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THE WEIGHTED POWER DIFFERENCE MEAN AND ITS GENERALIZATION

MASATOSHI ITO

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Abstract. Pal, Singh, Moslehian and Aujla obtained the inequalities for a convex function and introduced the weighted logarithmic mean for two positive numbers or bounded linear operators on a complex Hilbert space. Furuichi and Minculete refined the inequalities by Pal et al.

In this paper, based on their results, we newly introduce the weighted power difference mean as a generaliation of the weighted logarithmic mean. We show relations among the weighted power, power difference and arithmetic means. Moreover, we obtain its generalization by considering the notion of a transpose symmetric path of t-weighted operator means.

1. Introduction

The celebrated arithmetic-logarithmic-geometric-harmonic mean inequality

$$\frac{2ab}{a+b} \leqslant \sqrt{ab} \leqslant \frac{a-b}{\log a - \log b} \leqslant \frac{a+b}{2} \quad \text{for } a,b>0$$

are generalized to many directions. As one of them, we consider the weighted means. For $a,b>0$ and $t \in [0,1]$,

$$A_t(a,b) = (1-t)a + tb \quad \text{(arithmetic mean)},$$
$$G_t(a,b) = a^{1-t}b^t \quad \text{(geometric mean)},$$
$$H_t(a,b) = \{(1-t)a^{-1} + tb^{-1}\}^{-1} \quad \text{(harmonic mean)}.$$

There exists some definitions of the weighted logarithmic mean. Here, we consider the weighted logarithmic mean $LM_t(a,b)$ for $t \in [0,1]$ as

$$LM_t(a,b) = \frac{1}{\log a - \log b}\left\{\frac{1-t}{t}a^{1-t}(a^t - b^t) + \frac{t}{1-t}b^t(a^{1-t} - b^{1-t})\right\}$$

introduced by Pal, Singh, Moslehian and Aujla [7], which is based on the Hermite-Hadamard inequality for convex functions. In [7], they showed that the inequalities

$$H_t(a,b) \leqslant G_t(a,b) \leqslant LM_t(a,b) \leqslant A_t(a,b) \tag{1.1}$$

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always hold for $t \in [0,1]$. If the weight parameter t is equal to $\frac{1}{2}$, then the weighted means coincide with the original (non-weighted) ones, and then we abbreviate the weight t as $A(a,b) = A_{\frac{1}{2}}(a,b)$.

The following one-parameter weighted means are known, which extend the weighted arithmetic, geometric and harmonic means. For $a,b > 0$, $t \in [0,1]$ and $q \in \mathbb{R}$,

$$P_{t,[q]}(a,b) = \begin{cases} \{(1-t)a^q + tb^q\}^{\frac{1}{q}} & \text{if } q \neq 0, \\ a^{1-t}b^t & \text{if } q = 0, \end{cases} \quad \text{(power mean)},$$

$$K_{t,[q]}(a,b) = (1-q)a^{1-t}b^t + q\{(1-t)a + tb\} \quad \text{(Heron mean)},$$

$$\overline{HZ}_{t,[q]}(a,b) = (1-t)a^{(1-t)+qt}b^{(1-q)t} + ta^{(1-q)(1-t)}b^{t+q(1-t)} \quad \text{(Heinz mean)}.$$

It is well known that $P_{t,[q]}(a,b)$ is monotone increasing on $q \in \mathbb{R}$. We remark that the non-weighted Heinz mean $\overline{HZ}_{[q]}(a,b) = \frac{1}{2}(a^{\frac{1+q}{2}}b^{\frac{1-q}{2}} + a^{\frac{1-q}{2}}b^{\frac{1+q}{2}})$ is often expressed by $HZ_{[r]}(a,b) = \frac{1}{2}(a^r b^{1-r} + a^{1-r}b^r)$, that is, $HZ_{[r]}(a,b) = \overline{HZ}_{[|2r-1|]}(a,b)$ for $r \in [0,1]$. We can consider these means for two positive bounded linear operators on a complex Hilbert space by Kubo-Ando theory on operator means [6]. Details are in section 3.

Moreover, for $q \in \mathbb{R}$,

$$J_{[q]}(a,b) = \begin{cases} \dfrac{q}{q+1}\dfrac{a^{q+1}-b^{q+1}}{a^q-b^q} & \text{if } q \neq 0,-1, \\ \dfrac{a-b}{\log a - \log b} & \text{if } q = 0, \\ \dfrac{ab(\log a - \log b)}{a-b} & \text{if } q = -1, \end{cases} \quad \text{(power difference mean)}$$

is known. We note that $J_{[q]}(a,b)$ is monotone increasing on $q \in \mathbb{R}$. The power difference mean $J_{[q]}(a,b)$ includes non-weighted arithmetic, logarithmic, geometric and harmonic means by putting $q = 1, 0, \frac{-1}{2}, -2$, respectively. But it seems that the weighted power difference mean has not introduced yet.

In this paper, firstly we introduce the weighted power difference mean for two positive numbers, and also we show the relations among the weighted power, power difference and arithmetic means in section 2. Secondly, we consider weighted operator means on a complex Hilbert space. Recently, in [4], we introduced the notion of a transpose symmetric path of t-weighted operator means, and also we got relations among some weighted operator means. By using this concept, we discuss a generalization of the weighted power difference mean in sections 3 and 4.

2. The weighted power difference mean

Furuichi and Minculete [2] obtained a refinement of the result in [7] for a convex function.

THEOREM 2.A. ([2, Theorem 2.1]) *Let* $0 < \alpha \leqslant \beta$. *For every convex Riemann integrable function* $f : [\alpha,\beta] \to \mathbb{R}$ *and* $t \in [0,1]$, *we have*

$$f(\alpha\nabla_t\beta) \leqslant R^{(1)}_{f,t}(\alpha,\beta) \leqslant C_{f,t}(\alpha,\beta) \leqslant R^{(2)}_{f,t}(\alpha,\beta) \leqslant f(\alpha)\nabla_t f(\beta),$$

where $\alpha\nabla_t\beta = A_t(\alpha,\beta) = (1-t)\alpha + t\beta$,

$$C_{f,t}(\alpha,\beta) = \left(\int_0^1 f(\alpha\nabla_{tx}\beta)\,dx\right)\nabla_t\left(\int_0^1 f((1-t)(\beta-\alpha)x+\alpha\nabla_t\beta)\,dx\right),$$
$$R^{(1)}_{f,t}(\alpha,\beta) = f(\alpha\nabla_{\frac{t}{2}}\beta)\nabla_t f(\alpha\nabla_{\frac{1+t}{2}}\beta),$$
$$R^{(2)}_{f,t}(\alpha,\beta) = \big(f(\alpha)\nabla_t f(\beta)\big)\nabla f(\alpha\nabla_t\beta).$$

We remark that we can slightly extend the assumption of Theorem 2.A by considering parallel translation of f and the property $\alpha\nabla_t\beta = \beta\nabla_{1-t}\alpha$ for $t \in [0,1]$.

THEOREM 2.1. *Let* $\alpha,\beta \in \mathbb{R}$, *and let* $I = \big[\min\{\alpha,\beta\},\max\{\alpha,\beta\}\big]$. *For every convex Riemann integrable function* $f : I \to \mathbb{R}$ *and* $t \in [0,1]$, *we have*

$$f(\alpha\nabla_t\beta) \leqslant R^{(1)}_{f,t}(\alpha,\beta) \leqslant C_{f,t}(\alpha,\beta) \leqslant R^{(2)}_{f,t}(\alpha,\beta) \leqslant f(\alpha)\nabla_t f(\beta), \tag{2.1}$$

where $C_{f,t}(\alpha,\beta)$, $R^{(1)}_{f,t}(\alpha,\beta)$ *and* $R^{(2)}_{f,t}(\alpha,\beta)$ *are as stated in Theorem 2.A.*

In [7], the weighted logarithmic mean was defined as $LM_t(a,b) = C_{f,t}(\log a,\log b)$, where $f(x) = e^x$. It was shown in [2] that Theorem 2.A implies the following inequalities on the weighted logarithmic mean.

PROPOSITION 2.B. ([2, Corollary 2.2]) *Let* $a,b > 0$. *Then the inequalities*

$$G_t(a,b) \leqslant A_t(G_{\frac{t}{2}}(a,b),G_{\frac{1+t}{2}}(a,b)) \leqslant LM_t(a,b) \leqslant A(A_t(a,b),G_t(a,b)) \leqslant A_t(a,b)$$

hold for $t \in [0,1]$.

Here, we try to introduce a generalization of $LM_t(a,b)$. Let $f_q(x) = (1+qx)^{\frac{1}{q}}$, $\alpha = \frac{a^q-1}{q}$, $\beta = \frac{b^q-1}{q}$ for $a,b > 0$ and $q \in \mathbb{R}\setminus\{0\}$. Then f_q can be defined on $I = \big[\min\{\alpha,\beta\},\max\{\alpha,\beta\}\big]$. We remark that $\lim_{q\to 0} f_q(x) = \lim_{q\to 0}(1+qx)^{\frac{1}{q}} = e^x$ and $\lim_{q\to 0}\frac{x^q-1}{q} = \log x$.

If $q \neq -1$, we have

$$\begin{aligned}
&\int_0^1 f_q(\alpha \nabla_{tx} \beta)\, dx \\
&= \int_0^1 \{1 + q\alpha + qt(\beta - \alpha)x\}^{\frac{1}{q}}\, dx \\
&= \frac{1}{qt(\beta - \alpha)} \frac{q}{q+1} \left(\{1 + q\alpha + qt(\beta - \alpha)\}^{\frac{q+1}{q}} - (1 + q\alpha)^{\frac{q+1}{q}} \right) \\
&= \frac{q}{q+1} \frac{\{(1-t)a^q + tb^q\}^{\frac{q+1}{q}} - a^{q+1}}{t(b^q - a^q)},
\end{aligned} \tag{2.2}$$

and also by putting $u = 1 - x$, (2.2) ensures

$$\int_0^1 f_q((1-t)(\beta - \alpha)x + \alpha \nabla_t \beta)\, dx = \int_0^1 f_q(\beta \nabla_{(1-t)u} \alpha)\, du$$
$$= \frac{q}{q+1} \frac{\{tb^q + (1-t)a^q\}^{\frac{q+1}{q}} - b^{q+1}}{(1-t)(a^q - b^q)} = \frac{q}{q+1} \frac{b^{q+1} - \{(1-t)a^q + tb^q\}^{\frac{q+1}{q}}}{(1-t)(b^q - a^q)}.$$

Therefore we get

$$\begin{aligned}
C_{f_q,t}(\alpha, \beta) = \frac{q}{q+1} \Bigg(&\frac{1-t}{t} \frac{\{(1-t)a^q + tb^q\}^{\frac{q+1}{q}} - a^{q+1}}{b^q - a^q} \\
&+ \frac{t}{1-t} \frac{b^{q+1} - \{(1-t)a^q + tb^q\}^{\frac{q+1}{q}}}{b^q - a^q} \Bigg).
\end{aligned}$$

If $q = -1$, we have

$$\begin{aligned}
C_{f_{-1},t}(\alpha, \beta) = &\frac{1-t}{t} \frac{\log\{(1-t)a^{-1} + tb^{-1}\} - \log a^{-1}}{b^{-1} - a^{-1}} \\
&+ \frac{t}{1-t} \frac{\log b^{-1} - \log\{(1-t)a^{-1} + tb^{-1}\}}{b^{-1} - a^{-1}}
\end{aligned}$$

by the similar calculation.

Therefore we can define the weighted power difference mean $J_{t,[q]}(a,b)$ as follows: For $t \in [0,1]$ and $q \in \mathbb{R}$,

$$J_{t,[q]}(a,b) = \begin{cases} C_{f_q,t}\left(\dfrac{a^q - 1}{q}, \dfrac{b^q - 1}{q} \right), \text{ where } f_q(x) = (1 + qx)^{\frac{1}{q}} \text{ if } q \neq 0, \\ C_{f_0,t}(\log a, \log b), \text{ where } f_0(x) = e^x \text{ if } q = 0. \end{cases}$$

Of course, we can verify that $J_{\frac{1}{2},[q]}(a,b) = J_{[q]}(a,b)$ for $q \in \mathbb{R}$. By Theorem 2.1, we obtain a generalization of Proposition 2.B.

PROPOSITION 2.2. *Let $a,b>0$. Then the inequalities*

$$\begin{aligned} P_{t,[q]}(a,b) &\leqslant A_t(P_{\frac{t}{2},[q]}(a,b), P_{\frac{1+t}{2},[q]}(a,b)) \\ &\leqslant J_{t,[q]}(a,b) \leqslant A(A_t(a,b), P_{t,[q]}(a,b)) \leqslant A_t(a,b) \end{aligned}$$

hold for $t\in[0,1]$ and $q\leqslant 1$.

Proof. Let $f_q(x)=(1+qx)^{\frac{1}{q}}$, $\alpha=\frac{a^q-1}{q}$, $\beta=\frac{b^q-1}{q}$ for $a,b>0$ and $q\neq 0$. Then $f_q(x)$ is convex for $x>\frac{-1}{q}$ if $0<q\leqslant 1$, and $f_q(x)$ is convex for $x<\frac{-1}{q}$ if $q<0$. Here we consider applying Theorem 2.1. We have

$$\begin{aligned} f_q(\alpha)\nabla_t f_q(\beta) &= (1-t)(1+q\alpha)^{\frac{1}{q}} + t(1+q\beta)^{\frac{1}{q}} \\ &= (1-t)a+tb = A_t(a,b) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} f_q(\alpha\nabla_t\beta) &= \{1+(1-t)q\alpha+tq\beta\}^{\frac{1}{q}} \\ &= \{(1-t)a^q+tb^q\}^{\frac{1}{q}} = P_{t,[q]}(a,b). \end{aligned} \tag{2.4}$$

By using (2.3) and (2.4), we get

$$\begin{aligned} R^{(1)}_{f_q,t}(\alpha,\beta) &= (1-t)f_q(\alpha\nabla_{\frac{t}{2}}\beta) + tf_q(\alpha\nabla_{\frac{1+t}{2}}\beta) \\ &= (1-t)\left\{\left(1-\frac{t}{2}\right)a^q+\frac{t}{2}b^q\right\}^{\frac{1}{q}} + t\left\{\left(1-\frac{1+t}{2}\right)a^q+\frac{1+t}{2}b^q\right\}^{\frac{1}{q}} \\ &= A_t(P_{\frac{t}{2},[q]}(a,b), P_{\frac{1+t}{2},[q]}(a,b)) \end{aligned}$$

and

$$\begin{aligned} R^{(2)}_{f_q,t}(\alpha,\beta) &= \frac{1}{2}\{f_q(\alpha)\nabla_t f_q(\beta)\} + \frac{1}{2}f_q(\alpha\nabla_t\beta) \\ &= \frac{1}{2}\{(1-t)a+tb\} + \frac{1}{2}\{(1-t)a^q+tb^q\}^{\frac{1}{q}} = A(A_t(a,b), P_{t,[q]}(a,b)). \end{aligned}$$

Therefore Theorem 2.1 ensures the desired inequalities. The case $q=0$ is obtained by considering the limit as $q\to 0$. □

We remark that we easily obtain the inequalities for $q\geqslant 1$.

PROPOSITION 2.3. *Let $a,b>0$. Then the inequalities*

$$\begin{aligned} A_t(a,b) &\leqslant A(A_t(a,b), P_{t,[q]}(a,b)) \leqslant J_{t,[q]}(a,b) \\ &\leqslant A_t(P_{\frac{t}{2},[q]}(a,b), P_{\frac{1+t}{2},[q]}(a,b)) \leqslant P_{t,[q]}(a,b) \end{aligned}$$

hold for $t\in[0,1]$ and $q\geqslant 1$.

Proof. The reverse inequalities of (2.1) hold for a concave function f by replacing f by $-f$ in Theorem 2.1. Therefore we have the desired inequalities since $f_q(x)=(1+qx)^{\frac{1}{q}}$ is concave for $x>\frac{-1}{q}$ if $q\geqslant 1$. □

3. A transpose symmetric path of t-weighted operator means including the weighted power difference mean

In what follows, we discuss operator means and a generalization of the weighted power difference mean by using our concept in [4]. Of course, the results for operator means are valid for numerical means.

Here, an operator means a bounded linear operator on a complex Hilbert space $\mathscr{H}$. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in \mathscr{H}$, and T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. A real-valued function f defined on $J \subset \mathbb{R}$ is said to be operator monotone if $A \leqslant B$ implies $f(A) \leqslant f(B)$ for selfadjoint operators A and B whose spectra $\sigma(A), \sigma(B) \subset J$, where $A \leqslant B$ means $B - A \geqslant 0$.

On operator means for two positive operators, Kubo and Ando [6] obtained that there exists a one-to-one correspondence between an operator mean $\mathfrak{M}$ and an operator monotone function $f \geqslant 0$ on $[0,\infty)$ with $f(1) = 1$ via $f(x)I = \mathfrak{M}(I, xI)$ as follows:

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \tag{3.1}$$

if $A > 0$ and $B \geqslant 0$. An operator mean is also expressed as $\mathfrak{M}(A,B) = A\,\sigma_f\,B$ by using infix notation. We remark that f is called the representing function of $\mathfrak{M}$, and also it is permitted to consider binary operations given by (3.1) even if f is a general real-valued function. By (3.1), we can consider weighted operator means for two strictly positive operators. For example, for $A, B > 0$ and $t \in [0,1]$, $\mathfrak{A}_t(A,B) = (1-t)A + tB$ (arithmetic mean) and $\mathfrak{G}_t(A,B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}}$ (geometric mean). We remark that their representing functions are $A_t(1,x)$, $G_t(1,x)$ (denoted by $A_t(x)$, $G_t(x)$), respectively. Similarly, we can introduce the operator mean $\mathfrak{M}$ corresponding to the representing function $M(1,x)$ (denoted by $M(x)$) by the numerical mean M if $M(1,x)$ is operator monotone.

For all numerical means stated in section 1, we can consider their operator versions under suitable conditions of q. Concretely, we can consider the weighted power mean $\mathfrak{P}_{\mathfrak{t},[\mathfrak{q}]}$ and the power difference mean $\mathfrak{J}_{[\mathfrak{q}]}$ for two positive operators. In fact, $P_{t,[q]}(x)$ is an operator monotone function on $[0,\infty)$ for $t \in [0,1]$ and $q \in [-1,1]$, and also $J_{[q]}(x)$ is operator monotone on $[0,\infty)$ for $q \in [-2,1]$ (see [3], for instance). We remark that the weighted Heinz mean $\overline{HZ}_{t,[q]}$ and its operator version $\overline{\mathfrak{H}\mathfrak{Z}}_{t,[q]}$ for $q \in [0,1]$ are introduced in [4] by using (3.2) stated below, and also the operator weighted logarithmic mean $\mathfrak{L}\mathfrak{M}_\mathfrak{t}$ is considered in [7] (see also [4]).

An operator mean $\mathfrak{M}$ is said to be symmetric if $\mathfrak{M}(A,B) = \mathfrak{M}(B,A)$ (symmetry) holds. A weighted operator mean $\mathfrak{M}_t$ is said to be transpose symmetric if $\mathfrak{M}_t(A,B) = \mathfrak{M}_{1-t}(B,A)$ (transpose symmetry) holds for all $t \in [0,1]$. The weighted means in section 1 are not symmetric except the case $t = \frac{1}{2}$, but transpose symmetric. For two operator means $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$, $\mathfrak{M} \leqslant \widetilde{\mathfrak{M}}$ (resp. $\mathfrak{M} = \widetilde{\mathfrak{M}}$) means that $\mathfrak{M}(A,B) \leqslant \widetilde{\mathfrak{M}}(A,B)$ (resp. $\mathfrak{M}(A,B) = \widetilde{\mathfrak{M}}(A,B)$) for all $A, B > 0$.

For an operator mean $\mathfrak{M}$ and its representing function f, the operator means whose representing functions are $xf(x^{-1})$, $f(x^{-1})^{-1}$ and $\frac{x}{f(x)}$ are called transpose, adjoint and dual of $\mathfrak{M}$, and they are denoted by $\mathfrak{M}^\circ$, $\mathfrak{M}^*$ and $\mathfrak{M}^\perp$, respectively.

We easily obtain that $\mathfrak{M}^{\circ}(A,B) = \mathfrak{M}(B,A)$ for $A,B > 0$. An operator mean $\mathfrak{M}$ is symmetric if and only if $\mathfrak{M} = \mathfrak{M}^{\circ}$ if and only if $f(x) = xf(x^{-1})$ for all $x > 0$. An operator mean $\mathfrak{M}$ is said to be selfadjoint if $\mathfrak{M} = \mathfrak{M}^*$ holds, and $\mathfrak{M}$ is selfadjoint if and only if $f(x) = f(x^{-1})^{-1}$ for all $x > 0$, that is, f is selfadjoint. We also note that a weighted operator mean $\mathfrak{M}_t$ with the representing function f_t is transpose symmetric if and only if $\mathfrak{M}_t = \mathfrak{M}^{\circ}_{1-t}$ for all $t \in [0,1]$ if and only if $f_t(x) = xf_{1-t}(x^{-1})$ for all $x > 0$ and $t \in [0,1]$, that is, f_t is transpose symmetric.

Recently, in [4], we discussed the definition of weighted means and introduced the notion of a transpose symmetric path of t-weighted $\mathfrak{M}$-means.

DEFINITION 3.1. ([4]) Let $\mathfrak{M}$ be a symmetric operator mean and $A,B > 0$. If the following conditions hold, then $\mathfrak{M}_t$ is said to be a weighted $\mathfrak{M}$-mean, and a one-parameter family $\{\mathfrak{M}_t\}_{t\in[0,1]}$ is said to be a transpose symmetric path of t-weighted $\mathfrak{M}$-means.

(i) $\mathfrak{M}_t$ is an operator mean for all fixed $t \in [0,1]$.

(ii) $\mathfrak{M}_0(A,B) = A$, $\mathfrak{M}_{\frac{1}{2}}(A,B) = \mathfrak{M}(A,B)$ and $\mathfrak{M}_1(A,B) = B$.

(iii) $\mathfrak{M}_t(A,B) = \mathfrak{M}_{1-t}(B,A)$ for all $t \in [0,1]$ (transpose symmetry).

(iv) $\mathfrak{M}_t$ is t-weighted for all fixed $t \in [0,1]$, that is, $f_t'(1) = t$ for the representing function f_t of $\mathfrak{M}_t$.

In [4], we considered the function $n_t[\varphi_s] : [0,\infty) \to [0,\infty)$ defined by

$$n_t[\varphi_s](x) = (1-t)\varphi_{1-s}(x^t) + tx^t\varphi_s(x^{1-t}) \quad \text{for } \{\varphi_s\} \in \mathscr{R} \text{ and } t,s \in [0,1], \tag{3.2}$$

where $\mathscr{R} = \left\{\{f_t\}_{t\in[0,1]} : f_t \text{ is the representing function of } \mathfrak{M}_t \in \{\mathfrak{M}_t\}_{t\in[0,1]}\right\}$ for a transpose symmetric path $\{\mathfrak{M}_t\}_{t\in[0,1]}$ of t-weighted $\mathfrak{M}$-means, and $\{f_t\}_{t\in[0,1]}$ is denoted by $\{f_t\}$ briefly. We showed that $n_t[\varphi_s]$ makes a transpose symmetric path of t-weighted $\mathfrak{N}[\varphi_s]$-means.

THEOREM 3.A. ([4]) *Let $\{\varphi_s\} \in \mathscr{R}$ and $n_t[\varphi_s]$ be as in (3.2). Let $\mathfrak{N}_t[\varphi_s]$ be the binary operation whose representing function is $n_t[\varphi_s]$, and also $\mathfrak{N}[\varphi_s] = \mathfrak{N}_{\frac{1}{2}}[\varphi_s]$. Then the family $\{\mathfrak{N}_t[\varphi_s]\}_{t\in[0,1]}$ is a transpose symmetric path of t-weighted $\mathfrak{N}[\varphi_s]$-means.*

Here, as a generalization of $n_t[\varphi_s]$, we introduce the function $n_t[\varphi_s,\gamma_p] : [0,\infty) \to [0,\infty)$ defined by

$$n_t[\varphi_s,\gamma_p](x) = (1-t)\varphi_{1-s}(\gamma_p(x)) + t\gamma_p(x)\varphi_s(\gamma_{1-p}(x^{-1})^{-1}) \tag{3.3}$$

for $\{\varphi_s\},\{\gamma_p\} \in \mathscr{R}$ and $t,s,p \in [0,1]$. Particularly, when φ is the representing function of a symmetric operator mean, we can define

$$n_t[\varphi,\gamma_p](x) = (1-t)\varphi(\gamma_p(x)) + t\gamma_p(x)\varphi(\gamma_{1-p}(x^{-1})^{-1}) \tag{3.4}$$

for $t,p \in [0,1]$ as the case $s = \frac{1}{2}$ in (3.3), where we do not have to consider a one paremeter family $\{\varphi_s\}$. We get the condition that $n_t[\varphi_s,\gamma_p]$ makes a transpose symmetric path of t-weighted operator means as follows:

THEOREM 3.1. *Let* $\{\varphi_s\},\{\gamma_p\}\in\mathscr{R}$ *and* $n_t[\varphi_s,\gamma_p]$ *be as in* (3.3). *Let* $\mathfrak{N}_t[\varphi_s,\gamma_p]$ *be the binary operation whose representing function is* $n_t[\varphi_s,\gamma_p]$. *The the following assertions hold for* $t,s,p\in[0,1]$.

(i) $\mathfrak{N}_t[\varphi_s,\gamma_p]$ *is an operator mean.*

(ii) $\mathfrak{N}_t[\varphi_s,\gamma_p]$ *is* $\{(1-s)p+st\}$*-weighted. Moreover, for* $s\in[0,1)$, $\mathfrak{N}_t[\varphi_s,\gamma_p]$ *is* t*-weighted if and only if* $p=t$.

(iii) *Let* $p=t$ *or* $p=\frac{1}{2}$. *Then* $\mathfrak{N}_t[\varphi_s,\gamma_p]$ *is transpose symmetric.*

(iv) *Let* $\mathfrak{N}[\varphi_s,\gamma]=\mathfrak{N}_{\frac{1}{2}}[\varphi_s,\gamma_{\frac{1}{2}}]$. *Then the family* $\{\mathfrak{N}_t[\varphi_s,\gamma_t]\}_{t\in[0,1]}$ *is a transpose symmetric path of* t*-weighted* $\mathfrak{N}[\varphi_s,\gamma]$*-means.*

Proof. We can easily verify the case $s=1$ since $\mathfrak{N}_t[\varphi_1,\gamma_p]=\mathfrak{A}_t$, so we assume $s\neq 1$.

(i) We get operator monotonicity of $n_t[\varphi_s,\gamma_p]$ since

$$\begin{aligned}\gamma_p(x)\,\sigma_{\varphi_s}x&=\gamma_p(x)\,\varphi_s\big(\gamma_p(x)^{-1}x\big)=\gamma_p(x)\,\varphi_s\big(\{x\gamma_{1-p}(x^{-1})\}^{-1}x\big)\\&=\gamma_p(x)\,\varphi_s\big(\gamma_{1-p}(x^{-1})^{-1}\big)\end{aligned}\tag{3.5}$$

ensures operator monotonicity of $\gamma_p(x)\,\varphi_s\big(\gamma_{1-p}(x^{-1})^{-1}\big)$, where σ_{φ_s} means an operator mean with the representing function φ_s. We also have $n_t[\varphi_s,\gamma_p](1)=1$ obviously.

(ii) We have

$$\begin{aligned}n_t'[\varphi_s,\gamma_p](x)=(1-t)\varphi_{1-s}'\big(\gamma_p(x)\big)\gamma_p'(x)+t\Big\{&\gamma_p'(x)\varphi_s\big(\gamma_{1-p}(x^{-1})^{-1}\big)\\&+\gamma_p(x)\varphi_s'\big(\gamma_{1-p}(x^{-1})^{-1}\big)\gamma_{1-p}(x^{-1})^{-2}\gamma_{1-p}'(x^{-1})x^{-2}\Big\},\end{aligned}$$

so that we obtain

$$n_t'[\varphi_s,\gamma_p](1)=(1-t)(1-s)p+t\{p+s(1-p)\}=(1-s)p+st.$$

Therefore $\mathfrak{N}_t[\varphi_s,\gamma_p]$ is t-weighted if and only if $(1-s)p+st=t$, that is, $p=t$.

(iii) We have

$$\begin{aligned}&xn_{1-t}[\varphi_s,\gamma_{1-p}](x^{-1})\\&=x\big\{t\varphi_{1-s}\big(\gamma_{1-p}(x^{-1})\big)+(1-t)\gamma_{1-p}(x^{-1})\varphi_s\big(\gamma_p(x)^{-1}\big)\big\}\\&=(1-t)x\gamma_{1-p}(x^{-1})\varphi_s\big(\gamma_p(x)^{-1}\big)+tx\varphi_{1-s}\big(\gamma_{1-p}(x^{-1})\big)\\&=(1-t)\gamma_p(x)\varphi_s\big(\gamma_p(x)^{-1}\big)+tx\gamma_{1-p}(x^{-1})\varphi_s\big(\gamma_{1-p}(x^{-1})^{-1}\big)\\&=(1-t)\varphi_{1-s}\big(\gamma_p(x)\big)+t\gamma_p(x)\varphi_s\big(\gamma_{1-p}(x^{-1})^{-1}\big)\\&=n_t[\varphi_s,\gamma_p](x)\end{aligned}$$

since φ_s and γ_p are transpose symmetric. Therefore, if $p=t$ or $p=\frac{1}{2}$ holds, then $\mathfrak{N}_t[\varphi_s,\gamma_p]$ is transpose symmetric.

(iv) We easily get that $n_0[\varphi_s,\gamma_0](x)=1$ and $n_1[\varphi_s,\gamma_1](x)=x$, and also $n_{\frac{1}{2}}[\varphi_s,\gamma_{\frac{1}{2}}](x)$ is the representing function of $\mathfrak{N}[\varphi_s,\gamma]$ obviously. Therefore we can verify that $\mathfrak{N}_t[\varphi_s,\gamma_t]$ has four properties in Definition 3.1 by (i), (ii) and (iii). □

We have the following property for the weighted operator means in Theorem 3.1. It is immediately obtained by (3.3) and (3.5).

THEOREM 3.2. *Let* $\{\varphi_s\},\{\widetilde{\varphi}_s\},\{\gamma_p\},\{\widetilde{\gamma}_p\}\in\mathscr{R}$. *If* $\varphi_s\leqslant\widetilde{\varphi}_s$ *and* $\gamma_p\leqslant\widetilde{\gamma}_p$ *for all* $s,p\in[0,1]$, *then*

$$\mathfrak{N}_t[\varphi_s,\gamma_p]\leqslant\mathfrak{N}_t[\widetilde{\varphi}_s,\widetilde{\gamma}_p]$$

holds for $t,s,p\in[0,1]$.

4. Relations among the weighted means

In [4], we obtained the following result by considering (3.2). Recall that LM_t is t-weighted as in [4, pp. 180–181].

THEOREM 4.A. ([4]) *For* $t,s\in[0,1]$, *the inequalities*

$$\mathfrak{H}_t\leqslant\mathfrak{G}_t\leqslant\overline{\mathfrak{H}\mathfrak{Z}}_{t,[s]}\leqslant\mathfrak{N}_t[LM_s]\leqslant\mathfrak{K}_{t,[s]}\leqslant\mathfrak{A}_t \tag{4.1}$$

hold. In particular, for $s=\frac{1}{2}$, *we have*

$$\mathfrak{H}_t\leqslant\mathfrak{G}_t\leqslant\overline{\mathfrak{H}\mathfrak{Z}}_{t,[\frac{1}{2}]}\leqslant\mathfrak{L}\mathfrak{M}_t\leqslant\mathfrak{K}_{t,[\frac{1}{2}]}\leqslant\mathfrak{A}_t. \tag{4.2}$$

We recognize that (4.2) is the operator version of Proposition 2.B, so that (4.1) and Proposition 2.2 are different generalizations of (4.2). Here, we try to get a generalization including both (4.1) and Proposition 2.2.

Put $\varphi(x)=J_{[q]}(x)$ and $\gamma_t(x)=P_{t,[q]}(x)$ for $t\in[0,1]$ and $q\in[-1,1]$ in (3.4). Then the representing function of $\mathfrak{N}_t[J_{[q]},P_{t,[q]}]$ $(q\neq 0,-1)$ is

$$\begin{aligned}
&n_t[J_{[q]},P_{t,[q]}](x)\\
&=(1-t)\frac{q}{q+1}\frac{\{(1-t)+tx^q\}^{\frac{q+1}{q}}-1}{\{(1-t)+tx^q\}-1}\\
&\quad+t\{(1-t)+tx^q\}^{\frac{1}{q}}\frac{q}{q+1}\frac{\{t+(1-t)x^{-q}\}^{\frac{-(q+1)}{q}}-1}{\{t+(1-t)x^{-q}\}^{-1}-1}\\
&=\frac{q}{q+1}\left((1-t)\frac{\{(1-t)+tx^q\}^{\frac{q+1}{q}}-1}{t(x^q-1)}\right.\\
&\qquad\left.+t\{(1-t)+tx^q\}^{\frac{1}{q}}\frac{x^{q+1}\{(1-t)+tx^q\}^{\frac{-(q+1)}{q}}-1}{x^q\{(1-t)+tx^q\}^{-1}-1}\right)
\end{aligned}$$

$$= \frac{q}{q+1}\left((1-t)\frac{\{(1-t)+tx^q\}^{\frac{q+1}{q}}-1}{t(x^q-1)}\right.$$
$$\left. + t\{(1-t)+tx^q\}^{\frac{1}{q}}\frac{x^{q+1}\{(1-t)+tx^q\}^{\frac{-1}{q}}-\{(1-t)+tx^q\}}{x^q-\{(1-t)+tx^q\}}\right)$$
$$= \frac{q}{q+1}\left(\frac{1-t}{t}\frac{\{(1-t)+tx^q\}^{\frac{q+1}{q}}-1}{x^q-1}+\frac{t}{1-t}\frac{x^{q+1}-\{(1-t)+tx^q\}^{\frac{q+1}{q}}}{x^q-1}\right)$$
$$= J_{t,[q]}(x).$$

Similarly, for $q=0,-1$, we can get that

$$n_t[J_{[0]},P_{t,[0]}](x) = n_t[LM,G_t](x) = LM_t(x) = J_{t,[0]}(x)$$

and

$$\begin{aligned} &n_t[J_{[-1]},P_{t,[-1]}](x) \\ &= \frac{1-t}{t}\frac{\log\{(1-t)+tx^{-1}\}}{x^{-1}-1}+\frac{t}{1-t}\frac{\log x^{-1}-\log\{(1-t)+tx^{-1}\}}{x^{-1}-1} \\ &= J_{t,[-1]}(x). \end{aligned}$$

Therefore, for operators on a complex Hilbert space, we can define the weighted power difference mean $\mathfrak{J}_{t,[q]}$ by $\mathfrak{N}_t[J_{[q]},P_{t,[q]}]$ for $t\in[0,1]$ and $q\in[-1,1]$. We note that the following Proposition 4.1 holds.

PROPOSITION 4.1. *For $t\in[0,1]$ and $q\in[-1,1]$, $\mathfrak{P}_{t,[q]}\leqslant\mathfrak{J}_{t,[q]}\leqslant\mathfrak{A}_t$ holds.*

Moreover, by using the representing function of the weighted power difference mean $\mathfrak{J}_{t,[q]}$, we introduce

$$\mathfrak{J}_{t,[s,q]} = \mathfrak{N}_t[J_{s,[q]},P_{t,[q]}] \quad \text{for } t,s\in[0,1] \text{ and } q\in[-1,1]$$

as a generalization of $\mathfrak{J}_{t,[q]}$. Then we obtain a generalization of (4.1) in Theorem 4.A and Proposition 2.2 as follows:

THEOREM 4.2. *Let $A,B>0$ and $t,s\in[0,1]$. If $q\in[-1,1]$, then*

$$\begin{aligned} \mathfrak{H}_t(A,B) &\leqslant \mathfrak{P}_{t,[q]}(A,B) \leqslant \mathfrak{A}_t\left(\mathfrak{P}_{(1-s)t,[q]}(A,B),\mathfrak{P}_{s+(1-s)t,[q]}(A,B)\right) \\ &\leqslant \mathfrak{J}_{t,[s,q]}(A,B) \leqslant \mathfrak{A}_s\left(\mathfrak{P}_{t,[q]}(A,B),\mathfrak{A}_t(A,B)\right) \leqslant \mathfrak{A}_t(A,B) \end{aligned} \tag{4.3}$$

holds, and also if $q\in[0,1]$, then

$$\begin{aligned} \mathfrak{G}_t(A,B) &\leqslant \overline{\mathfrak{H}\mathfrak{Z}}_{t,[s]}(A,B) \leqslant \mathfrak{G}_s\left(\mathfrak{P}_{t,[q]}(A,B),\mathfrak{P}_{t,[s]}(A,B)\right) \\ &\leqslant \mathfrak{A}_t\left(\mathfrak{P}_{(1-s)t,[q]}(A,B),\mathfrak{P}_{s+(1-s)t,[q]}(A,B)\right) \leqslant \mathfrak{J}_{t,[s,q]}(A,B) \end{aligned} \tag{4.4}$$

holds.

We remark that (4.3) leads (4.1) in Theorem 4.A by putting $q=0$, and also (4.3) leads the operator version of Proposition 2.2 by putting $s=\frac{1}{2}$.

In order to prove Proposition 4.1 and Theorem 4.2, we show the following lemma.

LEMMA 4.3. *Let $t,s\in[0,1]$ and $q\in[-1,1]$. Then the following equations hold for $x>0$.*

(i) $P_{t,[q]}\left(P_{(1-s)t,[q]}(x),P_{s+(1-s)t,[q]}(x)\right)=P_{t,[q]}(x)$.

(ii) $n_t[P_{s,[q]},P_{t,[q]}](x)=A_t\left(P_{(1-s)t,[q]}(x),P_{s+(1-s)t,[q]}(x)\right)$.

(iii) $n_t[A_s,A_t](x)=A_t(x)$.

Proof. We can assume $q\neq 0$ since the case $q=0$ holds by considering the limit as $q\to 0$ or similar argument to the case $q\neq 0$. We get (i) since

$$\begin{aligned}
&P_{t,[q]}\left(P_{(1-s)t,[q]}(x),P_{s+(1-s)t,[q]}(x)\right)\\
&=\left[(1-t)\{1-(1-s)t+(1-s)tx^q\}+t\{1-(s+(1-s)t)+(s+(1-s)t)x^q\}\right]^{\frac{1}{q}}\\
&=\{(1-t)+tx^q\}^{\frac{1}{q}}=P_{t,[q]}(x).
\end{aligned}$$

We obtain (ii) since $\{t+(1-t)x^{-q}\}^{-1}=x^q\{(1-t)+tx^q\}^{-1}$ leads that

$$\begin{aligned}
&n_t[P_{s,[q]},P_{t,[q]}](x)\\
&=(1-t)\left[s+(1-s)\{(1-t)+tx^q\}\right]^{\frac{1}{q}}\\
&\quad+t\{(1-t)+tx^q\}^{\frac{1}{q}}\left[(1-s)+s\{t+(1-t)x^{-q}\}^{-1}\right]^{\frac{1}{q}}\\
&=(1-t)\{1-(1-s)t+(1-s)tx^q\}^{\frac{1}{q}}+t\left[(1-s)\{(1-t)+tx^q\}+sx^q\right]^{\frac{1}{q}}\\
&=(1-t)\{1-(1-s)t+(1-s)tx^q\}^{\frac{1}{q}}+t\left[1-\{s+(1-s)t\}+\{s+(1-s)t\}x^q\right]^{\frac{1}{q}}\\
&=A_t\left(P_{(1-s)t,[q]}(x),P_{s+(1-s)t,[q]}(x)\right).
\end{aligned}$$

By putting $q=1$ in (ii), we have (iii) since

$$\begin{aligned}
&n_t[A_s,A_t](x)=A_t\left(A_{(1-s)t}(x),A_{s+(1-s)t}(x)\right)\\
&=(1-t)\{1-(1-s)t+(1-s)tx\}+t\left[(1-s)(1-t)+\{s+(1-s)t\}x\right]\\
&=(1-t)-(1-s)(1-t)t+(1-s)(1-t)tx+(1-s)(1-t)t+\{s+(1-s)t\}tx\\
&=(1-t)+tx=A_t(x).
\end{aligned}$$

Therefore the proof is complete. □

Proof of Proposition 4.1. We can obtain the result by Proposition 2.2, but we give a direct proof here. It is known that $\mathfrak{P}_{[q]}\leqslant\mathfrak{I}_{[q]}$ holds for $q\in[-1,1]$ (see [1, 8], for instance).

By putting $s = \frac{1}{2}$ in Lemma 4.3, we have

$$P_{t,[q]}\left(P_{\frac{t}{2},[q]}(x), P_{\frac{1+t}{2},[q]}(x)\right) = P_{t,[q]}(x),$$

$$n_t[P_{[q]}, P_{t,[q]}](x) = A_t\left(P_{\frac{t}{2},[q]}(x), P_{\frac{1+t}{2},[q]}(x)\right)$$

and

$$n_t[A, A_t](x) = A_t(x).$$

Therefore for $A, B > 0$,

$$\begin{aligned}\mathfrak{P}_{t,[q]}(A,B) &= \mathfrak{P}_{t,[q]}\left(\mathfrak{P}_{\frac{t}{2},[q]}(A,B), \mathfrak{P}_{\frac{1+t}{2},[q]}(A,B)\right)\\ &\leqslant \mathfrak{A}_t\left(\mathfrak{P}_{\frac{t}{2},[q]}(A,B), \mathfrak{P}_{\frac{1+t}{2},[q]}(A,B)\right) = \mathfrak{N}_t[P_{[q]}, P_{t,[q]}](A,B)\\ &\leqslant \mathfrak{N}_t[J_{[q]}, P_{t,[q]}](A,B) = \mathfrak{J}_{t,[q]}(A,B)\end{aligned}$$

and

$$\mathfrak{J}_{t,[q]} = \mathfrak{N}_t[J_{[q]}, P_{t,[q]}] \leqslant \mathfrak{N}_t[A, A_t] = \mathfrak{A}_t$$

hold by Theorem 3.2. □

Proof of Theorem 4.2. The first inequality in (4.3) is well known and the first inequality in (4.4) is easily obtained. We can assume $q \neq 0$ since the case $q = 0$ holds by considering the limit as $q \to 0$ or similar argument to the case $q \neq 0$. By Lemma 4.3,

$$P_{t,[q]}\left(P_{(1-s)t,[q]}(x), P_{s+(1-s)t,[q]}(x)\right) = P_{t,[q]}(x),$$

$$n_t[P_{s,[q]}, P_{t,[q]}](x) = A_t\left(P_{(1-s)t,[q]}(x), P_{s+(1-s)t,[q]}(x)\right)$$

and

$$n_t[A_s, A_t](x) = A_t(x)$$

hold, and also we obtain

$$\begin{aligned}&n_t[A_s, P_{t,[q]}](x)\\ &= (1-t)\left[s + (1-s)\{(1-t)+tx^q\}^{\frac{1}{q}}\right]\\ &\quad + t\{(1-t)+tx^q\}^{\frac{1}{q}}\left[(1-s) + s\{t+(1-t)x^{-q}\}^{\frac{-1}{q}}\right]\\ &= s(1-t) + (1-s)(1-t)\{(1-t)+tx^q\}^{\frac{1}{q}} + (1-s)t\{(1-t)+tx^q\}^{\frac{1}{q}} + stx\\ &= (1-s)\{(1-t)+tx^q\}^{\frac{1}{q}} + s\{(1-t)+tx\}\\ &= A_s\left(P_{t,[q]}(x), A_t(x)\right).\end{aligned}$$

Then for $A, B > 0$ and $q \in [-1,1] \setminus \{0\}$,

$$\begin{aligned}\mathfrak{P}_{t,[q]}(A,B) &= \mathfrak{P}_{t,[q]}\left(\mathfrak{P}_{(1-s)t,[q]}(A,B), \mathfrak{P}_{s+(1-s)t,[q]}(A,B)\right)\\ &\leqslant \mathfrak{A}_t\left(\mathfrak{P}_{(1-s)t,[q]}(A,B), \mathfrak{P}_{s+(1-s)t,[q]}(A,B)\right)\end{aligned} \tag{4.5}$$

holds, and also

$$\mathfrak{N}_t[P_{s,[q]},P_{t,[q]}] \leqslant \mathfrak{N}_t[J_{s,[q]},P_{t,[q]}] \leqslant \mathfrak{N}_t[A_s,P_{t,[q]}] \leqslant \mathfrak{N}_t[A_s,A_t],$$

that is,

$$\begin{aligned} &\mathfrak{A}_t\left(\mathfrak{P}_{(1-s)t,[q]}(A,B),\mathfrak{P}_{s+(1-s)t,[q]}(A,B)\right) \leqslant \mathfrak{J}_{t,[s,q]}(A,B) \\ &\leqslant \mathfrak{A}_s\left(\mathfrak{P}_{t,[q]}(A,B),\mathfrak{A}_t(A,B)\right) \leqslant \mathfrak{A}_t(A,B) \end{aligned} \tag{4.6}$$

holds by Proposition 4.1 and Theorem 3.2. Therefore we obtain (4.3) by (4.5) and (4.6).

Moreover, by putting $q=0$ in (ii) in Lemma 4.3, we have

$$\begin{aligned} &n_t[G_s,G_t](x) = A_t\left(G_{(1-s)t}(x),G_{s+(1-s)t}(x)\right) \\ &= (1-t)x^{(1-s)t}+tx^{s+(1-s)t} = x^{(1-s)t}\{(1-t)+tx^s\} = \overline{HZ}_{t,[s]}(x), \end{aligned}$$

and also we obtain

$$\begin{aligned} &n_t[G_s,P_{t,[q]}](x) \\ &= (1-t)\{(1-t)+tx^q\}^{\frac{1-s}{q}}+t\{(1-t)+tx^q\}^{\frac{1}{q}}\{t+(1-t)x^{-q}\}^{\frac{-s}{q}} \\ &= (1-t)\{(1-t)+tx^q\}^{\frac{1-s}{q}}+tx^s\{(1-t)+tx^q\}^{\frac{1-s}{q}} \\ &= \{(1-t)+tx^q\}^{\frac{1-s}{q}}\{(1-t)+tx^s\}^{\frac{s}{s}} \\ &= G_s\left(P_{t,[q]}(x),P_{t,[s]}(x)\right). \end{aligned}$$

Then for $A,B>0$ and $q\in(0,1]$, we get

$$\mathfrak{N}_t[G_s,G_t] \leqslant \mathfrak{N}_t[G_s,P_{t,[q]}] \leqslant \mathfrak{N}_t[P_{s,[q]},P_{t,[q]}] \leqslant \mathfrak{N}_t[J_{s,[q]},P_{t,[q]}],$$

that is,

$$\begin{aligned} &\overline{\mathfrak{H}\mathfrak{Z}}_{t,[s]}(A,B) \leqslant \mathfrak{G}_s\left(\mathfrak{P}_{t,[q]}(A,B),\mathfrak{P}_{t,[s]}(A,B)\right) \\ &\leqslant \mathfrak{A}_t\left(\mathfrak{P}_{(1-s)t,[q]}(A,B),\mathfrak{P}_{s+(1-s)t,[q]}(A,B)\right) \leqslant \mathfrak{J}_{t,[s,q]}(A,B) \end{aligned}$$

by Proposition 4.1 and Theorem 3.2, so that we obtain (4.4). □

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Masatoshi Ito
Maebashi Institute of Technology
460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, Japan
e-mail: m-ito@maebashi-it.ac.jp