity theory of PDEs with nonstandard growth conditions [6, 16]. The grand Morrey spaces $M_u^{p)}(\mathbb{R}^n)$ represent a natural synthesis of the grand Lebesgue space structure with the Morrey space framework, providing refined control over both local and global integrability properties [8].

The motivation for studying bilinear fractional integrals on grand Morrey spaces stems from several sources. From the PDE perspective, these estimates are crucial for understanding the regularity of products of solutions to fractional elliptic equations. In harmonic analysis, they provide sharp endpoint estimates that complement the classical multilinear theory. Moreover, the grand Morrey space setting captures borderline phenomena that arise in critical scaling problems.

Our main contribution is the establishment of boundedness results for the bilinear fractional integral operator

$$I_\alpha(f_1,f_2)(x) = \int_{\mathbb{R}^{2n}} \frac{f_1(y_1)f_2(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}dy_1dy_2$$

on grand Morrey spaces and grand Hardy-Morrey spaces. The principal challenge lies in adapting the extrapolation machinery to handle the nonlinear nature of the bilinear operator while maintaining precise control over the interplay between the fractional parameter $\alpha$ and the various integrability indices.

Our approach synthesizes recent advances in two directions: the extrapolation theory for grand Morrey spaces developed by Ho [7] and the multilinear analysis on generalized Orlicz spaces by Wang [19]. A key observation is that the extrapolation method of [7] does not rely on the linearity of the operator, allowing us to treat bilinear operators within the same framework.

Our extrapolation argument ultimately descends from the pioneering work of Rubio de Francia on factorization and $A_p$ weights [13, 14, 15]. We adapt his scheme, via the grand-Morrey variant of Ho [7, 8], to the bilinear setting developed in Theorem 3.3.

Recent developments in the theory of grand function spaces include extensions to other classical scales, such as the grand Triebel–Lizorkin–Morrey spaces introduced by Ho [9], which further demonstrate the versatility and importance of the grand space framework in modern harmonic analysis.

The paper is organized as follows. Section 2 establishes the necessary preliminaries on grand Morrey spaces and weighted norm inequalities. Section 3 contains our main results on the boundedness of bilinear fractional integrals on grand Morrey spaces. Section 4 extends these results to the grand Hardy-Morrey space setting. Finally, Section 5 presents applications to bilinear Sobolev embeddings.

## 2. Preliminaries

### 2.1. Grand Lebesgue and grand Morrey spaces

We begin by recalling the definition of grand Lebesgue spaces. Let $\mathscr{M}(\mathbb{R}^n)$ denote the class of Lebesgue measurable functions on $\mathbb{R}^n$. For $f \in \mathscr{M}(\mathbb{R}^n)$ and a measurable set $B \subset \mathbb{R}^n$, we define the distribution function

$$d_{f,B}(s) = \frac{1}{|B|}|\{x \in B : |f(x)| > s\}|$$

and the non-increasing rearrangement

$$f_B^*(t) = \inf\{s > 0 : d_{f,B}(s) \leqslant t\}, \quad t > 0.$$

DEFINITION 2.1. Let $p \in (0,\infty)$ and $B \subset \mathbb{R}^n$ be a measurable set with $0 < |B| < \infty$. The grand Lebesgue space $L^{p)}(B)$ consists of all $f \in \mathscr{M}(\mathbb{R}^n)$ such that

$$\|f\|_{L^{p)}(B)} = \sup_{0<t<1} (1-\ln t)^{-\frac{1}{p}} \left(\int_0^t (f_B^*(s))^p ds\right)^{\frac{1}{p}} < \infty.$$

The small Lebesgue space $L^{(p}(B)$ is defined as the predual of $L^{p')}(B)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. For $p \in (1,\infty)$, these spaces satisfy the duality relation

$$\int_B |f(x)g(x)|dx \leqslant C\|f\|_{L^{p)}(B)}\|g\|_{L^{(p'}(B)}$$

for some constant $C > 0$.

DEFINITION 2.2. Let $p \in (0,\infty)$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. The grand Morrey space $M_u^{p)}(\mathbb{R}^n)$ consists of all $f \in \mathscr{M}(\mathbb{R}^n)$ such that

$$\|f\|_{M_u^{p)}(\mathbb{R}^n)} = \sup_{B(x,r)\subset\mathbb{R}^n} \frac{1}{u(x,r)}\|f\|_{L^{p)}(B(x,r))} < \infty,$$

where the supremum is taken over all balls $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$.

We will also use the notation $u(B) = u(x,r)$ for a ball $B = B(x,r)$. The function $u$ controls the local-to-global behavior of functions in the space. Classical examples include:

- $u(x,r) = r^{n(1-\lambda/p)}$ for $0 \leqslant \lambda < p$ embeds $M^u_{p)}(\mathbb{R}^n)$ isometrically into the classical Morrey space $M^{p,\lambda}(\mathbb{R}^n)$; the inclusion is strict unless $\lambda = 0$.
- Because the Luxemburg-type norm defining $L^{p)}(B)$ requires $|B| < \infty$, the grand Lebesgue space $L^{p)}(\mathbb{R}^n)$ cannot be meaningfully defined on the entire Euclidean space. This motivates the introduction of grand Morrey spaces, which circumvent this limitation by localizing the $L^{p)}$ norm on balls and then taking the supremum over all radii.

- For the same reason, there is no meaningful 'grand Hardy space' on $\mathbb{R}^n$; only its Morrey-type grand analogue $HM_p^u(\mathbb{R}^n)$ exists.

### 2.2. Muckenhoupt weights and weighted inequalities

DEFINITION 2.3. For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ belongs to the Muckenhoupt class $A_p$ if

$$[\omega]_{A_p} = \sup_B \left( \frac{1}{|B|} \int_B \omega(x) dx \right) \left( \frac{1}{|B|} \int_B \omega(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $p' = \frac{p}{p-1}$.

A weight $\omega$ belongs to $A_1$ if there exists $C > 0$ such that for almost every $x \in B$,

$$\frac{1}{|B|} \int_B \omega(y) dy \leqslant C\omega(x)$$

for all balls $B$ containing $x$. We denote $A_\infty = \bigcup_{p \geqslant 1} A_p$.

### 2.3. Small block spaces

The predual structure of grand Morrey spaces is characterized through small block spaces.

DEFINITION 2.4. Let $p \in (1,\infty)$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. A measurable function $b$ is a small $(p,u)$-block if there exists a ball $B$ such that $\operatorname{supp} b \subset B$ and

$$\|b\|_{L^{(p'}(B)} \leqslant \frac{1}{u(B)|B|}.$$

The small block space $\mathscr{B}_u^{(p'}(\mathbb{R}^n)$ consists of all $f \in \mathscr{M}(\mathbb{R}^n)$ that can be represented as

$$f = \sum_{i=1}^{\infty} \lambda_i b_i$$

where each $b_i$ is a small $(p,u)$-block and $\sum_{i=1}^{\infty} |\lambda_i| < \infty$. The norm is defined as

$$\|f\|_{\mathscr{B}_u^{(p'}(\mathbb{R}^n)} = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i| : f = \sum_{i=1}^{\infty} \lambda_i b_i \right\}.$$

## 3. Bilinear fractional integrals on grand Morrey spaces

### 3.1. Statement of main results

Our main result establishes the boundedness of bilinear fractional integral operators on grand Morrey spaces.

THEOREM 3.1. *Let* $0<\alpha<2n$, $p_1,p_2\in(1,\frac{n}{\alpha})$, *and define* $q$ *by*

$$\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}.$$

*Let* $u:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ *be such that there exist* $q_0\in(1,q)$ *and* $\theta\in(1,(q/q_0)')$ *satisfying:*

$$\begin{gathered}
C<(u(x,r)r^{\alpha})^{q_0}r^{n(\theta-1)}, \quad r\leqslant 1, x\in\mathbb{R}^n,\\
Cr^{-nq_0/q}<(u(x,r)r^{\alpha})^{q_0}r^{n(\theta-1)}, \quad r>1, x\in\mathbb{R}^n,\\
\sum_{k=0}^{\infty}(u(2^kB)|2^kB|^{\alpha/n})^{p_0}|2^kB|^{(p_0/q_0)/\theta}\leqslant C(u(B)|B|^{\alpha/n})^{p_0}|B|^{(p_0/q_0)/\theta}, \quad \forall B,
\end{gathered}$$

*where* $\frac{1}{p_0}=\frac{1}{q_0}+\frac{\alpha}{n}$ *and* $C>0$ *is independent of* $x$, $r$, *and* $B$. *Then the bilinear fractional integral operator*

$$I_\alpha(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}\frac{f_1(y_1)f_2(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}dy_1dy_2$$

*is bounded from* $M_u^{p_1)}(\mathbb{R}^n)\times M_u^{p_2)}(\mathbb{R}^n)$ *to* $M_u^{q)}(\mathbb{R}^n)$ *with*

$$\|I_\alpha(f_1,f_2)\|_{M_u^{q)}(\mathbb{R}^n)}\leqslant C\|f_1\|_{M_u^{p_1)}(\mathbb{R}^n)}\|f_2\|_{M_u^{p_2)}(\mathbb{R}^n)}.$$

### 3.2. Weighted norm inequalities

We first establish the necessary weighted norm inequalities for bilinear fractional integrals.

LEMMA 3.2. *Let* $0<\alpha<2n$, $p_0\in(1,\frac{n}{\alpha})$, $\frac{1}{p_0}=\frac{1}{q_0}+\frac{\alpha}{n}$, *and* $\omega\in A_1$. *Then there exists* $C>0$ *such that*

$$\left(\int_{\mathbb{R}^n}|I_\alpha(f_1,f_2)(x)|^{q_0}\omega(x)dx\right)^{1/q_0}\leqslant C\prod_{j=1}^{2}\left(\int_{\mathbb{R}^n}|f_j(x)|^{p_0}\omega(x)^{p_0/q_0}dx\right)^{1/p_0}.$$

*Proof.* The proof follows by combining the pointwise estimate

$$I_\alpha(f_1,f_2)(x)\leqslant CI_{\alpha/2}(|f_1|)(x)I_{\alpha/2}(|f_2|)(x)$$

with the weighted norm inequality for the linear fractional integral operator. For $\omega \in A_1$, we have $\omega^{p_0/q_0} \in A_{p_0}$ since

$$[\omega^{p_0/q_0}]_{A_{p_0}} \leqslant C[\omega]_{A_1}^{p_0/q_0}.$$

By the weighted boundedness of $I_{\alpha/2}$ (see [12]), we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |I_\alpha(f_1,f_2)(x)|^{q_0}\omega(x)dx\right)^{1/q_0}\\
&\leqslant C\left(\int_{\mathbb{R}^n} |I_{\alpha/2}(|f_1|)(x)|^{2q_0}|I_{\alpha/2}(|f_2|)(x)|^{2q_0}\omega(x)dx\right)^{1/q_0}\\
&\leqslant C\|I_{\alpha/2}(|f_1|)\|_{L^{2p_0}(\omega^{p_0/q_0})}\|I_{\alpha/2}(|f_2|)\|_{L^{2p_0}(\omega^{p_0/q_0})}\\
&\leqslant C\prod_{j=1}^{2}\|f_j\|_{L^{p_0}(\omega^{p_0/q_0})},
\end{aligned}$$

where we used Hölder's inequality with exponents $2q_0/q_0 = 2$ in the second step. □

### 3.3. Extrapolation on grand Morrey spaces

The key technical tool is an extrapolation theorem adapted to the bilinear setting.

THEOREM 3.3. *Let* $\alpha \in [0,2n)$, $0 \leqslant p_0 \leqslant q_0 < \infty$ *with* $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$. *Let* $p_0 < p_1, p_2$ *with* $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $\theta \in (1,(q/q_0)')$. *Suppose* $u$ *satisfies the conditions in Theorem 3.1.*

*Let* $(f_1,f_2) \in M_u^{p_1)}(\mathbb{R}^n) \times M_u^{p_2)}(\mathbb{R}^n)$ *and* $g$ *be measurable. If for any* $\omega \in \{M_\theta h : h \in b_{u^{q_0}}^{((q/q_0)')}\}$,

$$\left(\int_{\mathbb{R}^n} |g(x)|^{q_0}\omega(x)dx\right)^{1/q_0} \leqslant C\prod_{j=1}^{2}\left(\int_{\mathbb{R}^n} |f_j(x)|^{p_0}\omega(x)^{p_0/q_0}dx\right)^{1/p_0} < \infty,$$

*then* $g \in M_u^{q)}(\mathbb{R}^n)$ *and*

$$\|g\|_{M_u^{q)}(\mathbb{R}^n)} \leqslant C_0\|f_1\|_{M_u^{p_1)}(\mathbb{R}^n)}\|f_2\|_{M_u^{p_2)}(\mathbb{R}^n)}$$

*for some* $C_0 > 0$ *independent of* $f_1, f_2, g$.

*Proof.* Let $h \in b_{u^{q_0}}^{((q/q_0)')}$. By definition, there exists a ball $B$ such that $\operatorname{supp} h \subset B$ and

$$\|h\|_{L^{((q/q_0)')'}(B)} \leqslant \frac{1}{(u(B)|B|^{\alpha/n})^{q_0}|B|^{\theta-1}}.$$

Since $h$ is a small block, we have $|h| \leqslant M_\theta h$. As $\theta < (q/q_0)'$, Theorem 2.6 of [7] implies that there exist $\{\lambda_k\}_{k=0}^\infty$ and $\{d_k\}_{k=0}^\infty$ with

$$M_\theta h = \sum_{k=0}^{\infty} \lambda_k d_k,$$

where $\lambda_k = C_0 \frac{v(2^{k+1}B)}{v(B)}$, $v(B) = u(B)^{q_0}|B|^{\theta-1}$, and $d_k = \lambda_k^{-1}\chi_{2^{k+1}B\setminus 2^kB}M(|h|^\theta)$ satisfies $\{d_k\}_{k=0}^\infty \subset b_v^{((q/q_0)')/\theta}$.

For $(p_0/q_0)/\theta < 1$, we obtain

$$(M_\theta h)^{p_0/q_0} = \left(\sum_{k=0}^\infty \lambda_k d_k\right)^{(p_0/q_0)/\theta} \leqslant \sum_{k=0}^\infty |\lambda_k|^{(p_0/q_0)/\theta}|d_k|^{(p_0/q_0)/\theta}.$$

Since $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$, we have $\frac{q}{q_0}\cdot\frac{q_0}{p_0} = \frac{p_1}{p_0}\cdot\frac{p_2}{p_0}$. This yields

$$v(B)^{(p_0/q_0)/\theta}|B|^{(p_0/q_0)/\theta} = u(B)^{p_0}|B|.$$

Therefore, $|d_k|^{(p_0/q_0)/\theta} \in b_{u^{p_0}}^{((p_j/p_0)')}$ for $j = 1,2$. By the third condition on $u$,

$$\sum_{k=0}^\infty |\lambda_k|^{(p_0/q_0)/\theta} < C$$

for some $C > 0$ independent of $h$. Thus $(M_\theta h)^{p_0/q_0} \in \mathscr{B}_{u^{p_0}}^{((p_j/p_0)')}(\mathbb{R}^n)$ with uniformly bounded norm.

Now, applying Proposition 2.3 of [7] and the hypothesis,

$$\begin{aligned}\int_{\mathbb{R}^n}|g(x)|^{q_0}|h(x)|dx &\leqslant \int_{\mathbb{R}^n}|g(x)|^{q_0}M_\theta h(x)dx \\ &\leqslant C\prod_{j=1}^2\left(\int_{\mathbb{R}^n}|f_j(x)|^{p_0}(M_\theta h(x))^{p_0/q_0}dx\right)^{q_0/p_0} \\ &\leqslant C\prod_{j=1}^2 \||f_j|^{p_0}\|^{q_0/p_0}_{M_{u^{p_0}}^{p_j/p_0)}(\mathbb{R}^n)}\|(M_\theta h)^{p_0/q_0}\|^{q_0/p_0}_{\mathscr{B}_{u^{p_0}}^{((p_j/p_0)')}(\mathbb{R}^n)} \\ &\leqslant C\prod_{j=1}^2\|f_j\|^{q_0}_{M_u^{p_j)}(\mathbb{R}^n)}.\end{aligned}$$

Taking the supremum over $h \in b_{u^{q_0}}^{((q/q_0)')}$ and applying Proposition 2.2 of [7], we conclude that $|g|^{q_0} \in M_{u^{q_0}}^{q/q_0)}(\mathbb{R}^n)$ with

$$\||g|^{q_0}\|_{M_{u^{q_0}}^{q/q_0)}(\mathbb{R}^n)} \leqslant C\prod_{j=1}^2\|f_j\|^{q_0}_{M_u^{p_j)}(\mathbb{R}^n)}.$$

By the scaling property of grand Morrey spaces, $\|g\|_{M_u^{q)}(\mathbb{R}^n)} \leqslant C\prod_{j=1}^2\|f_j\|_{M_u^{p_j)}(\mathbb{R}^n)}$. □

The extrapolation technique employed in this theorem has its roots in the seminal work of Rubio de Francia [13, 14, 15] on factorization and weighted norm inequalities, which has been adapted here to the grand Morrey space setting for bilinear operators.

### 3.4. Proof of main theorem

*Proof of Theorem* 3.1. We establish the boundedness of the bilinear fractional integral operator $I_\alpha$ by applying the extrapolation machinery developed in Theorem 3.3. The strategy is to verify that $I_\alpha(f_1,f_2)$ satisfies the weighted norm inequality required in the hypothesis of the extrapolation theorem.

Let $(f_1,f_2)\in M_u^{p_1)}(\mathbb{R}^n)\times M_u^{p_2)}(\mathbb{R}^n)$. To apply Theorem 3.3 with $g=I_\alpha(f_1,f_2)$, we must show that for every weight $\omega$ of the form $\omega=M_\theta h$ where $h\in b_{u^{q_0}}^{((q/q_0)')}$, the weighted norm inequality

$$\left(\int_{\mathbb{R}^n}|I_\alpha(f_1,f_2)(x)|^{q_0}\omega(x)dx\right)^{1/q_0}\leqslant C\prod_{j=1}^{2}\left(\int_{\mathbb{R}^n}|f_j(x)|^{p_0}\omega(x)^{p_0/q_0}dx\right)^{1/p_0}$$

holds with a constant $C$ independent of the particular choice of $h$.

First, we examine the structure of the weight $\omega=M_\theta h$. Since $h$ belongs to the small block space $b_{u^{q_0}}^{((q/q_0)')}$, there exists a ball $B$ such that $\operatorname{supp} h\subset B$ and

$$\|h\|_{L^{((q/q_0)')'}(B)}\leqslant\frac{1}{(u(B)|B|^{\alpha/n})^{q_0}|B|^{\theta-1}}.$$

The operator $M_\theta$ is the fractional maximal function defined by $M_\theta f=[M(|f|^\theta)]^{1/\theta}$, where $M$ is the Hardy-Littlewood maximal operator.

A crucial observation is that $M_\theta h$ belongs to the Muckenhoupt class $A_1$. This follows from Theorem 9.2.8 of [5], which states that if $0<\theta<1$ and $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $[M(|f|^\theta)]^{1/\theta}\in A_1$ with

$$\left[[M(|f|^\theta)]^{1/\theta}\right]_{A_1}\leqslant C_\theta,$$

where $C_\theta$ depends only on $\theta$ and the dimension $n$. Since our parameter $\theta>1$, we use the fact that $M_\theta h=M(M_{(\theta-1)}h)$, and the composition preserves the $A_1$ property.

With $\omega=M_\theta h\in A_1$ established, we can now apply Lemma 3.2. Setting $\omega=M_\theta h$ in the lemma yields

$$\left(\int_{\mathbb{R}^n}|I_\alpha(f_1,f_2)(x)|^{q_0}M_\theta h(x)dx\right)^{1/q_0}\leqslant C\prod_{j=1}^{2}\left(\int_{\mathbb{R}^n}|f_j(x)|^{p_0}(M_\theta h(x))^{p_0/q_0}dx\right)^{1/p_0},$$

where the constant $C$ depends on $[M_\theta h]_{A_1}$, which is uniformly bounded for all $h\in b_{u^{q_0}}^{((q/q_0)')}$.

This establishes that the hypothesis of Theorem 3.3 is satisfied with $g=I_\alpha(f_1,f_2)$. The extrapolation theorem then guarantees that $I_\alpha(f_1,f_2)\in M_u^{q)}(\mathbb{R}^n)$ with

$$\|I_\alpha(f_1,f_2)\|_{M_u^{q)}(\mathbb{R}^n)}\leqslant C_0\|f_1\|_{M_u^{p_1)}(\mathbb{R}^n)}\|f_2\|_{M_u^{p_2)}(\mathbb{R}^n)},$$

where $C_0$ depends on the parameters $n$, $\alpha$, $p_1$, $p_2$, $q$, $\theta$, and the constants in the conditions on $u$, but is independent of $f_1$ and $f_2$.

To complete the proof, we verify that the parameter constraints are consistent. The relation $\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$ combined with $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$ ensures that the scaling relations in the extrapolation theorem are satisfied. The condition $\theta \in (1, (q/q_0)')$ guarantees that the small block decomposition in the proof of Theorem 3.3 converges. Finally, the three conditions on $u$ ensure that the necessary summability and growth properties hold throughout the extrapolation process. $\square$

## 4. Grand Hardy-Morrey spaces

### 4.1. Definition and basic properties

Let $\mathscr{F} = \{\|\cdot\|_{\alpha,\beta}\}$ be a finite collection of semi-norms on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and define

$$\mathscr{S}_{\mathscr{F}} = \{\psi \in \mathscr{S} : \|\psi\|_{\alpha,\beta} \leqslant 1 \text{ for all } \|\cdot\|_{\alpha,\beta} \in \mathscr{F}\}.$$

For $f \in \mathscr{S}'(\mathbb{R}^n)$, the grand maximal function is

$$\mathscr{M}_{\mathscr{F}} f(x) = \sup_{\psi \in \mathscr{S}_{\mathscr{F}}} \sup_{t>0} |(f * \psi_t)(x)|,$$

where $\psi_t(x) = t^{-n}\psi(x/t)$.

DEFINITION 4.1. Let $p \in (0,1]$ and $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. The grand Hardy-Morrey space $HM_u^{p)}(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{HM_u^{p)}(\mathbb{R}^n)} = \|\mathscr{M}_{\mathscr{F}} f\|_{M_u^{p)}(\mathbb{R}^n)} < \infty.$$

The grand Hardy-Morrey spaces extend both the classical Hardy spaces $H^p(\mathbb{R}^n)$ (when $u \equiv 1$) and the Hardy-Morrey spaces. The use of the grand maximal function $\mathscr{M}_{\mathscr{F}}$ ensures that these spaces have desirable properties such as completeness and appropriate duality relations.

THEOREM 4.2. *Under the same conditions as Theorem 3.1, the bilinear fractional integral operator $I_\alpha$ is bounded from $HM_u^{p_1)}(\mathbb{R}^n) \times HM_u^{p_2)}(\mathbb{R}^n)$ to $HM_u^{q)}(\mathbb{R}^n)$.*

*Proof.* The proof proceeds by establishing that the grand maximal function of $I_\alpha(f_1, f_2)$ satisfies the same estimates as $I_\alpha(f_1, f_2)$ itself, allowing us to reduce to the case already treated in Theorem 3.1.

For $(f_1, f_2) \in HM_u^{p_1)}(\mathbb{R}^n) \times HM_u^{p_2)}(\mathbb{R}^n)$, we need to show

$$\|\mathscr{M}_{\mathscr{F}} I_\alpha(f_1, f_2)\|_{M_u^{q)}(\mathbb{R}^n)} \leqslant C\|f_1\|_{HM_u^{p_1)}(\mathbb{R}^n)} \|f_2\|_{HM_u^{p_2)}(\mathbb{R}^n)}.$$

By the definition of the grand Hardy-Morrey norm, this is equivalent to proving

$$\|\mathscr{M}_{\mathscr{F}} I_\alpha(f_1, f_2)\|_{M_u^{q)}(\mathbb{R}^n)} \leqslant C\|\mathscr{M}_{\mathscr{F}} f_1\|_{M_u^{p_1)}(\mathbb{R}^n)} \|\mathscr{M}_{\mathscr{F}} f_2\|_{M_u^{p_2)}(\mathbb{R}^n)}.$$

To establish this, we follow the extrapolation approach of Theorem 3.1. For any $h \in b_{u^{q_0}}^{((q/q_0)')}$, we must verify

$$\left(\int_{\mathbb{R}^n} (\mathcal{M}_{\mathcal{F}} I_\alpha(f_1,f_2)(x))^{q_0} M_\theta h(x)dx\right)^{1/q_0} \leqslant C\prod_{j=1}^{2}\left(\int_{\mathbb{R}^n} (\mathcal{M}_{\mathcal{F}} f_j(x))^{p_0}(M_\theta h(x))^{p_0/q_0}dx\right)^{1/p_0}.$$

The key observation is that for weighted Hardy spaces, there exists a fundamental inequality relating the grand maximal function of $I_\alpha(f_1,f_2)$ to the grand maximal functions of $f_1$ and $f_2$. Specifically, by Theorem 8.1 of [18], for $\omega \in A_1$, we have

$$\int_{\mathbb{R}^n} (\mathcal{M}_{\mathcal{F}} I_\alpha(f_1,f_2)(x))^{q_0}\omega(x)dx \leqslant C\int_{\mathbb{R}^n} (I_\alpha(\mathcal{M}_{\mathcal{F}} f_1,\mathcal{M}_{\mathcal{F}} f_2)(x))^{q_0}\omega(x)dx,$$

where the constant $C$ depends on $[\omega]_{A_1}$ but is independent of $f_1$ and $f_2$.

This inequality is non-trivial and relies on the vector-valued extension of the Calderón-Zygmund theory. The proof uses the fact that the kernel of $I_\alpha$ satisfies appropriate smoothness conditions that allow the maximal function to be controlled by the fractional integral of the maximal functions.

Combining this with Lemma 3.2 applied to $\mathcal{M}_{\mathcal{F}} f_1$ and $\mathcal{M}_{\mathcal{F}} f_2$, we obtain

$$\begin{aligned}&\left(\int_{\mathbb{R}^n} (\mathcal{M}_{\mathcal{F}} I_\alpha(f_1,f_2)(x))^{q_0} M_\theta h(x)dx\right)^{1/q_0}\\ &\leqslant C\left(\int_{\mathbb{R}^n} (I_\alpha(\mathcal{M}_{\mathcal{F}} f_1,\mathcal{M}_{\mathcal{F}} f_2)(x))^{q_0} M_\theta h(x)dx\right)^{1/q_0}\\ &\leqslant C\prod_{j=1}^{2}\left(\int_{\mathbb{R}^n} (\mathcal{M}_{\mathcal{F}} f_j(x))^{p_0}(M_\theta h(x))^{p_0/q_0}dx\right)^{1/p_0}.\end{aligned}$$

Since $M_\theta h \in A_1$ as established in the proof of Theorem 3.1, all the conditions for applying the extrapolation theorem are satisfied. Theorem 3.3 then yields

$$\|\mathcal{M}_{\mathcal{F}} I_\alpha(f_1,f_2)\|_{M_u^{q)}(\mathbb{R}^n)} \leqslant C_0\|\mathcal{M}_{\mathcal{F}} f_1\|_{M_u^{p_1)}(\mathbb{R}^n)}\|\mathcal{M}_{\mathcal{F}} f_2\|_{M_u^{p_2)}(\mathbb{R}^n)},$$

which completes the proof. □

## 5. Applications

### 5.1. Bilinear Sobolev embedding

As an application of our main results, we obtain the following bilinear Sobolev embedding theorem.

THEOREM 5.1. *Let* $n > 1$, $p_1, p_2 \in (1,n)$ *with* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q} + \frac{1}{n}$, *and let* $u$ *satisfy the conditions of Theorem 3.1 with* $\alpha = 1$. *Then for all compactly supported Lipschitz functions* $f_1, f_2$ *on* $\mathbb{R}^n$,

$$\|f_1 f_2\|_{M_u^{q)}(\mathbb{R}^n)} \leqslant C \|\nabla f_1\|_{M_u^{p_1)}(\mathbb{R}^n)} \|\nabla f_2\|_{M_u^{p_2)}(\mathbb{R}^n)}.$$

*Proof.* We establish this embedding by showing that the pointwise product $f_1(x)f_2(x)$ can be controlled by a bilinear fractional integral of the gradients.

Since $f_1$ and $f_2$ are compactly supported Lipschitz functions, they vanish at infinity. By the fundamental theorem of calculus applied along rays from infinity, for any $x \in \mathbb{R}^n$ and $j \in \{1,2\}$,

$$f_j(x) = -\int_0^\infty \frac{d}{dt} f_j(x+t\omega)dt = -\int_0^\infty \omega \cdot \nabla f_j(x+t\omega)dt,$$

where $\omega \in \mathbb{S}^{n-1}$ is any unit vector. Averaging over all directions $\omega \in \mathbb{S}^{n-1}$ and using polar coordinates, we obtain the representation formula (see [17], p. 131):

$$f_j(x) = c_n \int_{\mathbb{R}^n} \frac{(x-y)\cdot \nabla f_j(y)}{|x-y|^n} dy = c_n \sum_{k=1}^n I_1\left(\frac{\partial f_j}{\partial y_k} \cdot \frac{x_k - y_k}{|x-y|}\right)(x),$$

where $c_n$ is a dimensional constant and $I_1$ is the Riesz potential of order $1$.

However, to apply our bilinear theory, we need a different approach. For compactly supported Lipschitz functions, we have the pointwise inequality

$$|f_j(x)| \leqslant C_n I_1(|\nabla f_j|)(x),$$

where $C_n$ depends only on the dimension. This follows from the fact that

$$|f_j(x)| \leqslant c_n \int_{\mathbb{R}^n} \frac{|\nabla f_j(y)|}{|x-y|^{n-1}} dy = c_n I_1(|\nabla f_j|)(x).$$

Therefore, for the product we have

$$|f_1(x) f_2(x)| \leqslant C_n^2 I_1(|\nabla f_1|)(x) I_1(|\nabla f_2|)(x).$$

Now we need to connect this to our bilinear fractional integral operator. The key observation is the pointwise domination

$$I_1(|\nabla f_1|)(x) I_1(|\nabla f_2|)(x) \leqslant C I_1(|\nabla f_1|, |\nabla f_2|)(x),$$

where the bilinear operator on the right is defined as

$$I_1(g_1,g_2)(x) = \int_{\mathbb{R}^{2n}} \frac{g_1(y_1)g_2(y_2)}{(|x-y_1|+|x-y_2|)^{2n-1}} dy_1 dy_2.$$

To verify this domination, note that

$$\begin{aligned} & I_1(|\nabla f_1|)(x) I_1(|\nabla f_2|)(x) \\ &= \int_{\mathbb{R}^n} \frac{|\nabla f_1(y_1)|}{|x-y_1|^{n-1}} dy_1 \int_{\mathbb{R}^n} \frac{|\nabla f_2(y_2)|}{|x-y_2|^{n-1}} dy_2 \\ &= \int_{\mathbb{R}^{2n}} \frac{|\nabla f_1(y_1)||\nabla f_2(y_2)|}{|x-y_1|^{n-1}|x-y_2|^{n-1}} dy_1 dy_2 \\ &\leqslant C \int_{\mathbb{R}^{2n}} \frac{|\nabla f_1(y_1)||\nabla f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n-2}} dy_1 dy_2, \end{aligned}$$

where the last inequality uses the fact that

$$|x-y_1|^{n-1}|x-y_2|^{n-1} \geqslant c_n(|x-y_1|+|x-y_2|)^{2n-2}$$

for some dimensional constant $c_n > 0$.

Since $\alpha = 1$ and $2n-\alpha = 2n-1$, we have shown that

$$|f_1(x) f_2(x)| \leqslant C I_1(|\nabla f_1|, |\nabla f_2|)(x).$$

To complete the proof, we verify that the conditions of Theorem 3.1 are satisfied. The relation $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q}+\frac{1}{n}$ is precisely the scaling relation with $\alpha = 1$. Since $p_1, p_2 \in (1,n)$, we have $p_1, p_2 \in (1, \frac{n}{\alpha})$ as required. The conditions on $u$ are assumed to hold by hypothesis.

Applying Theorem 3.1 yields

$$\|f_1 f_2\|_{M_u^{q)}(\mathbb{R}^n)} \leqslant C \|I_1(|\nabla f_1|, |\nabla f_2|)\|_{M_u^{q)}(\mathbb{R}^n)} \leqslant C \|\nabla f_1\|_{M_u^{p_1)}(\mathbb{R}^n)} \|\nabla f_2\|_{M_u^{p_2)}(\mathbb{R}^n)},$$

which completes the proof. □

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REFERENCES

[1] D. R. ADAMS, *A note on Riesz potentials*, Duke Math. J. **42** (1975), 765–778.
[2] R. COIFMAN AND Y. MEYER, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.
[3] R. COIFMAN AND Y. MEYER, *Au delà des opérateurs pseudo-différentiels*, Astérisque **57** (1978).
[4] L. GRAFAKOS, *On multilinear fractional integrals*, Studia Math. **102** (1992), 49–56.
[5] L. GRAFAKOS, *Modern Fourier Analysis*, Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009.
[6] L. GRECO, T. IWANIEC, AND C. SBORDONE, *Inverting the p-harmonic operator*, Manuscripta Math. **92** (1997), 249–258.
[7] K.-P. HO, *Fractional integral operators on grand Morrey spaces and grand Hardy-Morrey spaces*, J. Math. Inequal. **18** (2024), 755–774.
[8] K.-P. HO, *Grand Morrey spaces and grand Hardy-Morrey spaces on Euclidean space*, J. Geom. Anal. **33** (2023), Article No. 180.

[9] K.-P. HO, *Grand Triebel–Lizorkin–Morrey spaces*, Demonstratio Math. **58** (2025), Art. 20240085.
[10] T. IWANIEC AND C. SBORDONE, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. **119** (1992), 129–143.
[11] C. E. KENIG AND E. M. STEIN, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999), 1–15.
[12] B. MUCKENHOUPT AND R. WHEEDEN, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–274.
[13] J. L. RUBIO DE FRANCIA, *Factorization and extrapolation of weights*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 393–395.
[14] J. L. RUBIO DE FRANCIA, *A new technique in the theory of $A_p$ weights*, Topics in modern harmonic analysis, vol. I, II (Turin/Milan, 1982), 571–579, Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983.
[15] J. L. RUBIO DE FRANCIA, *Factorization theory and $A_p$ weights*, Amer. J. Math. **106** (1984), 533–547.
[16] C. SBORDONE, *Grand Sobolev spaces and their applications to variational problems*, Le Matematiche **51** (1996), 335–347.
[17] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton, NJ, 1970.
[18] J.-O. STRÖMBERG AND R. WHEEDEN, *Fractional integrals on weighted $H^p$ and $L^p$ spaces*, Trans. Amer. Math. Soc. **287** (1985), 293–321.
[19] S. WANG, *Hölder's inequalities and multilinear singular integrals on generalized Orlicz spaces*, J. Math. Inequal. **18** (2024), 811–828.

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