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# A HARDY TYPE INEQUALITY ON THE UNIT SPHERE

XING YIN AND SONGTING YIN*

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*Abstract.* We establish another Hardy type inequality on the unit sphere and also obtain the corresponding sharp constant.

## 1. Introduction

The classical Hardy inequality states that for $n \geqslant 3$

$$\int_{\mathbb{R}^n} |\nabla f|^2 dx \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} dx,$$

where $f \in C_c^\infty(\mathbb{R}^n \backslash \{0\})$. The constant $\frac{(n-2)^2}{4}$ is optimal and never archived. There has been a lot of research concerning Hardy inequality on the Euclidean space because of its applications to partial differential equations involving singular potentials. We see [2, 3, 8, 9, 12] and the references therein.

The validity of Hardy inequality on a manifold and its best constants allows people to obtain qualitative properties on the manifold. In [4], G. Carron studied the weighted $L^2$-Hardy inequalities on a Riemannian manifold under some geometric assumptions on the weighted function and obtained

$$\int_M \rho^\alpha |\nabla f|^2 dV \geqslant \frac{(C+\alpha-1)^2}{4} \int_M \rho^\alpha \frac{f^2}{\rho^2} dV,$$

where the weight function $\rho$ satisfies $|\nabla \rho| = 1$ and $\Delta \rho \geqslant \frac{C}{\rho}$ in the sense of distribution. Particularly, in [10] I. Kombe and M. Özaydin obtained the improved Hardy inequalities in the Poincaré conformal disc model

$$\int_{\mathbb{B}^n} |\nabla f|^2 dV \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{B}^n} \frac{f^2}{\rho^2} dV,$$

where $f \in C_c^\infty(\mathbb{B}^n \backslash \{0\})$ and $\rho = \log \frac{1+|x|}{1-|x|}$ is the geodesic distance. Furthermore, the constant $\frac{(n-2)^2}{4}$ is sharp. Along this line, we refer to [5, 6, 7, 10] and so on.

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* Corresponding author.

However, to our knowledge, there are relatively few literatures discussing Hardy inequality on the sphere. In [13], Y. Xiao first obtained a Hardy inequality on the sphere as follows:

$$C_1\int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV \geqslant \frac{(n-2)^2}{4}\left(\int_{\mathbb{S}^n}\frac{f^2}{d(x,p)^2}dV + \int_{\mathbb{S}^n}\frac{f^2}{(\pi - d(x,p))^2}dV\right),$$

where $d(x,p)$ is the geodesic distance from $x$ to $p$ on $\mathbb{S}^n$, and the constant $\frac{(n-2)^2}{4}$ is sharp. By similar approach used in Xiao's paper, S. Yin [15] established another type of Hardy inequality on the unit sphere:

$$\frac{n-2}{2}\int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV \geqslant \frac{(n-2)^2}{4}\int_{\mathbb{S}^n}\frac{f^2}{\tan^2 d(p,x)}dV,$$

where $p$ is a fixed point in $\mathbb{S}^n$, and the constant $\frac{(n-2)^2}{4}$ is sharp. For more results, we refer to [1, 11, 17] and other related sources.

It is worth mentioning that Yin, in [16], gave a refinement of Hardy type inequality with the best constant on the unit sphere as follows:

$$\begin{aligned}&\frac{(n-1)(n-2)}{\pi^2}\int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV\\ \geqslant&\frac{(n-2)^2}{4}\left(\int_{B_p(\frac{\pi}{2})}\frac{f^2}{d^2(p,x)}dV + \int_{B_q(\frac{\pi}{2})}\frac{f^2}{d^2(q,x)}dV\right),\end{aligned}$$

where $p$ and $q$ are the antipodal points, and $B_p(\frac{\pi}{2})$ (resp. $B_q(\frac{\pi}{2})$) denotes the hemisphere centered at $p$ (resp. $q$). Moreover, the constant $\frac{(n-2)^2}{4}$ is sharp in the sense that

$$\frac{(n-2)^2}{4} = \inf_{f\in(\mathbb{S}^n)\setminus\{0\}}\frac{\frac{(n-1)(n-2)}{\pi^2}\int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n}|\nabla f|^2 dV}{\int_{B_p(\frac{\pi}{2})}\frac{f^2}{d^2(p,x)}dV + \int_{B_q(\frac{\pi}{2})}\frac{f^2}{d^2(q,x)}dV}$$

In this short paper we will change $B_p(\frac{\pi}{2})$ and $B_q(\frac{\pi}{2})$ above to $B_p(r_0)$ and $B_q(\pi - r_0)$ for $0 < r_0 < \pi$, and obtain another type Hardy inequality on the sphere. Our main theorem is as follows:

THEOREM 1.1. *Let $\mathbb{S}^n$ be the unit sphere with $n \geqslant 3$. Then for any function $f \in C^\infty(\mathbb{S}^n)$ and $0 < r_0 < \pi$, it holds that*

$$\begin{aligned}&C_1(n,r_0)\int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n}|\nabla f|^2 dV + C_2(n,r_0)\int_{\partial B_p(r_0)} f^2 dv\\ \geqslant&\frac{(n-2)^2}{4}\left(\int_{B_p(r_0)}\frac{f^2}{d^2(p,x)}dV + \int_{B_q(\pi - r_0)}\frac{f^2}{d^2(q,x)}dV\right),\end{aligned}$$

*where $C_1(n,r_0) > 0, C_2(n,r_0)$ are two constant depending only on n and $r_0$, p and q are the antipodal points, $B_p(r_0)$ (resp. $B_q(\pi - r_0)$) denotes the geodesic ball centered* *at $p$ (resp. $q$), and $d\nu = \nabla r_p \rfloor dV$ is the induced volume form on $\partial B_p(r_0)$. Moreover, the constant $\frac{(n-2)^2}{4}$ is sharp in the sense that*

$$\frac{(n-2)^2}{4} = \inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{C_1(n,r_0)\int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV + C_2(n,r_0)\int_{\partial B_p(r_0)} f^2 d\nu}{\int_{B_p(r_0)} \frac{f^2}{d^2(p,x)} dV + \int_{B_q(\pi - r_0)} \frac{f^2}{d^2(q,x)} dV}$$

*In particular, when $r_0 = 0$ or $r_0 = \pi$, the inequality becomes*

$$C\int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla f|^2 dV \geqslant \frac{(n-2)^2}{4}\int_{\mathbb{S}^n} \frac{f^2}{d^2(p,x)} dV$$

*for some positive number $C$, and the constant $\frac{(n-2)^2}{4}$ remains sharp.*

In Theorem 1.1, the first term in the left-hand side of the inequality cannot be removed because it will lead to a contradiction if $f$ is a nonzero constant. When $r_0 = \frac{\pi}{2}$, the third term in the left-hand side equals zero and the result reduces to the main result in [16].

## 2. The proof of Theorem 1.1

*Proof.* Let $r_p(x) = d(p,x)$ denote the distance function from the fixed point $p \in \mathbb{S}^n$. Next we follow the arguments in [10] (see also [13]) Let $f = r_p^\alpha \varphi$ with $\alpha < 0$. Then $\nabla f = \varphi \nabla r_p^\alpha + r_p^\alpha \nabla \varphi$ and

$$\begin{aligned} |\nabla f|^2 &= \varphi^2 |\nabla r_p^\alpha|^2 + r_p^{2\alpha}|\nabla\varphi|^2 + r_p^\alpha \varphi\langle \nabla r_p^\alpha, \nabla\varphi\rangle \\ &\geqslant \varphi^2\alpha^2 r_p^{2\alpha-2} + \frac{1}{4}\langle \nabla r_p^{2\alpha}, \nabla\varphi^2\rangle \\ &= \varphi^2\alpha^2 r_p^{2\alpha-2} + \frac{1}{4}\mathrm{div}(\varphi^2\nabla r_p^{2\alpha}) - \frac{1}{4}\varphi^2\Delta r_p^{2\alpha}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \Delta r_p^{2\alpha} &= \mathrm{div}(\nabla r_p^{2\alpha}) = \mathrm{div}(2\alpha r_p^{2\alpha-1}\nabla r_p) \\ &= 2\alpha r_p^{2\alpha-1}\Delta r_p + 2\alpha(2\alpha-1) r_p^{2\alpha-2} \\ &= 2(n-1)\alpha r_p^{2\alpha-1}\cot r_p + 2\alpha(2\alpha-1) r_p^{2\alpha-2}. \end{aligned} \tag{2.2}$$

The last equality holds because $\Delta r_p = (n-1)\cot r_p$ in the sphere. Therefore, from (2.1) and (2.2), we have

$$|\nabla f|^2 \geqslant \frac{1}{4}\mathrm{div}(\varphi^2\nabla r_p^{2\alpha}) + \frac{\alpha}{2}\frac{f^2}{r_p^2} - \frac{(n-1)\alpha}{2}\frac{f^2}{r_p}\cot r_p.$$

Integrating both sides of the inequality above on $B_p(r_0)$ $(0 < r_0 < \pi)$ gives

$$\begin{aligned} \int_{B_p(r_0)} |\nabla f|^2 dV \geqslant & \frac{1}{4}\int_{B_p(r_0)} \mathrm{div}(\varphi^2\nabla r_p^{2\alpha}) dV + \frac{\alpha}{2}\int_{B_p(r_0)} \frac{f^2}{r_p^2} dV \\ & - \frac{(n-1)\alpha}{2}\int_{B_p(r_0)} \frac{f^2}{r_p}\cot r_p dV. \end{aligned} \tag{2.3}$$

Let $q$ be the antipodal point of $p$. Then $r_q(x)=d(q,x)=\pi-r_p$ for any $x\in\mathbb{S}^n$. Set $f=r_q^{\alpha}\phi$. Then by similar arguments, we also have

$$\begin{aligned}\int_{B_q(\pi-r_0)}|\nabla f|^2dV \geqslant & \frac{1}{4}\int_{B_q(\pi-r_0)}\operatorname{div}(\phi^2\nabla r_q^{2\alpha})dV+\frac{\alpha}{2}\int_{B_q(\pi-r_0)}\frac{f^2}{r_q^2}dV\\ &-\frac{(n-1)\alpha}{2}\int_{B_q(\pi-r_0)}\frac{f^2}{r_q}\cot r_q dV. \end{aligned}\tag{2.4}$$

Note that $\partial B_p(r_0)=\partial B_q(\pi-r_0)$, $\varphi=\phi$ and $\nabla r_p=-\nabla r_q$ on $\partial B_p(r_0)$. By Stokes theorem, we obtain

$$\begin{aligned}&\int_{B_p(r_0)}\operatorname{div}(\varphi^2\nabla r_p^{2\alpha})dV+\int_{B_q(\pi-r_0)}\operatorname{div}(\phi^2\nabla r_q^{2\alpha})dV\\ =&\int_{\partial B_p(r_0)}\langle\varphi^2\nabla r_p^{2\alpha},\mathbf{n}\rangle d\nu+\int_{\partial B_q(\pi-r_0)}\langle\phi^2\nabla r_q^{2\alpha},\mathbf{n}\rangle d\nu\\ =&2\alpha\int_{\partial B_p(r_0)}\frac{f^2}{r_p}\langle\nabla r_p,\mathbf{n}\rangle d\nu+2\alpha\int_{\partial B_p(r_0)}\frac{f^2}{r_q}\langle\nabla r_q,\mathbf{n}\rangle d\nu. \end{aligned}\tag{2.5}$$

where $\mathbf{n}$ is a fixed normal vector along $\partial B_p(r_0)$ and $d\nu$ is the induced volume form with respect to $\mathbf{n}$. Choose $\mathbf{n}=\nabla r_p$. Then

$$\int_{B_p(r_0)}\operatorname{div}(\varphi^2\nabla r_p^{2\alpha})dV+\int_{B_q(\pi-r_0)}\operatorname{div}(\phi^2\nabla r_q^{2\alpha})dV=2\alpha\left(\frac{1}{r_0}-\frac{1}{\pi-r_0}\right)\int_{\partial B_p(r_0)}f^2d\nu.$$

Therefore, it follows from (2.3)–(2.5) that

$$\begin{aligned}\int_{\mathbb{S}^n}|\nabla f|^2dV \geqslant & \frac{\alpha}{2}\int_{B_p(r_0)}\frac{f^2}{r_p^2}dV+\frac{\alpha}{2}\int_{B_q(\pi-r_0)}\frac{f^2}{r_q^2}dV-\frac{(n-1)\alpha}{2}\int_{B_p(r_0)}\frac{f^2}{r_p}\cot r_p dV\\ &-\frac{(n-1)\alpha}{2}\int_{B_q(\pi-r_0)}\frac{f^2}{r_q}\cot r_q dV+\frac{\alpha}{2}\left(\frac{1}{r_0}-\frac{1}{\pi-r_0}\right)\int_{\partial B_p(r_0)}f^2d\nu, \end{aligned}$$

which shows

$$\begin{aligned}&\int_{\mathbb{S}^n}|\nabla f|^2dV-\frac{(n-1)\alpha}{2}\int_{B_p(r_0)}f^2\frac{1-r_p\cot r_p}{r_p^2}dV\\ &\quad-\frac{(n-1)\alpha}{2}\int_{B_q(\pi-r_0)}f^2\frac{1-r_q\cot r_q}{r_q^2}dV\\ \geqslant&-\frac{(n-2)\alpha}{2}\int_{B_p(r_0)}\frac{f^2}{r_p^2}dV-\frac{(n-2)\alpha}{2}\int_{B_q(\pi-r_0)}\frac{f^2}{r_q^2}dV\\ &\quad+\frac{\alpha}{2}\left(\frac{1}{r_0}-\frac{1}{\pi-r_0}\right)\int_{\partial B_p(r_0)}f^2d\nu \end{aligned}$$

Notice that $\frac{1-r\cot r}{r^2}$ is increasing in $r$ for $r\in(0,\pi)$. Letting $\alpha=-\frac{n-2}{2}$, we deduce that there exists a positive constant $C(n,r_0)$, depending only on $n$ and $r_0$, such tat

$$\int_{\mathbb{S}^n}|\nabla f|^2dV+C(n,r_0)\int_{\mathbb{S}^n}f^2dV$$
$$\geqslant\frac{(n-2)^2}{4}\left(\int_{B_p(r_0)}\frac{f^2}{r_p^2}dV+\int_{B_q(\pi-r_0)}\frac{f^2}{r_q^2}dV\right)-\frac{n-2}{4}\left(\frac{1}{r_0}-\frac{1}{\pi-r_0}\right)\int_{\partial B_p(r_0)}f^2dv.$$

In what follows, we show the constant $\frac{(n-2)^2}{4}$ is sharp. The skill is borrowed from [14] (see also [13]). Let $\eta : R\to[0,1]$ be a smooth function such that $0\leqslant\eta\leqslant 1$ and

$$\eta(t)=\begin{cases}1, & t\in[-1,1];\\ 0, & |t|\geqslant 2.\end{cases}$$

Let $H(t)=1-\eta(t)$, and for sufficient small $\varepsilon>0$ we construct

$$f_{1,\varepsilon}(r)=\begin{cases}0, & r_0=0;\\ H(\frac{r_p}{\varepsilon})r_p^{\frac{2-n}{2}}, & 0<r_p\leqslant r_0,\end{cases}$$

$$f_{2,\varepsilon}(r)=\begin{cases}0, & r_q=0;\\ H(\frac{r_q}{\varepsilon})r_q^{\frac{2-n}{2}}, & 0<r_q\leqslant \pi-r_0.\end{cases}$$

Then $f_{1,\varepsilon}(r)$ and $f_{2,\varepsilon}(r)$ are defined in $B_p(r_0)$ and $B_q(\pi-r_0)$, respectively. Define

$$f_\varepsilon(r)=f_{1,\varepsilon}(r)+f_{2,\varepsilon}(r).$$

Then we find that $f_\varepsilon(r)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$. Compute

$$\begin{aligned}\int_{\mathbb{S}^n}f_\varepsilon^2dV=&\int_{B_p(r_0)}f_\varepsilon^2dV+\int_{B_q(\pi-r_0)}f_\varepsilon^2dV=\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{r_0}H^2\left(\frac{r_p}{\varepsilon}\right)r_p^{2-n}(\sin r_p)^{n-1}dr\\ &+\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\pi-r_0}H^2\left(\frac{r_q}{\varepsilon}\right)(r_q)^{2-n}(\sin(r_q))^{n-1}dr\\ \leqslant&\mathrm{Vol}(\mathbb{S}^{n-1})\left[\int_\varepsilon^{r_0}r_p^{2-n}(\sin r_p)^{n-1}dr+\int_\varepsilon^{\pi-r_0}r_q^{2-n}(\sin r_q)^{n-1}dr\right]\\ \leqslant&\mathrm{Vol}(\mathbb{S}^{n-1})\left[\int_\varepsilon^{r_0}r_p^{2-n}r_p^{n-1}dr+\int_\varepsilon^{\pi-r_0}r_q^{2-n}r_q^{n-1}dr\right]\\ =&\left(\frac{r_0^2}{2}+\frac{\pi-r_0^2}{2}-\varepsilon^2\right)\mathrm{Vol}(\mathbb{S}^{n-1}).\end{aligned}\tag{2.6}$$

On the other hand, we get

$$\begin{aligned}\int_{B_p(r_0)}\frac{f_\varepsilon^2}{r_p^2}dV=&\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{r_0}H^2\left(\frac{r_p}{\varepsilon}\right)r_p^{-n}(\sin r_p)^{n-1}dr\\ \geqslant&\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{r_0}H^2\left(\frac{r_p}{\varepsilon}\right)r_p^{-n}(\sin r_p)^{n-1}dr\\ =&\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{r_0}r_p^{-n}(\sin r_p)^{n-1}dr,\end{aligned}$$

$$\begin{aligned}\int_{B_q(\pi-r_0)} \frac{f_\varepsilon^2}{r_q^2} dV =& \mathrm{Vol}(\mathbb{S}^{n-1}) \int_\varepsilon^{\pi-r_0} H^2\left(\frac{r_q}{\varepsilon}\right) r_q^{-n} (\sin r_q)^{n-1} dr \\ \geqslant& \mathrm{Vol}(\mathbb{S}^{n-1}) \int_{2\varepsilon}^{\pi-r_0} r_q^{-n} (\sin r_q)^{n-1} dr.\end{aligned}$$

Therefore, combing the above two inequalities, we obtain

$$\begin{aligned}&\int_{B_p(r_0)} \frac{f_\varepsilon^2}{r_p^2} dV + \int_{B_q(\pi-r_0)} \frac{f_\varepsilon^2}{r_q^2} dV \\ \geqslant& \mathrm{Vol}(\mathbb{S}^{n-1}) \left[\int_{2\varepsilon}^{r_0} r^{-n} (\sin r)^{n-1} dr + \int_{2\varepsilon}^{\pi-r_0} r^{-n} (\sin r)^{n-1} dr\right]. \qquad (2.7)\end{aligned}$$

Next we are to estimate

$$\int_{\mathbb{S}^n} |\nabla f_\varepsilon|^2 dV = \int_{B_p(r_0)} |\nabla f_\varepsilon|^2 dV + \int_{B_q(\pi-r_0)} |\nabla f_\varepsilon|^2 dV.$$

A straightforward calculation yields

$$\begin{aligned}&\left(\int_{B_p(r_0)} |\nabla f_\varepsilon|^2 dV\right)^{\frac{1}{2}} \\ =& \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}} \left(\int_\varepsilon^{r_0} \left| H'\left(\frac{r_p}{\varepsilon}\right) \frac{1}{\varepsilon} r_p^{\frac{2-n}{2}} + \frac{2-n}{2} H\left(\frac{r_p}{\varepsilon}\right) r_p^{-\frac{n}{2}}\right|^2 (\sin r_p)^{n-1} dr\right)^{\frac{1}{2}} \\ \leqslant& \frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}}{\varepsilon} \left(\int_\varepsilon^{r_0} \left| H'\left(\frac{r_p}{\varepsilon}\right)\right|^2 r_p^{2-n} (\sin r_p)^{n-1} dr\right)^{\frac{1}{2}} \\ &+ \frac{n-2}{2} \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}} \left(\int_\varepsilon^{r_0} H^2\left(\frac{r_p}{\varepsilon}\right) r_p^{-n} (\sin r_p)^{n-1} dr\right)^{\frac{1}{2}} \\ =& \frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}}{\varepsilon} \left(\int_\varepsilon^{2\varepsilon} \left| H'\left(\frac{r_p}{\varepsilon}\right)\right|^2 r_p^{2-n} (\sin r_p)^{n-1} dr\right)^{\frac{1}{2}} \\ &+ \frac{n-2}{2} \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}} \left(\int_\varepsilon^{r_0} H^2\left(\frac{r_p}{\varepsilon}\right) r_p^{-n} (\sin r_p)^{n-1} dr\right)^{\frac{1}{2}} \\ \leqslant& \frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}}{\varepsilon} \max_{t\in[0,2]} H'(t) \left(\int_\varepsilon^{2\varepsilon} r_p dr\right)^{\frac{1}{2}} \\ &+ \frac{n-2}{2} \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}} \left(\int_\varepsilon^{r_0} r_p^{-n} (\sin r_p)^{n-1} dr\right)^{\frac{1}{2}} \\ =& \sqrt{\frac{3}{2}} \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}} \max_{t\in[0,2]} H'(t) + \frac{n-2}{2} \mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}} \left(\int_\varepsilon^{r_0} r_p^{-n} (\sin r_p)^{n-1} dr\right)^{\frac{1}{2}},\end{aligned}$$

$$
\begin{aligned}
&\left(\int_{B_q(\pi-r_0)}|\nabla f_\varepsilon|^2dV\right)^{\frac{1}{2}}\\
=&\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_\varepsilon^{\pi-r_0}\left|H'\left(\frac{r_q}{\varepsilon}\right)\frac{1}{\varepsilon}r_q^{\frac{2-n}{2}}+\frac{2-n}{2}H\left(\frac{r_q}{\varepsilon}\right)r_q^{-\frac{n}{2}}\right|^2(\sin r_q)^{n-1}dr\right)^{\frac{1}{2}}\\
\leqslant&\sqrt{\frac{3}{2}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\max_{t\in[0,2]}H'(t)+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_\varepsilon^{\pi-r_0}r_q^{-n}(\sin r_q)^{n-1}dr\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_{\mathbb{S}^n}|\nabla f_\varepsilon|^2dV\\
\leqslant&3\mathrm{Vol}(\mathbb{S}^{n-1})(\max_{t\in[0,2]}H'(t))^2+\frac{(n-2)^2}{4}\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{r_0}r^{-n}(\sin r)^{n-1}dr\\
&+\frac{(n-2)^2}{4}\mathrm{Vol}(\mathbb{S}^{n-1})\int_\varepsilon^{\pi-r_0}r^{-n}(\sin r)^{n-1}dr\\
&+\sqrt{\frac{3}{2}}(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})\max_{t\in[0,2]}H'(t)\left(\int_\varepsilon^{r_0}r^{-n}(\sin r)^{n-1}dr\right)^{\frac{1}{2}}\\
&+\sqrt{\frac{3}{2}}(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})\max_{t\in[0,2]}H'(t)\left(\int_\varepsilon^{\pi-r_0}r^{-n}(\sin r)^{n-1}dr\right)^{\frac{1}{2}}. \qquad (2.8)
\end{aligned}
$$

Finally, we compute

$$
\int_{\partial B_p(r_0)}f_\varepsilon^2d\nu=\int_{\partial B_p(r_0)}H^2(\frac{r_0}{\varepsilon})r_0^{2-n}d\nu=r_0^{2-n}\mathrm{Vol}(\partial B_p(r_0)). \qquad (2.9)
$$

Since $f_\varepsilon(r)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$, then, by (2.6)–(2.9), it holds that

$$
\begin{aligned}
C:=&\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\frac{\int_{\mathbb{S}^n}|\nabla f|^2dV+C(n,r_0)\int_{\mathbb{S}^n}f^2dV+\frac{n-2}{4}\left(\frac{1}{r_0}-\frac{1}{\pi-r_0}\right)\int_{\partial B_p(r_0)}f^2d\nu}{\int_{B_p(r_0)}\frac{f^2}{r_p^2}dV+\int_{B_q(\pi-r_0)}\frac{f^2}{r_q^2}dV}\\
\leqslant&\frac{\int_{\mathbb{S}^n}|\nabla f_\varepsilon|^2dV+C(n,r_0)\int_{\mathbb{S}^n}f_\varepsilon^2dV+\frac{n-2}{4}\left(\frac{1}{r_0}-\frac{1}{\pi-r_0}\right)\int_{\partial B_p(r_0)}f_\varepsilon^2d\nu}{\int_{B_p(r_0)}\frac{f_\varepsilon^2}{r_p^2}dV+\int_{B_q(\pi-r_0)}\frac{f_\varepsilon^2}{r_q^2}dV}\\
\leqslant&\frac{C(n,r_0)(\frac{r_0^2}{2}+\frac{\pi-r_0^2}{2}-\varepsilon^2)}{\int_{2\varepsilon}^{r_0}r^{-n}(\sin r)^{n-1}dr+\int_{2\varepsilon}^{\pi-r_0}r^{-n}(\sin r)^{n-1}dr}\\
&+\frac{\frac{n-2}{4}\left(\frac{1}{r_0}-\frac{1}{\pi-r_0}\right)r_0^{2-n}\frac{\mathrm{Vol}(\partial B_p(r_0))}{\mathrm{Vol}(\mathbb{S}^{n-1})}}{\int_{2\varepsilon}^{r_0}r^{-n}(\sin r)^{n-1}dr+\int_{2\varepsilon}^{\pi-r_0}r^{-n}(\sin r)^{n-1}dr}\\
&+\frac{3(\max_{t\in[0,2]}H'(t))^2}{\int_{2\varepsilon}^{r_0}r^{-n}(\sin r)^{n-1}dr+\int_{2\varepsilon}^{\pi-r_0}r^{-n}(\sin r)^{n-1}dr}
\end{aligned}
$$

$$+\frac{(n-2)^2}{4}\frac{\int_\varepsilon^{r_0} r^{-n}(\sin r)^{n-1}dr+\int_\varepsilon^{\pi-r_0} r^{-n}(\sin r)^{n-1}dr}{\int_{2\varepsilon}^{r_0} r^{-n}\sin r^{n-1}dr+\int_{2\varepsilon}^{\pi-r_0} r^{-n}(\sin r)^{n-1}dr}$$

$$+\frac{\sqrt{\frac{3}{2}}(n-2)\max_{t\in[0,2]}H'(t)\left[\left(\int_\varepsilon^{r_0} r^{-n}(\sin r)^{n-1}dr\right)^{\frac{1}{2}}+\left(\int_\varepsilon^{\pi-r_0} r^{-n}(\sin r)^{n-1}dr\right)^{\frac{1}{2}}\right]}{\int_{2\varepsilon}^{r_0} r^{-n}(\sin r)^{n-1}dr+\int_{2\varepsilon}^{\pi-r_0} r^{-n}(\sin r)^{n-1}dr}$$

$$:=I+II+III+IV+V.$$

Note that

$$\lim_{\varepsilon\to 0}\int_{2\varepsilon}^{r_0} r^{-n}(\sin r)^{n-1}dr=\lim_{\varepsilon\to 0}\int_{2\varepsilon}^{\pi-r_0} r^{-n}(\sin r)^{n-1}dr=+\infty,$$

which implies that

$$I=II=III=V=0.$$

By L'Hospital rule,

$$\lim_{\varepsilon\to 0}\frac{\int_\varepsilon^{r_0} r^{-n}(\sin r)^{n-1}dr+\int_\varepsilon^{\pi-r_0} r^{-n}(\sin r)^{n-1}dr}{\int_{2\varepsilon}^{r_0} r^{-n}\sin r^{n-1}dr+\int_{2\varepsilon}^{\pi-r_0} r^{-n}(\sin r)^{n-1}dr}=1.$$

This gives

$$C\leqslant\frac{(n-2)^2}{4},$$

which shows that the constant $\frac{(n-2)^2}{4}$ is sharp.

Finally, if $r_0=\pi$ (resp. $r_0=0$), it follows from (2.3) (resp. (2.4)) that

$$\begin{aligned}\int_{\mathbb{S}^n}|\nabla f|^2dV\geqslant&\frac{1}{4}\int_{\mathbb{S}^n}\operatorname{div}(\varphi^2\nabla r_p^{2\alpha})dV+\frac{\alpha}{2}\int_{\mathbb{S}^n}\frac{f^2}{r_p^2}dV-\frac{(n-1)\alpha}{2}\int_{\mathbb{S}^n}\frac{f^2}{r_p}\cot r_p dV\\=&\frac{\alpha}{2}\int_{\mathbb{S}^n}\frac{f^2}{r_p^2}dV-\frac{(n-1)\alpha}{2}\int_{\mathbb{S}^n}\frac{f^2}{r_p}\cot r_p dV.\end{aligned}$$

Letting $\alpha=-\frac{n-2}{2}$, we have

$$\int_{\mathbb{S}^n}|\nabla f|^2dV+\frac{(n-1)(n-2)}{4}\int_{\mathbb{S}^n}f^2\frac{1-r_p\cot r_p}{r_p^2}dV\geqslant\frac{(n-2)^2}{4}\int_{\mathbb{S}^n}\frac{f^2}{r_p^2}dV.$$

By mean value theorem, there exists a constant $r_0$ such that

$$\int_{\mathbb{S}^n}f^2\frac{1-r_p\cot r_p}{r_p^2}dV=C(r_0)\int_{\mathbb{S}^n}f^2dV,$$

and thus

$$\int_{\mathbb{S}^n}|\nabla f|^2dV+C(n,r_0)\int_{\mathbb{S}^n}f^2dV\geqslant\frac{(n-2)^2}{4}\int_{\mathbb{S}^n}\frac{f^2}{r_p^2}dV.$$

By similar arguments as above, we can also demonstrate that the constant $\frac{(n-2)^2}{4}$ is sharp. □

## REFERENCES


[1] A. ABOLARINWA, K. RAUF AND S. YIN, *Sharp $L^p$ Hardy type and uncertainty principle inequalities on the sphere*, J. Math. Inequal., 2019, **13**, 1011–1022.

[2] P. BARAS AND J. A. GOLDSTEIN, *The heat equation with a singular potential*, Trans. Am. Math. Soc., 1984, **284**, 121–139.

[3] H. BREZIS AND J. L. VÁZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madr., 1997, **10**, 443–469.

[4] G. CARRON, *Inégalités de Hardy sur les variétés Riemanniennes non-compactes*, J. Math. Pures Appl., 1997, **76**, 883–891.

[5] F. DU AND J. MAO, *Hardy and Rellich type inequalities on metric measure spaces*, J. Math. Anal. Appl., 2015, **429**, 354–365.

[6] L. D'AMBROSIO AND S. DIPIERRO, *Hardy inequalities on Riemannian manifolds and applications*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 2014, **31**, 449–475.

[7] G. GRILLO, *Hardy and Rellich-type inequalities for metrics defined by vector fields*, Potential Anal., 2003, **18**, 187–217.

[8] J. GARCIA AZORERO AND I. PERAL, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differ. Equ., 1998, **144**, 441–476.

[9] I. KOMBE, *The linear heat equation with a highly singular, oscillating potential*, Proc. Am. Math. Soc., 2004, **132**, 2683–2691.

[10] I. KOMBE AND M.ÖZAYDIN, *Improved Hardy and Rellich inequalities on Riemannian manifolds*, Trans. Am. Math. Soc., 2009, **361**: 6191–6203.

[11] X. SUN AND F. PAN, *Hardy type inequalities on the sphere*, J. Ineq. Appli., 2017, **148**, 1–8.

[12] J. L. VÁZQUEZ AND E. ZUAZUA, *The Hardy constant and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal., 2000, **173**, 103–153.

[13] Y. XIAO, *Some Hardy inequalities on the sphere*, J. Math. Inequal., 2016, **10**: 793–805.

[14] Q. YANG, D. SU AND Y. KONG, *Hardy inequalities on Riemannian manifolds with negative curvature*, Commun. Contemp. Math. **16**, 1350043 (2014) [24 pages].

[15] S. YIN, *A sharp Hardy type inequality on the sphere*, New York J. Math., 2018, **24**, 1101–1110.

[16] S. YIN, *A refinement of Hardy type inequality on the n-spheres*, ScienceAsia, 2022, **48**, 474–478.

[17] S. YIN AND A. ABDELHAKIM, *Optimal critical exponent $L^p$ inequalities of Hardy type on the sphere via Xiao's method*, J. Math. Inequal., 2022, **16** (1), 265–272.


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*Xing Yin*
*Department of Mathematics and Computer Science*
*Tongling University*
*Tongling, 244000 Anhui, China*
*e-mail:* 15756009928@vip.163.com

*Songting Yin*
*Department of Mathematics and Computer Science*
*Tongling University*
*Tongling, 244000 Anhui, China*
*e-mail:* yst419@163.com