Journal of
Mathematical
Inequalities
Volume 19, Number 3 (2025), 981–1006 doi:10.7153/jmi-2025-19-64

# COMPLETE MOMENT CONVERGENCE FOR $\rho^*$–MIXING LINEAR PROCESSES WITH RANDOM COEFFICIENTS AND ITS APPLICATIONS

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(*Communicated by X. Wang*)

*Abstract.* In this paper, we will study the complete moment convergence for the dependent linear processes under some suitable conditions, which $Y_t = \sum_{j=-\infty}^{\infty} A_j X_{t-j}$ be a dependent linear process, where the $\{X_n, n \in \mathbb{Z}\}$ is a sequence of $\rho^*$-mixing random variables, with stochastically dominated a random variable $X$, and $\{A_n, n \in \mathbb{Z}\}$ is a sequence independent random variables. As applications, we will present Marcinkiewicz-Zygumund strong laws and strong laws of large numbers for this linear processes. Finally, we also present some numerical simulations to demonstrate the finite sample performances of the theoretical results.

## 1. Introduction

Research on structured stochastic models, such as Markov chains, Gaussian processes, and linear models including autoregressive moving average systems has been extensive and well-developed. However, by mid-20th century, researchers recognized an important gap: many observed time series resisted classification within these specific frameworks, yet exhibited clear asymptotic independence characteristics. This realization spurred the development of a comprehensive theoretical framework for "mixing conditions" to address such cases. The present note offers a concise overview of this theoretical development.

The theory of strong mixing conditions constitutes an expansive field of study that far exceeds what can be adequately covered in this brief treatment. Limitations of space prevent citation of relevant journal literature (with one exception) and proper acknowledgment of the many researchers who have made significant contributions. What follows is necessarily a focused glimpse into one portion of this important theoretical domain.

In this paper, we will study focuses on exploring the complete moment convergence of dependent linear processes. Hence, we present the concept of which is defined as follows.

*Mathematics subject classification* (2020): 60F15, 60G07.
*Keywords and phrases*: $\rho^*$-mixing random variables, complete moment convergence, complete convergence, linear processes, random coefficients, Marcinkiewicz-Zygumund strong law of large numbers.
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Paper JMI-19-64

DEFINITION 1.1. Let $\{X_n,\ n \in \mathbb{Z}\}$ be a doubly infinite sequence of random variables and $\{a_n,\ n \in \mathbb{Z}\}$ be a sequence of absolutely summable real numbers. Denote

$$Y_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}, \quad t = 1, 2, \cdots, \tag{1.1}$$

then $\{Y_t,\ t \geqslant 1\}$ is called a linear process.

Linear process has a very important position in time series analysis, and a large number of literatures have discussed various properties of linear process, which has a wide range of applications in economics, engineering and physics, so many scholars are committed to studying the limit theorem of linear process when the error term of linear process satisfies different conditions. For example, [30] established asymptotically efficient selection for parameter estimation in linear processes; when the error term is a martingale and strong mixed error random variable sequence, [10] and [4] studied the corresponding central limit theorem (CLT) and functional center limit theorem for linear processes, respectively. Under some appropriate conditions, there are many limit results for linear processes. For example, [6] presented the principle of large deviations in linear processes, [45] established the CLT and the law of heavy logarithms, and so on.

In recent years, researchers have conducted extensive investigations into linear processes, yielding significant findings. For instance, [25] explored the linear processes governing somatic evolution; the asymptotic convergence and central limit theorems for linear processes in Hilbert spaces presented in [34]; [37] investigated the central limit theorems for linear processes generated by dependent random variables; [7] developed a unified framework for model-based multi-objective linear processes. Employing two types of testing data, [29] proposed a reliability estimation method based on a two-phase Wiener process with an evidential variable. [48] investigated the asymptotic behavior of maximum likelihood estimators for Ornstein-Uhlenbeck processes with large linear drift, specifically $dX_t = -\frac{1}{\varepsilon}(\theta X_t - \varepsilon^{1/2}\nu)dt + dB_t$ for $0 \leqslant t \leqslant T$, where $\theta, \nu \in \mathbb{R}$, and $\{B_t, t \geqslant 0\}$ represents a standard Brownian motion, analyzing aspects such as the law of iterated logarithm, consistency, and asymptotic distributions of the estimators. These research endeavors underscore the profound significance of linear processes in statistics and related disciplines.

Inspired by Definition 1.1, the concept of a linear process with random coefficients is introduced as follows:

DEFINITION 1.2. Let $\{X_n,\ n \in \mathbb{Z}\}$ and $\{A_n,\ n \in \mathbb{Z}\}$ be two sequences of random variables and

$$Y_t = \sum_{j=-\infty}^{\infty} A_j X_{t-j}, \quad t = 1, 2, \cdots. \tag{1.2}$$

Then $\{Y_t,\ t \geqslant 1\}$ is called a linear process with random coefficients.

[32] established the complete moment convergence for a class of linear processes with random coefficients generated by a specific type of random variables. [43] investigated the convergence properties of the CUSUM estimator for a mean shift in linear processes with random coefficients.

Additionally, various other studies have explored related aspects of linear processes. Notably, [17] conducted a comprehensive study on the complete moment convergence of dependent linear processes with random coefficients, the key findings of which are summarized below.

THEOREM 1.1. *Suppose that $\alpha > 0$, $1 < p < 2$, $Y_t = \sum_{j=-\infty}^{\infty} A_j X_{t-j}$ is a linear process with random coefficients, where $\{X_n, n \in \mathbb{Z}\}$ is a sequence of END random variables with mean zero and stochastically dominated by a random variable $X$ with $E|X|^p < \infty$. Furthermore, suppose that $\{A_n, n \in \mathbb{Z}\}$ is a sequence of END random variables with zero mean with*

$$\sum_{j=-\infty}^{\infty} \mathbb{E}|A_j| < \infty,$$

*and for some $p < q \leqslant 2$*

$$\sum_{j=-\infty}^{\infty} \mathbb{E}|A_j|^q < \infty.$$

*If $\{X_n, n \in \mathbb{Z}\}$ is independent of $\{A_n, n \in \mathbb{Z}\}$, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha - 2} \mathbb{E}\left(\left|\sum_{t=1}^{n} X_t\right| - \varepsilon n^{\alpha}\right)_+ < \infty,$$

*where $x_+ = xI(x \geqslant 0)$.*

The purpose of this paper is to study the complete moment convergence for $\rho^*$-mixing linear processes with random coefficients. Hence, we will initially introduce the concept of $\rho^*$-mixing random variables, details of which can be found in [5].

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, P)$. For any $S \subset \mathbb{N} = \{1, 2, \cdots\}$, define $\mathscr{F}_S = \sigma(X_i, i \in S)$. Given two $\sigma$-algebras $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{F}$, put

$$\rho(\mathscr{A}, \mathscr{B}) = \sup\left\{\frac{|EXY - EXEY|}{\sqrt{E(X-EX)^2 E(Y-EY)^2}} : X \in L_2(\mathscr{A}), Y \in L_2(\mathscr{B})\right\}.$$

Define the $\rho^*$-mixing coefficients by

$$\rho_n^* = \sup\{\rho(\mathscr{F}_S, \mathscr{F}_T) : S, T \subset \mathbb{N} \text{ such that } dist(S,T) \geqslant n\},$$

where $\operatorname{dist}(S,T) = \inf\{|s-t| : s \in S, t \in T\}$. Obviously, $0 \leqslant \rho_{n+1}^* \leqslant \rho_n^* \leqslant \rho_0^* = 1$.

DEFINITION 1.3. A sequence $\{X_n, n \geqslant 1\}$ of random variables is called $\rho^*$-mixing if there exists some $k \in \mathbb{N}$ such that $\rho_k^* < 1$.

Significant progress has been made in establishing limit theorems for $\rho^*$-mixing sequences, such as [35] derived Rosenthal-type maximal inequalities for $\rho^*$-mixing random variables; [9] established the Hajek-Renyi type inequality for $\rho^*$-mixing sequences; Kuczmaszewska (2008) as well as [41] independently proved the Chung-Teicher type strong law of large numbers for $\rho^*$-mixing random variables. Further developments include: [31] obtained complete $q$-th moment convergence results for arrays of rowwise $\rho^*$-mixing random variables; [24] investigated the asymptotic properties of CVaR estimators under $\rho^*$-mixing samples; [42] studied complete moment convergence for weighted sums of $\rho^*$-mixing sequences with applications to nonparametric regression models; [8] extended these results by establishing complete and complete moment convergence for weighted sums of $\rho^*$-mixing random variables under appropriate conditions, among other significant contributions; [23] studied complete and complete moment convergence for randomly weighted sums of $\rho^*$-mixing random variables and its applications, and so on.

In this article, we will use the concept of slowly varying function as follows.

DEFINITION 1.4. Let $k(x)$ be a real-valued function, which is positive and measurable on $(0, \infty)$. If for any $t > 0$,

$$\lim_{x \to \infty} \frac{k(tx)}{k(x)} = 1, \tag{1.3}$$

then $k(x)$ is said to be slowly varying at infinity.

Next, we restate the concept of stochastic domination. Some details in the definition of stochastic control can be found in [20].

DEFINITION 1.5. A sequence $\{Y_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable $Y$, if there exists a positive constant $C$ and for all $x \geqslant 0$, we have

$$\sup_{i \geqslant 1} P(|Y_i| > x) \leqslant CP(|Y| > x). \tag{1.4}$$

The paper is organized as follows: Section 2 presents several lemmas, while Section 3 provides the main results and their proof. Some numerical simulations are presented in Section 4.

Throughout this paper, we show some markers. Let $C$ denotes a positive constant not depending on $n$, which may be different in various places. $Z$ represents the set of integer. Let $I(A)$ be the indicator function of the set $A$. Denote $\log x = \ln \max(x, p)$ for any $p > 1$, we denote $||X||_p = (E|X|^p)^{1/p}$ and $x^+ = xI\ (x \geqslant 0)$.

## 2. Some lemmas and their proofs

In order to better illustrate our main results, we need the following lemmas as tools.

LEMMA 2.1. ([12]) *Suppose that random variables $\xi$ and $\eta$ are measurable with respect to $\mathscr{F}_1^k$ and $\mathscr{F}_{k+n}^{\infty}$. Moreover suppose that $||\xi||_2 < \infty$ and $||\eta||_2 < \infty$. Then*

$$|E\xi\eta - E\xi E\eta| \leqslant \rho(n)||\xi||_2||\eta||_2.$$

LEMMA 2.2. ([42]) *Suppose that $\{T_i,\ 1 \leqslant i \leqslant n\}$ and $\{L_i,\ 1 \leqslant i \leqslant n\}$ are two sequences of random variables. For any $l > t > 0$, $\varepsilon > 0$, and $a > 0$, denote $C_t = 1$ when $0 < t \leqslant 1$, or $C_t = 2^{t-1}$ when $t > 1$. Then*

$$\mathbb{E}\left(\left|\sum_{i=1}^{n}(T_i + L_i)\right| - \varepsilon a\right)_+^t \leqslant C_t\left(\varepsilon^{-l} + \tfrac{t}{l-t}\right)a^{t-l}\mathbb{E}\left(\left|\sum_{i=1}^{n} T_i\right|^l\right) + C_t\mathbb{E}\left(\left|\sum_{i=1}^{n} L_i\right|^t\right),$$

*and*

$$\mathbb{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(T_i + L_i)\right| - \varepsilon a\right)_+^t \leqslant C_t\left(\varepsilon^{-l} + \frac{t}{l-t}\right)a^{t-l}\mathbb{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} T_i\right|^l\right) + C_t\mathbb{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} L_i\right|^t\right).$$

LEMMA 2.3. ([1] and [2]) *Suppose that $\{Y_m, m \geqslant 1\}$ is a sequence of random variables, which is stochastically dominated by random variable $Y$. Then for any $m \geqslant 1$, $a > 0$ and $b > 0$, and $C_1$ and $C_2$ are two positive constants. Then*

$$\mathbb{E}\left[|Y_m|^a I\left(|Y_m| \leqslant b\right)\right] \leqslant C_1\left\{\mathbb{E}\left[|Y|^a I(|Y| \leqslant b)\right] + b^a P(|Y| > b)\right\},$$

$$\mathbb{E}\left[Y_m|^a I(|Y_m| > b)\right] \leqslant C_2\mathbb{E}\left[|Y|^a I(|Y| > b)\right].$$

LEMMA 2.4. ([46] and [47]) *Let $q > 1$, and $\{X_n, n \geqslant 1\}$ be a sequence of $\rho^*$-mixing random variables with $\mathbb{E}X_n = 0$ and $\mathbb{E}|X_n|^q < \infty$ for each $n \geqslant 1$. Then for each $n \geqslant 1$,*

$$\mathbb{E}\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|^q\right) \leqslant C_q\sum_{i=1}^{n}\mathbb{E}|X_i|^q,\ \ 1 < q \leqslant 2,$$

*and*

$$\mathbb{E}\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|^q\right) \leqslant C_q\left(\sum_{i=1}^{n}\mathbb{E}|X_i|^q + \left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2}\right),\ \ q > 2,$$

*where $C_q > 0$ is a constant depending only on $q$ and the $\rho^*$-mixing coefficients.*

We can get the following lemma form Lemma 2.4, the details are omitted.

LEMMA 2.5. *Let $q > 1$, and $\{X_n, n \geqslant 1\}$ be a sequence of $\rho^*$-mixing random variables with $\mathbb{E}X_n = 0$ and $\mathbb{E}|X_n|^q < \infty$ for each $n \geqslant 1$. Then for each $n \geqslant 1$,*

$$\mathbb{E}\left(\left|\sum_{i=1}^{j} X_i\right|^q\right) \leqslant C_q \sum_{i=1}^{n} \mathbb{E}|X_i|^q, \quad 1 < q \leqslant 2,$$

*and*

$$\mathbb{E}\left(\left|\sum_{i=1}^{j} X_i\right|^q\right) \leqslant C_q \left(\sum_{i=1}^{n} \mathbb{E}|X_i|^q + \left(\sum_{i=1}^{n} \mathbb{E}X_i^2\right)^{q/2}\right), \quad q > 2,$$

*where $C_q > 0$ is a constant depending only on $q$ and the $\rho^*$-mixing coefficients.*

LEMMA 2.6. ([3]) *If we let $k(x)$ be a slowly varying function at infinity, then the following hold:*

$(i)$ $\lim_{x\to\infty} \frac{k(x+u)}{k(x)} = 1$ *for each* $u > 0$.

$(ii)$ $\lim_{l\to\infty} \sup_{2^l \leqslant x \leqslant 2^{l+1}} \frac{k(x)}{k(2^l)} = 1$.

$(iii)$ $\lim_{x\to\infty} x^{\delta} k(x) = \infty, \ \lim_{x\to\infty} x^{-\delta} k(x) = 0$ *for each* $\delta > 0$.

*(iv) For each $q > 0$, $\varepsilon > 0$ and positive integer $l$, and $C_1$ and $C_2$ are two positive constants:*

$$C_1 2^{lq} k(\varepsilon 2^l) \leqslant \sum_{j=1}^{k} 2^{jq} k(\varepsilon 2^j) \leqslant C_2 2^{lq} k(\varepsilon 2^l).$$

*(v) For each $q < 0$, $\varepsilon > 0$ and positive integer $l$, and $C_3$ and $C_4$ are two positive constants:*

$$C_3 2^{lq} k(\varepsilon 2^l) \leqslant \sum_{j=l}^{\infty} 2^{jq} k(\varepsilon 2^j) \leqslant C_4 2^{lq} k(\varepsilon 2^l).$$

LEMMA 2.7. ([28]) *Suppose that $k(x)$ is a slowly varying at infinity, $C_1$–$C_4$ are four positive constants. Then for any $q > 0$,*

$$C_1 m^{-q} k(m) \leqslant \sum_{l=m}^{\infty} l^{-1-q} k(l) \leqslant C_2 m^{-q} k(m),$$

*and*

$$C_3 m^{q} k(m) \leqslant \sum_{l=1}^{m} l^{-1+q} k(l) \leqslant C_4 m^{q} k(m).$$

The following two lemmas are important to prove our results. The proofs can be referred to [16] and [17], respectively. We only prove Lemma 2.8.

LEMMA 2.8. *Let* $1 \leqslant q \leqslant 2$. *Suppose that* $\{X_m, m \geqslant 1\}$ *is a sequence of* $\rho^*$-*mixing random variables, which satisfies zero mean and* $\mathbb{E}|X_m|^q < \infty$. *Suppose that* $\{A_m, m \geqslant 1\}$ *is a sequence of random variables, which satisfies* $\mathbb{E}|A_m|^q < \infty$. *If we further assume* $\{X_m, m \geqslant 1\}$ *is independent of* $\{A_m, m \geqslant 1\}$, *and* $\sum_{i=-\infty}^{\infty}(\sum_{j=1-i}^{m-i} A_j)X_i < \infty$ *a.s., then for any* $m \geqslant 1$,

$$\mathbb{E}\left|\sum_{i=-\infty}^{\infty}\left(\sum_{j=1-i}^{m-i} A_j\right)X_i\right|^q \leqslant C_q \sum_{i=-\infty}^{\infty} \mathbb{E}\left|\sum_{j=1-i}^{m-i} A_j\right|^q \mathbb{E}|X_i|^q.$$

*Proof.* By applying Fatou's Lemma and the $C_r$-inequality, we obtain the following estimation:

$$\begin{aligned}
\mathbb{E}\left|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{n-i} A_j X_i\right|^q &= \mathbb{E}\lim_{m\to\infty}\left|\sum_{i=-m}^{m}\left(\sum_{j=1-i}^{n-i} A_j\right)X_i\right|^q \\
&\leqslant \liminf_{m\to\infty}\mathbb{E}\left|\sum_{i=-m}^{m}\left(\left(\sum_{j=1-i}^{n-i} A_j\right)^+ - \left(\sum_{j=1-i}^{n-i} A_j\right)^-\right)X_i\right|^q \\
&\leqslant 2^{q-1}\limsup_{m\to\infty}\mathbb{E}\left|\sum_{i=-m}^{m}\left(\sum_{j=1-i}^{n-i} A_j\right)^+ X_i\right|^q \\
&\quad +2^{q-1}\limsup_{m\to\infty}\mathbb{E}\left|\sum_{i=-m}^{m}\left(\sum_{j=1-i}^{n-i} A_j\right)^- X_i\right|^q \\
&= I_1 + I_2. \qquad (2.1)
\end{aligned}$$

For all $n \geqslant 1$ and any $j \in \mathbb{Z}$, the term $(\sum_{j=1-i}^{n-i} a_j)^+$ represents a positive constant. According to the Definition 1.3, we can get that the sequence $\{(\sum_{j=1-i}^{n-i} a_j)^+ X_i, i \in \mathbb{Z}\}$ also $\rho^*$-mixing random variables with zero mean. Consequently, by employing Lemma 2.5, we have:

$$\begin{aligned}
&\mathbb{E}\left|\sum_{i=-m}^{m}\left(\sum_{j=1-i}^{n-i} A_j\right)^+ X_i\right|^q \\
&= \mathbb{E}\left[\mathbb{E}\left|\sum_{i=-m}^{m}\left(\sum_{j=1-i}^{n-i} a_j\right)^+ X_i\right|^q \middle| A_{-m+1}=a_{-m+1},\ldots,A_{m+n}=a_{m+n}\right] \\
&\leqslant C_q \mathbb{E}\left[\sum_{i=-m}^{m}\mathbb{E}\left|\left(\sum_{j=1-i}^{n-i} a_j\right)^+ X_i\right|^q \middle| A_{-m+1}=a_{-m+1},\ldots,A_{m+n}=a_{m+n}\right] \\
&\leqslant C_q \sum_{i=-m}^{m}\mathbb{E}\left(\left(\sum_{j=1-i}^{n-i} A_j\right)^+\right)^q \mathbb{E}|X_i|^q.
\end{aligned}$$

Hence, for $I_1$, we establish the following upper bound from $C_r$-inequality:

$$I_1 \leqslant C_q \limsup_{m\to\infty} \sum_{i=-m}^{m} \mathbb{E}\left(\left(\left(\sum_{j=1-i}^{n-i} A_j\right)^+\right)^q\right) \mathbf{E}|X_i|^q$$
$$\leqslant C_q \sum_{i=-\infty}^{\infty} \mathbb{E}\left|\sum_{j=1-i}^{n-i} A_j\right|^p \mathbb{E}|X_i|^q.$$

Furthermore, since $\left\{\max_{1<k\leqslant n}\left(\sum_{j=1-i}^{k-i} a_j\right)^- X_i, i \in \mathbb{Z}\right\}$ constitutes a sequence of $\rho^*$-mixing random variables with zero mean, we can derive an analogous bound for $I_2$:

$$I_2 \leqslant C_q \sum_{i=-\infty}^{\infty} \mathbb{E}\left|\sum_{j=1-i}^{n-i} A_j\right|^q \mathbb{E}|X_i|^q.$$

Consequently, combining the results from above, we obtain the desired inequality (2.1). This completes the proof. □

LEMMA 2.9. *Let* $1 \leqslant q \leqslant 2$. *Suppose that* $\{X_m, m \geqslant 1\}$ *is a sequence of* $\rho^*$*-mixing random variables, which satisfies zero mean and* $\mathbb{E}|X_m|^q < \infty$. *Suppose that* $\{A_m, m \geqslant 1\}$ *is a sequence of random variables, which satisfies* $\mathbb{E}|A_m|^q < \infty$. *If we further assume* $\{X_m, m \geqslant 1\}$ *is independent of* $\{A_m, m \geqslant 1\}$, *and* $\sum_{i=-\infty}^{\infty}(\sum_{j=1-i}^{m-i} A_j)X_i < \infty$ *a.s., then for any* $m \geqslant 1$,

$$\mathbb{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=-\infty}^{\infty}\left(\sum_{j=1-i}^{m-i} A_j\right) X_i\right|^q\right) \leqslant C_q \sum_{i=-\infty}^{\infty} \mathbb{E}\left|\sum_{j=1-i}^{m-i} A_j\right|^q \mathbb{E}|X_i|^q.$$

## 3. Main results and theirs proofs

THEOREM 3.1. *Let* $\alpha > 0$, $1 \leqslant r < p < q \leqslant 2$ *and* $k(x) > 0$ *is a slowly varying function. Let* $Y_t = \sum_{j=-\infty}^{\infty} A_j X_{t-j}$ *be a linear process with random coefficients. Suppose that* $\{X_m, m \in \mathbb{Z}\}$ *is a sequence of* $\rho^*$*-mixing random variable, which also satisfies mean zero and stochastically dominated by a nonnegative random variable* $X$ *with* $\mathbb{E}\left[|X|^p k(|X|^{\frac{1}{\alpha}})\right] < \infty$. *If we suppose that* $\{A_m, m \in \mathbb{Z}\}$ *is a sequence of indpendent random variables with mean zero, which is independent of* $\{X_m, m \in \mathbb{Z}\}$,

$$\sum_{j=-\infty}^{\infty} \mathbb{E}|A_j| < \infty, \quad \sum_{j=-\infty}^{\infty} \mathbb{E}|A_j|^r < \infty, \tag{3.1}$$

*and*

$$\sum_{j=-\infty}^{\infty} \mathbb{E}|A_j|^q < \infty. \tag{3.2}$$

*Then for any* $\varepsilon > 0$*,*

$$\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left(\left|\sum_{t=1}^{m} Y_t\right| - \varepsilon m^{\alpha}\right)_+^r < \infty, \tag{3.3}$$

*which imply that*

$$\sum_{m=1}^{\infty} m^{\alpha p-2}k(m)P\left(\left|\sum_{t=1}^{m} Y_t\right| > \varepsilon m^{\alpha}\right) < \infty. \tag{3.4}$$

*Proof.* Since $\mathbb{E}X_i = 0$, then we can denote that for $i \in \mathbb{Z}$ and $m \geqslant 1$

$$X_i = X_i^{(1)} - \mathbb{E}X_i^{(1)} + X_i^{(2)} - \mathbb{E}X_i^{(2)},$$

where

$$X_i^{(1)} = X_i I(|X_i| \leqslant m^{\alpha}) + m^{\alpha} I(X_i > m^{\alpha}) - m^{\alpha} I(X_i < -m^{\alpha})$$

and

$$X_i^{(2)} = X_i - X_i^{(1)} = (X_i - m^{\alpha})I(X_i > m^{\alpha}) + (X_i + m^{\alpha})I(X_i < -m^{\alpha}).$$

Form above, we can get that

$$\begin{aligned}
\sum_{t=1}^{n} Y_t &= \sum_{t=1}^{n}\sum_{j=-\infty}^{\infty} A_j X_{t-j} = \sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{n-i} A_j X_i \\
&= \sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{n-i} A_j\left(X_i^{(1)} - \mathbb{E}X_i^{(1)}\right) + \sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{n-i} A_j\left(X_i^{(2)} - \mathbb{E}X_i^{(2)}\right).
\end{aligned}$$

We can know that for $i \in \mathbb{Z}$, $\left\{X_i^{(1)} - \mathbb{E}X_i^{(1)}\right\}$ and $\left\{X_i^{(2)} - \mathbb{E}X_i^{(2)}\right\}$ are both $\rho^*$-mixing random variables form Definition 1.3, with mixing coefficients $\alpha(n)$ and zero mean. We have by Lemma 2.2 that for $q > r > 0$

$$\begin{aligned}
&\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left(\left|\sum_{t=1}^{m} Y_t\right| - \varepsilon m^{\alpha}\right)_+^r \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\frac{1}{m^{\alpha(q-r)}}\mathbb{E}\left[\left|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{m-i} A_j\left(X_i^{(1)} - \mathbb{E}X_i^{(1)}\right)\right|^q\right] \\
&\quad + C\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left[\left|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{m-i} A_j\left(X_i^{(2)} - \mathbb{E}X_i^{(2)}\right)\right|^r\right] \\
&= C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-2}k(m)\mathbb{E}\left[\left|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{m-i} A_j\left(X_i^{(1)} - \mathbb{E}X_i^{(1)}\right)\right|^q\right] \\
&\quad + C\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left[\left|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{m-i} A_j\left(X_i^{(2)} - \mathbb{E}X_i^{(2)}\right)\right|^r\right] \\
&:= I_1 + I_2.
\end{aligned} \tag{3.5}$$

For $I_1$, by Lemmas 2.3–2.4, 2.6–2.8 and (3.2), then

$$\begin{aligned}
I_1 &\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-2}k(m)\sum_{i=-\infty}^{\infty}\mathbb{E}\left(\left|\sum_{j=1-i}^{m-i}A_j\right|^q\right)\mathbb{E}\left|X_i^{(1)}-\mathbb{E}X_i^{(1)}\right|^q \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-2}k(m)\sum_{i=-\infty}^{\infty}\mathbb{E}\left|X_i^{(1)}\right|^q\sum_{j=1-i}^{m-i}\mathbb{E}|A_j|^q \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-2}k(m)\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}\mathbb{E}|A_j|^q\mathbb{E}|X_i|^q I(|X_i|\leqslant m^{\alpha}) \\
&\quad +C\sum_{m=1}^{\infty} m^{\alpha p-2}k(m)\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}\mathbb{E}|A_j|^q P(|X_i|>m^{\alpha}) \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-1}k(m)\Big\{\mathbb{E}[|X|^q I(|X|\leqslant m^{\alpha})]+m^{\alpha q}P(|X|>m^{\alpha})\Big\}\sum_{j=-\infty}^{\infty}\mathbb{E}|A_j|^q \\
&\quad +C\sum_{m=1}^{\infty} m^{\alpha p-\alpha-2}k(m)\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}\mathbb{E}|A_j|^q\mathbb{E}[|X|I(|X|>n^{\alpha})] \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-1}k(m)\mathbb{E}[|X|^q I(|X|\leqslant m^{\alpha})] \\
&\quad +C\sum_{m=1}^{\infty} m^{\alpha p-\alpha-1}k(m)\mathbb{E}[|X|I(|X|>m^{\alpha})] \\
&:= I_3+I_4. \qquad (3.6)
\end{aligned}$$

For $I_3$, noting that $p<q$, then we get by Lemma 2.7 that,

$$\begin{aligned}
I_3 &= C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-1}k(m)\sum_{i=1}^{m}\mathbb{E}[|X|^q I((i-1)^{\alpha}<|X|\leqslant i^{\alpha})] \\
&= C\sum_{i=1}^{\infty}\mathbb{E}[|X|^q I((i-1)^{\alpha}<|X|\leqslant i^{\alpha})]\sum_{m=i}^{\infty} m^{\alpha p-\alpha q-1}k(m) \\
&\leqslant C\sum_{i=1}^{\infty} i^{\alpha p-\alpha q}k(i)\mathbb{E}[|X|^q I((i-1)^{\alpha}<|X|\leqslant i^{\alpha})] \\
&\leqslant C\mathbb{E}\left[|X|^p k\left(|X|^{\frac{1}{\alpha}}\right)\right]<\infty, \qquad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= C\sum_{l=1}^{\infty}\mathbb{E}[|X|I(l^{\alpha}<|X|\leqslant (l+1)^{\alpha})]\sum_{m=1}^{l} m^{\alpha p-\alpha-1}k(m) \\
&\leqslant C\sum_{l=1}^{\infty} l^{\alpha p-\alpha}k(l)\mathbb{E}[|X|I(l^{\alpha}<|X|\leqslant (l+1)^{\alpha})] \\
&\leqslant C\mathbb{E}\left[|X|^p k(|X|^{\frac{1}{\alpha}})\right]<\infty. \qquad (3.8)
\end{aligned}$$

Next, we will analysis $I_2$. Noting that $\left|X_i^{(2)}\right| = (|X_i| - n^\alpha)I(|X_i| > n^\alpha) \leqslant |X_i|I(|X_i| > n^\alpha)$ and $\mathbb{E}\left[|X|^p k(|X|^{\frac{1}{\alpha}})\right] < \infty$, then we can obtain by Lemmas 2.3–2.4, 2.6–2.8 and (3.1) that

$$\begin{aligned}
I_2 &\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}\mathbb{E}|A_j|^r\mathbb{E}[|X_i|^r I(|X_i| > m^\alpha)] \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}[|X|^r I(|X| > m^\alpha)]\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}\mathbb{E}|A_j|^r \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-1}k(m)\mathbb{E}[|X|^r I(|X| > n^\alpha)] \\
&= C\sum_{l=1}^{\infty}\mathbb{E}[|X|^r I(l^\alpha < |X| \leqslant (l+1)^\alpha)]\sum_{m=1}^{l} m^{\alpha p-\alpha r-1}k(m) \\
&\leqslant C\sum_{l=1}^{\infty} l^{\alpha p-\alpha r}k(l)\mathbb{E}[|\varepsilon|^r I(l^\alpha < |X| \leqslant (l+1)^\alpha)] \\
&\leqslant C\mathbb{E}\left[|X|^p k\left(|X|^{\frac{1}{\alpha}}\right)\right] < \infty. \qquad (3.9)
\end{aligned}$$

Thus, we can get (3.3) from (3.5)–(3.9) immediately. We will get (3.4) by (3.3) again that

$$\begin{aligned}
\infty &> \sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left(\left|\sum_{t=1}^{m} Y_t\right| - \varepsilon m^\alpha\right)_+^r \\
&= \sum_{m=1}^{\infty} m^{\alpha p-2}k(m)\int_0^\infty P\left(\frac{\left|\sum_{t=1}^{m} Y_t\right|}{m^\alpha} > t^{1/r} + \varepsilon\right)dt \\
&\geqslant \sum_{m=1}^{\infty} m^{\alpha p-2}k(m)\int_0^{\varepsilon^r} P\left(\frac{\left|\sum_{t=1}^{m} Y_t\right|}{m^\alpha} > 2\varepsilon\right)dt \\
&= \varepsilon^r\sum_{m=1}^{\infty} m^{\alpha p-2}k(m)P\left(\left|\sum_{t=1}^{m} Y_t\right| > 2\varepsilon m^\alpha\right).
\end{aligned}$$

Thus the proof is completed. □

THEOREM 3.2. *Let $\alpha > p$, $1 \leqslant r < p < q \leqslant 2$ and $k(x) > 0$ is a slowly varying function. Let $Y_t = \sum_{j=-\infty}^{\infty} A_j X_{t-j}$ be a linear process with random coefficients and $\{X_n, n \geqslant 1\}$ be a sequence of $\rho^*$-mixing random variable, which also satisfies mean zero and stochastically dominated by a random variable $X$ with $\mathbb{E}\left[|X|^p k(|X|^{\frac{1}{\alpha}})\right] < \infty$.*

*If we suppose that $\{A_m, m \in \mathbb{Z}\}$ is a sequence of indpendent random variables with mean zero, which is independent of $\{X_m, m \in \mathbb{Z}\}$,*

$$\mathbb{E}\left(\sum_{j=-\infty}^{\infty}|A_j|\right)^q < \infty. \tag{3.10}$$

*Then for any $\varepsilon > 0$,*

$$\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k}Y_t\right| - \varepsilon m^{\alpha}\right)_+^r < \infty. \tag{3.11}$$

*Furthermore, we have*

$$\sum_{m=1}^{\infty} m^{\alpha p-2}k(m)P\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k}Y_t\right| > \varepsilon m^{\alpha}\right) < \infty. \tag{3.12}$$

*Proof.* The proof is similar to Theorem 3.1, hence, we only provide a simple proof for this theorem. First and foremost, we have following by Lemma 2.2 that

$$\begin{aligned}
&\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k}Y_t\right| - \varepsilon m^{\alpha}\right)_+^r \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-2}k(m)\mathbb{E}\left[\max_{1\leqslant k\leqslant m}\left|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{k-i}A_j\left(X_i^{(1)}-\mathbb{E}X_i^{(1)}\right)\right|^q\right] \\
&\quad + C\sum_{m=1}^{\infty} m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left[\max_{1\leqslant k\leqslant m}\left|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{k-i}A_j\left(X_i^{(2)}-\mathbb{E}X_i^{(2)}\right)\right|^r\right] \\
&:= I_1' + I_2'.
\end{aligned} \tag{3.13}$$

Similar to the proof of $I_1$, by Lemmas 2.5, 2.9 and (3.10), we obtain

$$\begin{aligned}
I_1' &\leqslant \sum_{m=1}^{\infty} m^{\alpha p-\alpha q-2}k(m)\sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{m-j}\mathbb{E}\left|X_i^{(1)}-\mathbb{E}X_i^{(1)}\right|^q \\
&\leqslant C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q-1}k(m)\mathbb{E}[|X|^q I(|X|\leqslant m^{\alpha})] \\
&\quad + C\sum_{m=1}^{\infty} m^{\alpha p-1}k(m)\mathbb{P}(|X|>m^{\alpha}) \\
&:= I_3' + I_4'.
\end{aligned}$$

For $I_3'$, we have

$$\begin{aligned}
I_3' &\leqslant C\sum_{i=1}^{\infty}\mathbb{E}[|X|^q I((i-1)^{\alpha}<|X|\leqslant i^{\alpha})]\sum_{m=i}^{\infty} m^{\alpha p-\alpha q-1}k(m) \\
&\leqslant C\sum_{i=1}^{\infty} i^{\alpha p-\alpha q}k(i)\mathbb{E}\left[|X|^q I\left((i-1)^{\alpha}<|X|\leqslant i^{\alpha}\right)\right] \\
&\leqslant C\mathbb{E}\left[|X|^p k(|X|^{\frac{1}{\alpha}})\right] < \infty.
\end{aligned}$$

For $I_4'$, noting that $p>1$, we have

$$\begin{aligned} I_4' &\leqslant C\sum_{i=1}^{\infty}\mathbb{E}[|X|I((i-1)^{\alpha}<|X|\leqslant i^{\alpha})]\sum_{m=i}^{\infty}m^{\alpha p-\alpha-1}k(m) \\ &\leqslant C\sum_{i=1}^{\infty}i^{\alpha p-\alpha}k(i)\mathbb{E}[|X|I((i-1)^{\alpha}<|X|\leqslant i^{\alpha})] \\ &\leqslant C\mathbb{E}\left[|X|^p k(|X|^{\frac{1}{\alpha}})\right]<\infty. \end{aligned}$$

Finally, we prove $I_2'<\infty$. Note that $\sum_{j=-\infty}^{\infty}\mathbb{E}|A_j|<\infty$ by (3.10). By Lemma 2.9, we get

$$\begin{aligned} I_2' &\leqslant C\sum_{m=1}^{\infty}m^{\alpha p-\alpha r-2}k(m)\sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{m-j}\mathbb{E}\left|X_i^{(2)}-\mathbb{E}X_i^{(2)}\right|^r \\ &\leqslant C\sum_{m=1}^{\infty}m^{\alpha p-\alpha r-2}k(m)\mathbb{E}\left|X_i^{(2)}-\mathbb{E}\varepsilon_i^{(2)}\right|^r \\ &\leqslant C\sum_{m=1}^{\infty}m^{\alpha p-\alpha r-1}k(m)\mathbb{E}\left(|X|^r I(|X|>m^{\alpha})\right) \\ &= C\sum_{m=1}^{\infty}m^{\alpha p-\alpha r-1}k(m)\mathbb{E}\left[|X|^r\sum_{l=m}^{\infty}I(l^{\alpha}<|X|\leqslant(l+1)^{\alpha})\right] \\ &= C\sum_{l=1}^{\infty}\mathbb{E}\left[|X|^r I\left((l-1)^{\alpha}<|X|\leqslant l^{\alpha}\right)\right]\sum_{m=1}^{l}m^{\alpha p-\alpha r-1}k(m) \\ &\leqslant C\mathbb{E}\left[|X|^p k(|X|^{\frac{1}{\alpha}})\right]<\infty. \end{aligned}$$

Thus the proof is completed. □

REMARK 3.1. Let $\alpha=1/p$, $k(m)=1$, under the conditions of Theorem 3.1, we can get by (3.4) that

$$\sum_{m=1}^{\infty}\frac{1}{m}P\left(\left|\sum_{t=1}^{m}Y_t\right|>\varepsilon m^{1/p}\right)<\infty.$$

Then according to Marcinkiewica-Zygmund strong laws of large numbers for linear processes with random coefficients, we have

$$\frac{1}{m^{1/p}}\sum_{t=1}^{m}Y_t\xrightarrow{a.s.}0,\quad \text{as}\quad m\to\infty.$$

Similarly, let $\alpha=1/p$, $k(m)=1$, under the conditions of Theorem 3.2, we can get by (3.12) that

$$\sum_{m=1}^{\infty}\frac{1}{m}P\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k}Y_t\right|>\varepsilon m^{1/p}\right)<\infty,$$

and

$$\frac{1}{m^{1/p}}\sum_{t=1}^{m}Y_t \xrightarrow{a.s.} 0, \quad \text{as} \quad m\to\infty.$$

For $p=r=1$, we can get the following two corollaries from Theorem 3.1 and Theorem 3.2, respectively.

COROLLARY 3.1. *Let $\alpha>0$, $1<q\leqslant 2$ and $k(x)>0$ is a slowly varying function. Let $Y_t=\sum_{j=-\infty}^{\infty}A_jX_{t-j}$ be a linear process with random coefficients. Suppose that $\{X_m, m\in\mathbb{Z}\}$ is a sequence of $\rho^*$-mixing random variables, which also satisfies mean zero and stochastically dominated by a nonnegative random variable $X$ with $\mathbb{E}\left[|X|k(|X|^{\frac{1}{\alpha}})\right]<\infty$. If we suppose that $\{A_m, m\in\mathbb{Z}\}$ is a sequence of indpendent random variables with mean zero, which is independent of $\{X_m, m\in\mathbb{Z}\}$,*

$$\sum_{j=-\infty}^{\infty}\mathbb{E}|A_j|<\infty \quad \textit{and} \quad \sum_{j=-\infty}^{\infty}\mathbb{E}|A_j|^q<\infty, \tag{3.14}$$

*then for any $\varepsilon>0$,*

$$\sum_{m=1}^{\infty}m^{-2}k(m)\mathbb{E}\left(\left|\sum_{t=1}^{m}Y_t\right|-\varepsilon m^{\alpha}\right)_+<\infty. \tag{3.15}$$

*Furthermore, we have*

$$\sum_{m=1}^{\infty}m^{\alpha-2}k(m)P\left(\left|\sum_{t=1}^{m}Y_t\right|>\varepsilon m^{\alpha}\right)<\infty. \tag{3.16}$$

*Proof.* The proof is similar to that of Theorem 3.1 with $p=r=1$. We only need to show $I_2<\infty$, $I_3<\infty$ and $I_4<\infty$.

For $I_3$ and $I_4$, by Lemmas 2.5–2.6, we can get

$$\begin{aligned}
I_3 &\leqslant C\sum_{m=1}^{\infty}m^{\alpha-\alpha q-1}k(m)\mathbb{E}[|X|^qI(|X|\leqslant m^{\alpha})]\\
&= C\sum_{m=1}^{\infty}m^{\alpha-\alpha q-1}k(m)\sum_{i=1}^{m}\mathbb{E}[|X|^qI((i-1)^{\alpha}<|X|\leqslant i^{\alpha})]\\
&= C\sum_{i=1}^{\infty}\mathbb{E}[|X|^qI((i-1)^{\alpha}<|X|\leqslant i^{\alpha})]\sum_{m=i}^{\infty}m^{\alpha-\alpha q-1}k(m)\\
&\leqslant C\sum_{i=1}^{\infty}i^{\alpha-\alpha q}k(i)\mathbb{E}[|X|^qI((i-1)^{\alpha}<|X|\leqslant i^{\alpha})]\\
&\leqslant C\mathbb{E}\left[|X|k(|X|^{\frac{1}{\alpha}})\right]<\infty,
\end{aligned} \tag{3.17}$$

and

$$
\begin{aligned}
I_4 &\leqslant C\sum_{m=1}^{\infty} m^{-1}k(m)\sum_{l=m}^{\infty}\mathbb{E}[|X|I(l^{\alpha}<|X|\leqslant (l+1)^{\alpha})]\\
&= C\sum_{l=1}^{\infty}\mathbb{E}[|X|I(l^{\alpha}<|X|\leqslant (l+1)^{\alpha})]\sum_{m=1}^{l} m^{-1}k(m)\\
&\leqslant C\sum_{l=1}^{\infty}k(l)\mathbb{E}[|X|I(l^{\alpha}<|X|\leqslant (l+1)^{\alpha})]\\
&\leqslant C\mathbb{E}\left[|X|k(|X|^{\frac{1}{\alpha}})\right]<\infty. \qquad (3.18)
\end{aligned}
$$

For $I_2$, we can obtain by Lemmas 2.3–2.4, 2.6–2.8 and (3.1) that

$$
\begin{aligned}
I_2 &\leqslant C\sum_{m=1}^{\infty} m^{-2}k(m)\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{m-j}\mathbb{E}|A_j|^r\mathbb{E}[|X_i|^rI(|X_i|>m^{\alpha})]\\
&\leqslant C\sum_{m=1}^{\infty} m^{-1}k(m)\mathbb{E}[|X|I(|X|>m^{\alpha})]\\
&= C\sum_{l=1}^{\infty}\mathbb{E}[|X|I(l^{\alpha}<|X|\leqslant (l+1)^{\alpha})]\sum_{m=1}^{l} m^{-1}k(m)\\
&\leqslant C\sum_{l=1}^{\infty}k(l)\mathbb{E}[|X|I(l^{\alpha}<|X|\leqslant (l+1)^{\alpha})]\\
&\leqslant C\mathbb{E}\left[|X|k(|X|^{\frac{1}{\alpha}})\right]<\infty. \qquad (3.19)
\end{aligned}
$$

Thus, we can get (3.15) from (3.17)–(3.19) immediately. We can get (3.16) by (3.15) immediately. Thus the proof is completed. □

COROLLARY 3.2. *Let $\alpha>0$, $1<q\leqslant 2$ and $k(x)>0$ is a slowly varying function. Let $Y_t=\sum_{j=-\infty}^{\infty}A_jX_{t-j}$ be a linear process with random coefficients. Suppose that $\{X_m, m\geqslant 1\}$ be a sequence of $\rho^*$-mixing random variables, which also satisfies mean zero and stochastically dominated by a nonnegative random variable $X$ with $\mathbb{E}\left[|X|k(|X|^{\frac{1}{\alpha}})\right]<\infty$. If we suppose that $\{A_m, m\in\mathbb{Z}\}$ is a sequence of indpendent random variables with mean zero, which is independent of $\{X_m, m\in\mathbb{Z}\}$,*

$$
\mathbb{E}\left(\sum_{j=-\infty}^{\infty}|A_j|\right)^q<\infty, \qquad (3.20)
$$

*then for any $\varepsilon>0$,*

$$
\sum_{m=1}^{\infty} m^{-2}k(m)\mathbb{E}\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k}Y_t\right|-\varepsilon m^{\alpha}\right)_+<\infty, \qquad (3.21)
$$

*which imply that*

$$\sum_{m=1}^{\infty} m^{\alpha-2}k(m)P\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k}Y_t\right| > \varepsilon m^{\alpha}\right) < \infty. \tag{3.22}$$

*Proof.* The proof can be referred to Theorem 3.2 and Corollary 3.1. Thus, the details are omitted. □

REMARK 3.2. Let $\alpha = 1$, $k(m) = 1$, under the conditions of Corollary 3.1, then we can get by (3.16) that, for any $\varepsilon > 0$

$$\sum_{m=1}^{\infty} \frac{1}{m}P\left(\left|\sum_{t=1}^{m}Y_t\right| > \varepsilon m\right) < \infty.$$

Then according to Marcinkiewica-Zygmund strong laws of large numbers for linear processes with random coefficients,

$$\frac{1}{m}\sum_{t=1}^{m}Y_t \xrightarrow{a.s.} 0, \quad \text{as} \quad m \to \infty.$$

Similarly, let $\alpha = 1$, $l(m) = 1$, under the conditions of Corollary 3.2, we can get by (3.22) that

$$\sum_{m=1}^{\infty} \frac{1}{m}P\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k}Y_t\right| > \varepsilon m\right) < \infty,$$

and

$$\frac{1}{m}\sum_{t=1}^{m}Y_t \xrightarrow{a.s.} 0, \quad \text{as} \quad m \to \infty.$$

For $1 < p = r < 2$, we can get the following two corollaries from Theorem 3.1 and Theorem 3.2, respectively. The proof can be referred to theorems and corollaries above. Thus, the details are omitted.

COROLLARY 3.3. *Let* $\alpha > 0$, $1 < p < q \leqslant 2$, $0 < \gamma < q - p$ *and* $k(x) > 0$ *is a slowly varying function. Let* $Y_t = \sum_{j=-\infty}^{\infty} A_j X_{t-j}$ *be a linear process with random coefficients and the* $\{X_m, m \geqslant 1\}$ *be a sequence of* $\rho^*$*-mixing random variables, which also satisfies mean zero and stochastically dominated by a nonnegative random variable* $X$ *with* $\mathbb{E}\left[|X|^p k(|X|^{\frac{1}{\alpha}})\right] < \infty$. *If we suppose that* $\{A_m, m \in \mathbb{Z}\}$ *is a sequence of indpendent random variables with mean zero, which is independent of* $\{X_m, m \in \mathbb{Z}\}$,

$$\sum_{j=-\infty}^{\infty} \mathbb{E}|A_j| < \infty, \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \mathbb{E}|A_j|^p < \infty. \tag{3.23}$$

$$\sum_{j=-\infty}^{\infty} \mathbb{E}|A_j|^q < \infty, \tag{3.24}$$

*then for any* $\varepsilon > 0$,

$$\sum_{m=1}^{\infty} m^{-2}k(m)\mathbb{E}\left(\left|\sum_{t=1}^{m} Y_t\right| - \varepsilon m^{\alpha}\right)_+^p < \infty, \tag{3.25}$$

*which imply that*

$$\sum_{m=1}^{\infty} m^{\alpha p-2}k(m)P\left(\left|\sum_{t=1}^{m} Y_t\right| > \varepsilon m^{\alpha}\right) < \infty. \tag{3.26}$$

COROLLARY 3.4. *Let* $\alpha > 0$, $1 < p < q \leqslant 2$, $0 < \gamma < q-p$ *and* $k(x) > 0$ *is a slowly varying function. Let* $Y_t = \sum_{j=-\infty}^{\infty} A_j X_{t-j}$ *be a linear process with random coefficients and the* $\{X_m, m \geqslant 1\}$ *be a sequence of* $\rho^*$*-mixing random variables, which also satisfies mean zero and stochastically dominated by a nonnegative random variable* $X$ *with* $\mathbb{E}\left[|X|^p \log^q(1+\varepsilon)k\left(|\varepsilon|^{\frac{1}{\alpha}}\right)\right] < \infty$. *If we suppose that* $\{A_m, m \in \mathbb{Z}\}$ *is a sequence of indpendent random variables with mean zero, which is independent of* $\{X_m, m \in \mathbb{Z}\}$,

$$\mathbb{E}\left(\sum_{j=-\infty}^{\infty} |A_j|\right)^q < \infty, \tag{3.27}$$

*then for any* $\varepsilon > 0$,

$$\sum_{m=1}^{\infty} m^{-2}k(m)\mathbb{E}\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k} Y_t\right| - \varepsilon m^{\alpha}\right)_+^p < \infty, \tag{3.28}$$

*which imply that*

$$\sum_{m=1}^{\infty} m^{\alpha p-2}k(m)P\left(\max_{1\leqslant k\leqslant m}\left|\sum_{t=1}^{k} Y_t\right| > \varepsilon m^{\alpha}\right) < \infty. \tag{3.29}$$

REMARK 3.3. Let $\alpha = 1/p$, $k(m) = 1$, under the conditions of Corollary 3.3, we can get by (3.26) that, for any $\varepsilon > 0$,

$$\sum_{m=1}^{\infty} \frac{1}{m} P\left(\left|\sum_{t=1}^{k} X_t\right| > \varepsilon m^{1/p}\right) < \infty.$$

Then according to Marcinkiewica-Zygmund strong laws of large numbers for linear processes with random coefficients, we have

$$\frac{1}{m^{1/p}} \sum_{t=1}^{m} X_t \xrightarrow{a.s.} 0, \quad \text{as} \quad m \to \infty.$$

Similarly, let $\alpha = 1/p$, $k(m) = 1$, under the conditions of Corollary 3.4, we can get by (3.29) that

$$\sum_{m=1}^{\infty} \frac{1}{m} P\left( \max_{1 \leqslant k \leqslant m} \left| \sum_{t=1}^{k} Y_t \right| > \varepsilon m^{1/p} \right) < \infty,$$

and

$$\frac{1}{m^{1/p}} \sum_{t=1}^{m} Y_t \xrightarrow{a.s.} 0, \quad \text{as} \quad m \to \infty.$$

The next theorem is about the $L^p$ convergence for linear process with random coefficients.

THEOREM 3.3. *Let $1 \leqslant p \leqslant 2$. Let $Y_t = \sum_{j=-\infty}^{\infty} A_j X_{t-j}$ be a linear process with random coefficients and the $\{X_m, m \in \mathbb{Z}\}$ be a sequence of $\rho^*$-mixing random variables, which also satisfies mean zero and stochastically dominated by a nonnegative random variable $X$ with $\mathbb{E}|X|^p < \infty$. If we suppose that $\{A_m, m \in \mathbb{Z}\}$ is a sequence of indpendent random variables with mean zero, which is independent of $\{X_m, m \in \mathbb{Z}\}$,*

$$\sum_{j=-\infty}^{\infty} \mathbb{E}|A_j|^p < \infty, \tag{3.30}$$

*then*

$$\sup_{m \geqslant 1} \frac{1}{m^{r+1}} \mathbb{E} \left| \sum_{t=1}^{m} \sum_{j=-\infty}^{\infty} A_j X_{t-j} \right|^p \leqslant C < \infty. \tag{3.31}$$

*If we further assume that $pq > r+1$, then as $m \to \infty$,*

$$\mathbb{E} \left| \frac{1}{m^q} \sum_{t=1}^{m} \sum_{j=-\infty}^{\infty} A_j X_{t-j} \right|^p = O(m^{r+1-pq}), \tag{3.32}$$

*and thus*

$$\frac{1}{m^q} \sum_{t=1}^{m} \sum_{j=-\infty}^{\infty} A_j X_{t-j} \xrightarrow{L^p} 0. \tag{3.33}$$

*Proof.* By Lemmas 2.5, 2.8, $\mathbb{E}|\varepsilon|^p < \infty$ and (3.30), we can get that

$$\begin{aligned} \frac{1}{m^{r+1}} \mathbb{E} \left| \sum_{t=1}^{m} \sum_{j=-\infty}^{\infty} A_j X_{t-j} \right|^p &= \frac{1}{m^{r+1}} \mathbb{E} \left| \sum_{j=-\infty}^{\infty} \left( \sum_{t=1}^{m} X_{t-j} \right) A_j \right|^p \\ &\leqslant \frac{1}{m^{r+1}} \sum_{j=-\infty}^{\infty} \mathbb{E} \left| \sum_{t=1}^{n} X_{t-j} \right|^p \mathbb{E}|A_j|^p \\ &\leqslant C\mathbb{E}|X|^p \sum_{j=-\infty}^{\infty} \mathbb{E}|A_j|^p < \infty, \end{aligned}$$

which implies (3.31), and (3.32) follows from (3.31) immediately.

The proof is completed. □

REMARK 3.4. Note that $Y_m = \sum_{j=-\infty}^{\infty} A_j X_{m-j}$. Under the conditions of Theorem 3.3, we have by $C_r$-inequality and (3.31) that

$$\begin{aligned}
\sup_{m\geqslant 1} \frac{1}{m^{r+1}} \mathbb{E}|Y_m|^p &= \sup_{m\geqslant 1} \frac{1}{m^{r+1}} \mathbb{E}\left|\sum_{t=1}^{m}\sum_{j=-\infty}^{\infty} A_j X_{t-j} - \sum_{t=1}^{m-1}\sum_{j=-\infty}^{\infty} A_j X_{t-j}\right|^p \\
&\leqslant C \sup_{m\geqslant 1} \frac{1}{m^{r+1}} \mathbb{E}\left|\sum_{t=1}^{m}\sum_{j=-\infty}^{\infty} A_j X_{t-j}\right|^p + C \sup_{m\geqslant 1} \frac{1}{m^{r+1}} \mathbb{E}\left|\sum_{t=1}^{m}\sum_{j=-\infty}^{\infty} A_j X_{t-j}\right|^p \\
&\leqslant C < \infty.
\end{aligned}$$

If we further assume that $pq > r+1$, then as $m \to \infty$.

$$\mathbb{E}\left|\frac{1}{m^q}\sum_{j=-\infty}^{\infty} A_j X_{m-j}\right|^p = O(m^{r+1-pq}), \tag{3.34}$$

and thus

$$\frac{1}{m^q}\sum_{j=-\infty}^{\infty} A_j X_{m-j} \xrightarrow{L^p} 0. \tag{3.35}$$

## 4. Application

In this section, we will investigate the convergence of state observers for linear time-invariant systems using Theorem 3.3.

For $t > 0$, we consider a multi-input-single-output (MISO) linear-time-invariant system as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Dx(t), \end{cases} \tag{4.1}$$

where $A \in \mathbb{R}^{m_0\times m_0}$, $B \in \mathbb{R}^{m_0\times m_1}$ and $D \in \mathbb{R}^{1\times m_0}$ are known system matrices, $u(t) \in \mathbb{R}^{m_1}$ is the control input, $x(t) \in \mathbb{R}^{m_0}$ is the state and $y(t) \in \mathbb{R}$ is the system output. The initial state $x(0)$ is unknown. For some limited observations on $y(t)$, it is interesting to estimate $x(t)$. In the setup, the output $y(t)$ is only measured at a sequence of sampling time instants $\{t_i\}$ with measured values $\gamma(t_i)$ and the noise $\{d_i\}$ such that

$$y(t_i) = \gamma(t_i) + d_i, \quad 1 \leqslant i \leqslant n. \tag{4.2}$$

We would like to estimate the state $x(t)$ from information on $u(t)$, $\{t_i\}$ and $\{\gamma(t_i)\}$. Let $G'$ represents the transpose of $G$. In order to proceed, we need the following assumption.

ASSUMPTION 1. The system (4.1) is observable, i.e., the observability matrix

$$W_0' = [D', (DA)', \cdots, (DA^{m_0-1})']$$

has full rank.

It follows from (4.1) that

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$

Suppose that $\{t_i,\ 1 \leqslant i \leqslant n\}$ is a sequence of sampling times. We have that

$$\gamma(t_i) + d_i = y(t_i) = De^{A(t_i-t_n)}x(t_n) + D\int_{t_n}^{t_i} e^{A(t_i-\tau)}Bu(\tau)d\tau. \tag{4.3}$$

Denote

$$v(t_i,t_n) = D\int_{t_n}^{t_i} e^{A(t_i-\tau)}Bu(\tau)d\tau,\quad 1 \leqslant i \leqslant n.$$

Then we can get

$$De^{A(t_i-t_n)}x(t_n) = \gamma(t_i) - v(t_i,t_n) + d_i,\quad 1 \leqslant i \leqslant n. \tag{4.4}$$

Define

$$\Phi_n = \begin{pmatrix} De^{A(t_1-t_n)} \\ \vdots \\ De^{A(t_{n-1}-t_n)} \\ D \end{pmatrix},\quad \Gamma_n = \begin{pmatrix} \gamma(t_1) \\ \vdots \\ \gamma(t_{n-1}) \\ \gamma(t_n) \end{pmatrix},\quad V_n = \begin{pmatrix} v(t_1,t_n) \\ \vdots \\ v(t_{n-1},t_n) \\ 0 \end{pmatrix},\quad D_n = \begin{pmatrix} d_1 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix}.$$

So, (4.4) can be rewritten as

$$\Phi_n x(t_n) = \Gamma_n - V_n + D_n. \tag{4.5}$$

Suppose that $\Phi_n$ is full rank. Then the least-squares estimator of $x(t_n)$ is given by

$$\widetilde{x}(t_n) = (\Phi_n'\Phi_n)^{-1}\Phi_n'(\Gamma_n - V_n). \tag{4.6}$$

From the equations (4.5) and (4.6), the estimation error for $x(t_n)$ at sampling time $t_n$ is

$$e(t_n) = \widetilde{x}(t_n) - x(t_n) = \left(\frac{1}{n^q}\Phi_n'\Phi_n\right)^{-1}\frac{1}{n^q}\Phi_n'D_n \tag{4.7}$$

for some $1/2 < q < 1$. For convergence analysis, one must consider a typical entry in $(1/n^q)\Phi_n'D_n$. In [26], by the Cayley-Hamilton theorem, the matrix exponential can be expressed by a polynomial function of $A$ of order at most $m_0 - 1$ as follows:

$$e^{At} = \alpha_1(t)I + \cdots + \alpha_{m_0}(t)A^{m_0-1}, \tag{4.8}$$

where the time functions $\alpha_i(t)$ can be derived by the Sylvester interpolation method. Denote $\phi'(t_i-t_n)=[\alpha_1(t_i-t_n),\cdots,\alpha_{m_0}(t_i-t_n)]$ and

$$\Psi_n=\begin{pmatrix}\phi'(t_1-t_n)\\ \vdots\\ \phi'(t_{n-1}-t_n)\\ \phi'(0)\end{pmatrix}.$$

Then $\Phi_n=\Psi_n W_0$, which reduces to

$$\frac{1}{n^q}\Phi_n'\Phi_n=W_0'\frac{1}{n^q}\Psi_n'\Psi_n W_0,$$

and

$$\frac{1}{n^q}\Phi_n'D_n=\frac{1}{n^q}W_0'\Psi_n'D_n.$$

As a result, one has

$$e(t_n)=\left(\frac{1}{n^q}\Phi_n'\Phi_n\right)^{-1}\frac{1}{n^q}\Phi_n'D_n=W_0^{-1}\left(\frac{1}{n^q}\Psi_n'\Psi_n\right)^{-1}\frac{1}{n^q}\Psi_n'D_n. \tag{4.9}$$

By Assumption 4.1, it can be found that $W_0^{-1}$ exists. The convergence analysis will be established by the sufficient conditions: $\frac{1}{n^q}\Psi_n'D_n\to 0$ *a.s.*, and $\frac{1}{n^q}\Psi_n'\Psi_n\geqslant\lambda I$ *a.s.*, for some $\lambda>0$. So, we need the following persistent excitation (PE, for short) condition which has been used in [33] and [38].

ASSUMPTION 2. For some $1/2<q<1$,

$$\inf_{n\geqslant 1}\sigma_{\min}\left(\frac{1}{n^q}\Psi_n'\Psi_n\right)\geqslant M>0 \text{ a.s. for some } M>0, \tag{4.10}$$

where $\sigma_{min}(H)$ is the smallest eigenvalue of $H$ for a suitable symmetric $H$.

We focus on the convergence of partial sums of randomly weighted $\rho^*$-mixing random variables such that

$$\frac{1}{n^q}\sum_{j=1}^{n}A_jX_{n-j} \tag{4.11}$$

for some $1/2<q<1$. Since a typical entry of $\frac{1}{n^q}\Psi_n'D_n$ is

$$\frac{1}{n^q}\sum_{j=1}^{n}\alpha_k(t_j-t_n)d_j, \tag{4.12}$$

The convergence analysis of $e(t_n)$ corresponds to a specific case of $(4.11)$. It is noteworthy that when the sampling time sequence follows a stochastic process, the

$\alpha_k(t_j - t_n)$ terms in (4.12) manifest as randomly weighted noise driven by $\rho^*$-mixing random variables.

In their recent work, [38] investigated the convergence analysis of a state observer for a linear time-invariant system under *rho*$^*$ mixed sampling. [33] explored the convergence analysis of double index and stochastic weighted sums for $\rho^*$-mixing processes and applied them to state observers. In addition, [40] delved into the complete moment convergence of the randomly weighted sum of double indexes and applied it to state observers, and so on.

As a direct application of Theorem 3.3, we derive the following theorem.

THEOREM 4.1. *Suppose that Assumptions 1 and 2 hold. Let* $\{d_n,\ n \geqslant 1\}$ *be a sequence of zero mean independent random variables, and* $\{\alpha_k(t_j - t_n), 1 \leqslant j \leqslant n, n \geqslant 1\}$ *be an array of zero mean rowwise* $\rho^*$*-mixing random variables, which is stochastically dominated a random variable* $X_k$ *for each* $1 \leqslant k \leqslant m_0$*. Furthermore,* $\{\alpha_k(t_j - t_n), 1 \leqslant j \leqslant n, n \geqslant 1\}$ *is independent of* $\{d_n,\ n \geqslant 1\}$*. If*

$$\mathbb{E}|X_k|^2 < \infty, \textit{ for each } 1 \leqslant k \leqslant m_0$$

*and*

$$\sum_{j=1}^{n} \mathbb{E}|d_j|^2 < \infty,$$

*then*

$$\sup_{n \geqslant 1} n^{2q-1} \mathbb{E} e'(t_n) e(t_n) < \infty, \tag{4.13}$$

*and*

$$\mathbb{E} e'(t_n) e(t_n) \to 0, \textit{ as } n \to \infty. \tag{4.14}$$

*Proof.* By Assumptions 1 and 2, we can get that $W_0^{-1}$ and $(\frac{1}{n^q}\Psi_n'\Psi_n)^{-1}$ exist, and $\sigma_{\max}[(1/n^q)\Psi_n'\Psi_n]^{-1} \leqslant 1/M$ *a.s.*, where $\sigma_{\max}(\cdot)$ stands for the largest eigenvalue. Noting that

$$\frac{1}{n^q}\Psi_n' D_n = \begin{pmatrix} \frac{1}{n^q}\sum_{j=1}^{n} \alpha_1(t_j - t_n) d_j \\ \vdots \\ \frac{1}{n^q}\sum_{j=1}^{n} \alpha_{m_0}(t_j - t_n) d_j \end{pmatrix},$$

we only need to verify that for each $1 \leqslant k \leqslant m_0$,

$$n^{2q-r-1} \mathbb{E} \left| \frac{1}{n^q} \sum_{j=1}^{n} \alpha_k(t_j - t_n) d_j \right|^2 \leqslant C < \infty, \quad n \geqslant 1. \tag{4.15}$$

Applying Theorem 3.3 or Remark 3.3 with $X_{n-j} = \alpha_k(t_j - t_n)$, $A_j = d_j$ and $p = 2$, we can get (4.15) immediately. Thus, the proof is completed. □

## 5. Numerical simulation

In this section, numerical simulations will be performed to further verify the convergence of the equation (4.14). We will evaluate the approximate validity of the equation (4.14).

Based on the Taylor approximation algorithm proposed by [26], we obtain explicit expressions for $\alpha_i(t)$, where $1 \leqslant i \leqslant m_0$. To simplify the analysis, we consider two cases: $m_0 = 5$ and $m_0 = 10$. For each $1 \leqslant j \leqslant m_0$, we set $t_k = k/n$, and for each $n \geqslant 1$, $d_n$ is a sequence of independent identically distributed random variables, where each $d_n$ follows a standard normal distribution as specified in Theorem 4.1. By equation (4.9), we compute $e(t_n)$. We consider sample sizes of $n = 100$, $200$, $400$, $800$, $1600$, and $3200$. We choose $r = 0.2$, $r = 0.5$, and $r = 0.8$, and utilize Python software to compute the mean of $e'(t_n)e(t_n)$ for each $n$, repeating the calculation 1000 times.

For convenience, we define $D(n) = e'(t_n)e(t_n)$. Figures 1 and 2 depict the trend charts of the mean values of $D(n)$ for $m_0 = 5$ and $m_0 = 10$, respectively. It can be observed that regardless of the value of $r$, as the sample size $n$ increases, the mean values of $\mathbb{E}(n)$ gradually approach zero.

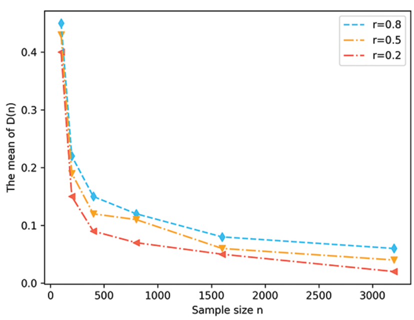


Figure 1: *The mean of $D(n)$ as $m_0 = 5$.*

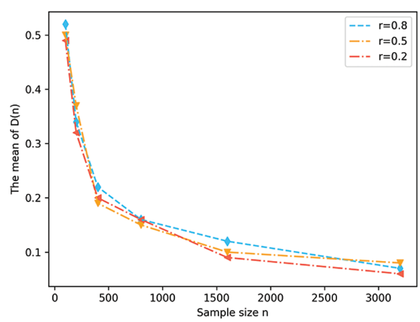


Figure 2: *The mean of $D(n)$ as $m_0 = 10$.*

*Acknowledgements.* Chao Lu is supported by the Natural Science Foundation of Anhui Province (2308085QA11); the Anhui Jianzhu University postdoctoral scientific research project (No. 2024QDHZ05); the University Key Project of Natural Science Foundation of Anhui Province (No. 2022AH051723); Visiting program for outstanding university young backbone teachers (No. gxgnfx2022061). This work also supported by the National Natural Science Foundation of China (No. 12301356, 12201600, 12471248 and 12301172); Cultivation Special Project of Basic-Clinical Collaborative Research Enhancement Program, The Third Affiliated Hospital of Anhui Medical University (Grant No. 2023sfy012).

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(Received October 22, 2024)

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