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INEQUALITIES INVOLVING A MEASURE OF BGM CLASS AND ZEROS OF CORRESPONDING ORTHOGONAL POLYNOMIALS

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Abstract. Let $\tilde{\Phi}_n$ be a quasi-orthogonal polynomial of order 1 on the unit circle, obtained from an orthogonal polynomial Φ_n with measure μ, which belongs to BGM class, if there exists another measure $\tilde{\mu}$ such that $\tilde{\Phi}_n$ is a monic orthogonal polynomial. This article aims to investigate various properties related to BGM class. At first, we study the behaviour of the zeros of Φ_n and $\tilde{\Phi}_n$. Along with numerical examples, we analyze the zeros of Φ_n, corresponding para-orthogonal polynomial and its linear combination. Further, comparison of the norm inequalities among Φ_n and $\tilde{\Phi}_n$ are obtained by involving their measures. This leads to the study of the Lubinsky type inequality for the measures μ and $\tilde{\mu}$, without using the ordering relation between μ and $\tilde{\mu}$. Additionally, similar type of inequalities for the kernel type polynomials related to μ and $\tilde{\mu}$ are obtained.

1. Introduction

Let assume that $\{\Phi_n\}_{n\geqslant 0}$ is the sequence of monic orthogonal polynomials with respect to the non-trivial positive Borel measure μ supported on the unit circle $\partial\mathbb{D} = \{z\in\mathbb{C}; |z|=1\}$. In other words, the polynomials satisfy the orthogonality condition:

$$\int_{\partial\mathbb{D}} \Phi_n(z)\overline{\Phi_m(z)}d\mu(z) = \kappa_n^{-2}\delta_{nm}.$$

Here $\kappa_n^{-1} = \|\Phi_n\|_\mu$, where $\|\cdot\|_\mu$ represents the $L^2(\partial\mathbb{D}, d\mu)$ norm. Through the use of a suitable sequence $\{\alpha_n\}_{n\geqslant 0}$, where each α_n belongs to the unit disk $\mathbb{D}$, we can recursively determine the monic orthogonal polynomials Φ_n. This process is achieved using the forward Szegő recurrence relations:

$$\begin{aligned} \Phi_n(z) &= z\Phi_{n-1}(z) - \bar{\alpha}_{n-1}\Phi_{n-1}^*(z), \quad n\geqslant 1, \\ \Phi_n^*(z) &= \Phi_{n-1}^*(z) - \alpha_{n-1}z\Phi_{n-1}(z), \quad n\geqslant 1, \end{aligned} \tag{1.1}$$

with the initial condition $\Phi_0(z)=1$ and $\Phi_n^*(z) = z^n\overline{\Phi_n(\frac{1}{\bar{z}})}$ is known as reversed polynomial. Verblunsky theorem means that given a sequence of complex numbers $\{\alpha_n\}_{n\geqslant 0}$ in the unit disk, there exists a probability measure supported on the unit circle such

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that the corresponding sequence of Verblunsky parameters is $\{\alpha_n\}_{n\geqslant 0}$. It is worth mentioning that these coefficients are sometimes referred to as reflection or Schur parameters [35]. The polynomial $\Phi_n(z)$ can be normalized to obtain the orthonormal polynomial denoted by $\phi_n(z) = \frac{\Phi_n(z)}{\|\Phi_n\|_\mu}$.

For several decades, the study of finite linear combinations of orthogonal polynomials on the real line $\{\mathcal{P}_n(x)\}_{n\geqslant 0}$ as well as on the unit circle $\{\Phi_n(z)\}_{n\geqslant 0}$ has been a vibrant area of research. The exploration of linear combinations of two consecutive degrees of orthogonal polynomials on the real line (OPRL) was initially studied by Riesz [30] while proving the Hamburger moment problem. The necessary and sufficient condition for the orthogonality of linear combinations of OPRL has been investigated in [1]. For the Jacobi polynomials $\mathcal{P}_n^{(\alpha,\beta)}(x)$, the behaviour of zeros and interlacing properties of Jacobi polynomials and quasi-Jacobi polynomials are discussed in [15]. The orthogonality of self-perturbations of those orthogonal polynomials, which are generated by Christoffel transformation of a measure on the real line, has been studied in [23]. In the same article, the recovery of the original orthogonal polynomials from quasi-type kernel polynomials and polynomials generated by linear spectral transformations is established. For the spectral transformation in other direction involving block Hessenberg matrices corresponding to matrix orthogonal polynomials, we refer to [18]. There is a close connection between the quadrature formula and linear combination of OPRL. More specifically, [28] provides the sufficient conditions on the coefficients $a_{n,t}$ for the polynomial $Q_n(x) = \sum_{t=0}^{k} a_{n,t}\mathcal{P}_{n-t}(x)$ to have n distinct zeros within the interval of orthogonality. These conditions ensure that when these zeros are utilized as nodes in an interpolatory quadrature formula, the weights of the quadrature formula remain positive.

In [6], the linear combination of two elements from a sequence of monic orthogonal polynomials on the unit circle (OPUC) is investigated. They introduced the sequence of monic polynomials

$$\tilde{\Phi}_n(z) = \Phi_n(z) - a_n\Phi_{n-1}(z), \quad n \geqslant 1,$$

and explored necessary conditions on the parameters a_n for which the sequence $\{\tilde{\Phi}_n(z)\}_{n\geqslant 0}$ becomes an OPUC. Later, in [5], the authors provided both necessary and sufficient conditions for the orthogonality of $\tilde{\Phi}_n(z)$. In the same paper, they also presented a description of the orthogonality measure associated with the new sequence $\{\tilde{\Phi}_n(z)\}_{n\geqslant 0}$. More generally, the orthogonality of a finite linear combination of orthogonal polynomials on the unit circle with respect to the Bernstein-Szegő measure is discussed in [26]. In [8], the necessary and sufficient conditions are established for a sequence of monic OPUC $\{\Psi_n\}_{n\geqslant 0}$ such that the convex linear combination of $\{\Phi_n\}_{n\geqslant 0}$ and $\{\Psi_n\}_{n\geqslant 0}$ becomes orthogonal with respect to a specific measure supported on the unit circle.

The CD kernel of the orthonormal polynomials ϕ_n is defined as:

$$\mathbb{K}_n(z,w,\mu) = \sum_{k=0}^{n} \overline{\phi_k(w)}\phi_k(z). \tag{1.2}$$

A more concise expression for this kernel is known as the Christoffel-Darboux formula [4, equation 3.2]. For any $z, w \in \mathbb{C}$, with $z\bar{w} \neq 1$, the Christoffel-Darboux kernel can be written as:

$$\mathbb{K}_n(z,w,\mu) = \frac{\overline{\phi_n^*(w)}\phi_n^*(z) - z\bar{w}\overline{\phi_n(w)}\phi_n(z)}{1 - z\bar{w}}. \tag{1.3}$$

Next, we state the class of measures which we call "BGM class", named after the first letter of the authors of the work [5] in which this class of measures is studied explicitly.

DEFINITION 1.1. Let $\{\Phi_n\}_{n\geqslant 0}$ be sequence of monic OPUC with respect to a non-trivial positive Borel measure μ on the unit circle. If there exists a sequence of constants $\{a_n\}_{n\geqslant 1}$, where $a_n \in \mathbb{C}$ and a measure $\tilde{\mu}$ on the unit circle such that the sequence of polynomials $\{\tilde{\Phi}_n\}_{n\geqslant 0}$ defined by

$$\tilde{\Phi}_n(z) = \Phi_n(z) - a_n\Phi_{n-1}(z), \quad n \geqslant 1, \tag{1.4}$$

is an OPUC with respect to $\tilde{\mu}$, then we say that the measure μ belongs to BGM class.

Note that similar concepts are studied by various authors, in the recent past, in different directions, which are beyond the scope of this manuscript. The primary aim of this manuscript is to explore the theory of quasi-orthogonal polynomials of order one on the unit circle, while also deriving estimates based on Verblunsky coefficients and the sequence of parameters a_n for polynomials within BGM class.

1.1. Organization

In Section 2, given the orthogonal polynomial Φ_n, we consider its quasi-orthogonal polynomial $\tilde{\Phi}_n$ from BGM class. We exhibited the expression which alternatively represents Φ_n as the linear combination of $\tilde{\Phi}_n$ and its reversed polynomial $\tilde{\Phi}_n^*$. This is achieved through a recurrence relation with appropriate variable coefficients. We conduct numerical experiments to observe the behavior of zeros of quasi-orthogonal polynomials of order one on the unit circle and discuss examples belonging to BGM class. The discussion about obtaining the expression for the positive chain sequence in terms of Verblunsky coefficients from Definition 1.1 is also presented. In Section 3, we provide a detailed description of the Lubinsky inequality involving the measures μ and $\tilde{\mu}$, for which we prove certain norm inequalities. We define the kernel-type polynomials and additional inequalities involving μ and $\tilde{\mu}$ are exhibited for kernel-type polynomials.

So far we have discussed the orthogonal polynomials on the real line. In this chapter, we will discuss the orthogonal polynomials on the unit circle, an analogue of the real line case. More precisely, the concept of quasi-orthogonality on the unit circle is studied in [5, 6].

2. Orthogonal polynomials in BGM class

It is clear from the Definition 1.1 that $a_n \not\equiv 0$, because $a_n \equiv 0$ gives the equivalence of $\tilde{\Phi}_n(z)$ and $\Phi_n(z)$ for each $n \in \mathbb{N}$. In fact, we can say more about the constant a_n's. The proof provided in [5, Theorem 4], which relies on the Szegő recursion for the polynomial $\tilde{\Phi}_n(z)$, establishes that $a_n \neq 0$ hold for $n \geqslant 1$.

The condition of non-zero a_n's plays a crucial role in order to prove that the polynomial Φ_n cannot be orthogonal to the constant function 1 with respect to $\tilde{\mu}$.

PROPOSITION 1. *If μ is in BGM class, then there does not exist any $N \in \mathbb{N}$ such that $\int \Phi_N(z) d\tilde{\mu}(z) = 0$.*

Proof. Suppose that there exists an $N \in \mathbb{N}$ such that

$$\int \Phi_N(z) d\tilde{\mu}(z) = 0.$$

Using (1.4) and the fact that $a_n \neq 0$ for each $n \geqslant 1$, we can deduce

$$\int \Phi_{N-1}(z) d\tilde{\mu}(z) = 0.$$

By applying reverse induction, we establish

$$\int \Phi_n(z) d\tilde{\mu}(z) = 0 \ \text{ for each } \ n = 0, 1, \cdots, N.$$

This eventually leads to the conclusion that $\tilde{\mu}(\partial\mathbb{D}) = 0$, which is a contradiction. □

In (1.4), we see that $\tilde{\Phi}_n$ is an nth degree monic polynomial written as a linear combination of known polynomials Φ_n and Φ_{n-1}. Proposition 2 deals with the expression the orthogonal polynomial Φ_n in terms of $\tilde{\Phi}_{n+1}$ and its reverse polynomial, which is useful to give the information that $z = \bar{a}_{n+1}^{-1}$ will not be the zero of $\tilde{\Phi}_{n+1}^*(z)$ and $\Phi_n(z)$ for any $n \in \mathbb{N}$.

PROPOSITION 2. *For any positive Borel measure μ in BGM class, we can have a sequence of constants $\{a_n\}_{n\geqslant 1}$ such that the following recurrence relations*

$$((z-a_{n+1})(1-\bar{a}_{n+1}z) - |\alpha_n|^2 z)\Phi_n(z) = (1-\bar{a}_{n+1}z)\tilde{\Phi}_{n+1}(z) + \bar{\alpha}_n \tilde{\Phi}_{n+1}^*(z), \ n \geqslant 0, \tag{2.1}$$

$$((z-a_{n+1})(1-\bar{a}_{n+1}z) - |\alpha_n|^2 z)\Phi_n^*(z) = \alpha_n z \tilde{\Phi}_{n+1}(z) + (z-a_{n+1})\tilde{\Phi}_{n+1}^*(z), \ n \geqslant 0,$$

hold. Moreover, $a_{n+1} = \frac{\tilde{\Phi}_{n+1}(0)+\bar{\alpha}_n}{\bar{\alpha}_{n-1}}$ for non-zero α_n's.

Proof. If μ is in BGM class, then by Definition 1.1, there exists a complex sequence $\{a_n\}_{n\geqslant 1}$ such that the polynomial sequence $\{\tilde{\Phi}_n\}_{n\geqslant 0}$ is orthogonal with respect to $\tilde{\mu}$. Using the Szegő recursion satisfied by Φ_n, we can write the expression (1.4) as

$$\tilde{\Phi}_{n+1}(z) = (z-a_{n+1})\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z). \tag{2.2}$$

Applying the reverse operation * on both sides of (1.4), we get

$$\tilde{\Phi}^*_{n+1}(z) = \Phi^*_{n+1}(z) - \bar{a}_{n+1} z \Phi^*_n(z). \tag{2.3}$$

Again we can use Szegő recursion satisfied by Φ^*_n to write the expression (2.3) as

$$\tilde{\Phi}^*_{n+1}(z) = -\alpha_n z \Phi_n(z) + (1 - \bar{a}_{n+1} z)\Phi^*_n(z). \tag{2.4}$$

We can write (2.2) and (2.4) in matrix form as follows:

$$\begin{pmatrix} \tilde{\Phi}_{n+1}(z) \\ \tilde{\Phi}^*_{n+1}(z) \end{pmatrix} = A(a_{n+1}, \alpha_n; z) \begin{pmatrix} \Phi_n(z) \\ \Phi^*_n(z) \end{pmatrix},$$

with

$$A(a_{n+1}, \alpha_n; z) = \begin{pmatrix} z - a_{n+1} & -\bar{\alpha}_n \\ -\alpha_n z & 1 - \bar{a}_{n+1} z \end{pmatrix}.$$

The matrix $A(a_{n+1}, \alpha_n; z)$ is invertible since $\det(A(a_{n+1}, \alpha_n; z)) = (z - a_{n+1})(1 - \bar{a}_{n+1} z)$ $- |\alpha_n|^2 z \neq 0$. Hence we have

$$\begin{pmatrix} \Phi_n(z) \\ \Phi^*_n(z) \end{pmatrix} = \frac{1}{(z - a_{n+1})(1 - \bar{a}_{n+1} z) - |\alpha_n|^2 z} \begin{pmatrix} 1 - \bar{a}_{n+1} z & \bar{\alpha}_n \\ \alpha_n z & z - a_{n+1} \end{pmatrix} \begin{pmatrix} \tilde{\Phi}_{n+1}(z) \\ \tilde{\Phi}^*_{n+1}(z) \end{pmatrix},$$

which gives the desired recurrence relations. If we substitute $z = 0$ in (2.3), then using [32, Lemma 1.5.1], we get $\tilde{\Phi}^*_{n+1}(0) = 1$. Hence, by plugging $z = 0$ in (2.1) and the fact that $\alpha_n = -\overline{\Phi_{n+1}(0)}$ we get $a_{n+1} = \frac{\tilde{\Phi}_{n+1}(0) + \bar{\alpha}_n}{\bar{\alpha}_{n-1}}$ for non-zero α_n's. □

COROLLARY 2.1. *If there does not exist any $n \in \mathbb{N}$ such that $\alpha_n = 0$, then $z = \bar{a}^{-1}_{n+1}$ is neither a zero of $\tilde{\Phi}^*_{n+1}(z)$ nor a zero of $\Phi_n(z)$. Moreover,*

$$\alpha_n = -\frac{\tilde{\Phi}^*_{n+1}\left(\frac{1}{\bar{a}_{n+1}}\right)}{\Phi_n\left(\frac{1}{\bar{a}_{n+1}}\right)} \bar{a}_{n+1}. \tag{2.5}$$

Proof. Considering $z = \bar{a}^{-1}_{n+1}$ in (2.1), and using the fact that $\alpha_n \neq 0$, we have

$$\Phi_n\left(\frac{1}{\bar{a}_{n+1}}\right) \frac{\alpha_n}{\bar{a}_{n+1}} + \tilde{\Phi}^*_{n+1}\left(\frac{1}{\bar{a}_{n+1}}\right) = 0.$$

If either of $\Phi_n\left(\frac{1}{\bar{a}_{n+1}}\right)$ or $\tilde{\Phi}^*_{n+1}\left(\frac{1}{\bar{a}_{n+1}}\right)$ is zero, then eventually the other one is also reduced to zero. This means both of them are non-zero leading to (2.5). □

Next, we consider a specific measure to derive the orthogonality measure for quasi-orthogonal polynomials on the unit circle.

Let $d\mu(z) = |dz|$ be the Lebesgue measure on the unit circle and the sequence $\{\Phi_n\}_{n\geqslant 0}$ with $\Phi_n(z) = z^n$ the corresponding OPUC. Then we can find the non-zero complex constants $a_n \equiv a$ such that

$$\tilde{\Phi}_n(z) = z^n - az^{n-1}, \quad \text{for } n \geqslant 1, \tag{2.6}$$

is an OPUC with respect to measure $d\tilde{\mu}(z) = \frac{|dz|}{|z-a|^2}$. Indeed,

$$\int_{\partial\mathbb{D}} \tilde{\Phi}_n(z)\overline{\tilde{\Phi}_m(z)}d\tilde{\mu}(z) = \int_{\partial\mathbb{D}} z^{n-m}|z-a|^2\frac{|dz|}{|z-a|^2} = \begin{cases} 0, & n \neq m \geqslant 1 \\ \neq 0, & n = m \geqslant 1 \end{cases}.$$

Since $\{\tilde{\Phi}_n\}_{n\geqslant 0}$ is a sequence of OPUC, then by [32, Theorem 1.7.1], $a_n \equiv a$ lies in the unit disc.

Note that for $a = 1$ in (2.6), we can recover the polynomial $\Phi_n(z) = z^n$ of degree n using [22, equation 3.9].

The method illustrated in the case of Lebesgue measure yields a new measure $d\tilde{\mu}(z) = \frac{|dz|}{|z-a|^2}$. Subsequently, we initiate with the measure constructed in the previous case and explore the measure associated with the linear combination of OPUC.

In the sequel, let $d\mu(z) = \frac{|dz|}{|z-a|^2}$ be a positive measure on the unit circle and the sequence $\{\Phi_n\}_{n\geqslant 0}$ defined by $\Phi_n(z) = z^{n-1}(z-a)$ for $n \geqslant 1$ is an OPUC with respect to μ. Then there exist $a_n \equiv b \neq 0$ in the unit disc such that

$$\tilde{\Phi}_n(z) = z^{n-1}(z-a) - bz^{n-2}(z-a) \tag{2.7}$$

is an OPUC with respect to measure $d\tilde{\mu}(z) = \frac{|dz|}{|z-b|^2|z-a|^2}$. Indeed, for $n,m \geqslant 2$, we have

$$\int_{\partial\mathbb{D}} \tilde{\Phi}_n(z)\overline{\tilde{\Phi}_m(z)}d\tilde{\mu}(z) = \int_{\partial\mathbb{D}} z^{n-m}|z-b|^2|z-a|^2\frac{|dz|}{|z-b|^2|z-a|^2} = \begin{cases} 0, & n \neq m \\ \neq 0, & n = m \end{cases}.$$

If $b = -\bar{a}$, then $\tilde{\Phi}_n(z) = z^{n-2}(z-a)(z+\bar{a})$ for $n \geqslant 2$ is an OPUC with respect to $d\tilde{\mu}(z) = \frac{|dz|}{|z+\bar{a}|^2|z-a|^2}$.

If $d\mu$ represents the canonical Christoffel transformation [17] of the Borel measure on the unit circle, given by $d\mu = |z-\tilde{\gamma}|^2 d\nu$, where $\tilde{\gamma} \in \mathbb{C}$, and if $\mathbb{K}_{n-1}(\tilde{\gamma},\tilde{\gamma},\nu) > 0$ for $n \geqslant 1$, then there exists a sequence of monic OPUC with respect to μ, denoted by $\Phi_n(z;\tilde{\gamma})$, such that the following relation holds:

$$\Phi_{n-1}(z;\tilde{\gamma}) = \frac{1}{z-\tilde{\gamma}}\left(\Psi_n(z) - \frac{\Psi_n(\tilde{\gamma})}{\mathbb{K}_{n-1}(\tilde{\gamma},\tilde{\gamma},\nu)}\mathbb{K}_{n-1}(z,\tilde{\gamma},\nu)\right), \tag{2.8}$$

where $\Psi_n(z)$ represents monic orthogonal polynomials with respect to ν, see [12, proposition 2.4].

By generalizing 2.6 and 2.7, we arrive at the general form of measures in BGM class, as outlined in [5, Theorem 14]. This form consists of measures expressed as

$$d\mu(z) = K\frac{|z-\bar{\beta}|^2}{|z-\chi_1|^2|z-\chi_2|^2}|dz|, \tag{2.9}$$

where $0 < |\beta| \leqslant 1$, $\chi_1, \chi_2 \in \mathbb{D}$ and $K > 0$. Such measures belong to BGM class. Additionally, the measure $d\tilde{\mu}$ is given by:

$$d\tilde{\mu}(z) = K\frac{1}{|z-\chi_1|^2|z-\chi_2|^2}|dz|. \tag{2.10}$$

In other words, $d\tilde{\mu}$ is a quadratic Bernstein-Szegő measure while $d\mu$ is a Christoffel transformation of $d\tilde{\mu}$.

Suppose μ is in BGM class, it follows from [5, Theorem 4] that the sequence $\{\tilde{\Phi}_n\}_{n\geqslant 0}$, where

$$\tilde{\Phi}_n(z) = z^{n-2}\tilde{\Phi}_2(z) = z^{n-2}(z-\chi_1)(z-\chi_2), \quad \text{for } n \geqslant 2, \tag{2.11}$$

is a monic OPUC with respect to the measure $\tilde{\mu}$. However, the explicit expression of the polynomial $\tilde{\Phi}_1$ is not available. Using the backward Szegő recursion, it is possible to determine $\tilde{\Phi}_1$ explicitly. The backward Szegő recurrence relation, given in [32, Theorem 1.5.4], is expressed as

$$z\tilde{\Phi}_n(z) = \tilde{\rho}_n^{-2}\left(\tilde{\Phi}_{n+1}(z) + \bar{\tilde{\alpha}}_n\tilde{\Phi}^*_{n+1}(z)\right), \tag{2.12}$$

where $\tilde{\rho}_n^2 + |\tilde{\alpha}_n|^2 = 1$, and $\tilde{\alpha}_n$ are the Verblunsky coefficients corresponding to $\tilde{\mu}$. For $n = 1$ and the polynomial $\tilde{\Phi}_2(z) = (z-\chi_1)(z-\chi_2)$, the inverse Szegő recursion (2.12) becomes

$$\begin{aligned} z\tilde{\Phi}_1(z) &= \tilde{\rho}_1^{-2}[(z-\chi_1)(z-\chi_2) + \bar{\tilde{\alpha}}_1(1-z\overline{\chi_1})(1-z\overline{\chi_2})] \\ &= \tilde{\rho}_1^{-2}[z^2 - \chi_1 z - \chi_2 z + \chi_1\chi_2 + \bar{\tilde{\alpha}}_1(1 - \overline{\chi_1}z - \overline{\chi_2}z + \overline{\chi_1\chi_2}z^2)] \\ &= \tilde{\rho}_1^{-2}[(1+\bar{\tilde{\alpha}}_1\overline{\chi_1\chi_2})z^2 - (\chi_1+\chi_2+\overline{\chi_1}\bar{\tilde{\alpha}}_1 + \overline{\chi_2}\bar{\tilde{\alpha}}_1)z + (\bar{\tilde{\alpha}}_1 + \chi_1\chi_2)]. \end{aligned} \tag{2.13}$$

Since $\tilde{\Phi}_1(z)$ is a monic polynomial and $z\tilde{\Phi}_1(z)$ has no constant term, we have $\bar{\tilde{\alpha}}_1 = -\chi_1\chi_2$ and $\tilde{\rho}_1^{-2} = (1-|\chi_1|^2|\chi_2|^2)^{-1}$. Thus, (2.13) can be rewritten as

$$\tilde{\Phi}_1(z) = z - \frac{\chi_1(1-|\chi_2|^2) + \chi_2(1-|\chi_1|^2)}{1-|\chi_1|^2|\chi_2|^2}. \tag{2.14}$$

Since the measure $d\mu$ is a Christoffel perturbation of the measure $d\tilde{\mu}$, the coefficients a_n in (1.4) can be expressed in terms of χ_1, χ_2, and β. This is achieved by comparing the coefficients of z^n in (1.4).

PROPOSITION 3. *Let μ be in BGM class and $\{\Phi_n\}_{n\geqslant 0}$ be a sequence of monic OPUC with respect to $d\mu$. Let $\tilde{\Phi}_n(z) = \Phi_n(z) - a_n\Phi_{n-1}(z)$ be a monic OPUC with respect to $\tilde{\mu}$.*

1. *Suppose $\mathbb{K}_n(z,w,\tilde{\mu})$ is the CD kernel with respect to $\tilde{\mu}$. Then*

$$\mathbb{K}_n(z,z,\tilde{\mu}) = \frac{1}{2\pi K}\frac{|1-\bar{z}\chi_1|^2|1-\bar{z}\chi_2|^2 - |z|^{2(n-1)}|z-\chi_1|^2|z-\chi_2|^2}{1-|z|^2}. \tag{2.15}$$

2. *The coefficients a_n can be explicitly written as*

$$a_n = \bar{\beta}\left[1 - \frac{(1-|\beta|^2)|\bar{\beta}^{n-2}(\bar{\beta}-\chi_1)(\bar{\beta}-\chi_2)|^2}{|1-\beta\chi_1|^2|1-\overline{\beta}\chi_2|^2 - |\beta|^{2(n-1)}|\beta-\overline{\chi_1}|^2|\beta-\overline{\chi_2}|^2}\right]. \tag{2.16}$$

Proof. If μ is in BGM class, then $\tilde{\Phi}_n(z) = z^{n-2}(z-\chi_1)(z-\chi_2)$ and $\tilde{\Phi}_n^*(z) = (1-z\overline{\chi_1})(1-z\overline{\chi_2})$ for $n \geqslant 2$.

1. For $z\overline{w} \neq 1$, using the CD formula, we can write

$$\begin{aligned}\mathbb{K}_n(z,w,\tilde{\mu}) &= \frac{\overline{\phi_n^*(w)}\phi_n^*(z) - z\overline{w}\overline{\phi_n(w)}\phi_n(z)}{1-z\overline{w}} \\ &= \frac{(1-\chi_1\overline{w})(1-\chi_2\overline{w})(1-z\overline{\chi_1})(1-z\overline{\chi_2})-(z\overline{w})^{n-1}(z-\chi_1)(z-\chi_2)(\overline{w}-\overline{\chi_1})(\overline{w}-\overline{\chi_2})}{2\pi K(1-z\overline{w})}.\end{aligned}$$

For $z = w$, we obtain the closed form

$$\mathbb{K}_n(z,z,\tilde{\mu}) = \frac{1}{2\pi K}\frac{|1-\bar{z}\chi_1|^2|1-\bar{z}\chi_2|^2 - |z|^{2(n-1)}|z-\chi_1|^2|z-\chi_2|^2}{1-|z|^2}.$$

2. Since $d\mu(z) = |z-\bar{\beta}|^2 d\tilde{\mu}(z)$, using (2.8), the polynomial Φ_n can be expressed as

$$\Phi_n(z) = \frac{1}{z-\bar{\beta}}\left(z^{n-1}(z-\chi_1)(z-\chi_2) - \frac{\bar{\beta}^{n-1}(\bar{\beta}-\chi_1)(\bar{\beta}-\chi_2)}{\mathbb{K}_n(\bar{\beta},\bar{\beta},\tilde{\mu})}\mathbb{K}_n(z,\bar{\beta},\tilde{\mu})\right). \tag{2.17}$$

Using the above expression of Φ_n, the polynomial given in (1.4) becomes:

$$\begin{aligned}&z^{n-2}(z-\bar{\beta})(z-\chi_1)(z-\chi_2) \\ &= z^{n-1}(z-\chi_1)(z-\chi_2) - \frac{\bar{\beta}^{n-1}(\bar{\beta}-\chi_1)(\bar{\beta}-\chi_2)}{\mathbb{K}_n(\bar{\beta},\bar{\beta},\tilde{\mu})}\mathbb{K}_n(z,\bar{\beta},\tilde{\mu}) \\ &\quad - a_n\left(z^{n-2}(z-\chi_1)(z-\chi_2) - \frac{\bar{\beta}^{n-2}(\bar{\beta}-\chi_1)(\bar{\beta}-\chi_2)}{\mathbb{K}_{n-1}(\bar{\beta},\bar{\beta},\tilde{\mu})}\mathbb{K}_{n-1}(z,\bar{\beta},\tilde{\mu})\right).\end{aligned} \tag{2.18}$$

By comparing the coefficients of z^n in (2.18), we get

$$-\bar{\beta}-\chi_1-\chi_2 = -\chi_1-\chi_2 - \frac{1}{2\pi K}\frac{\bar{\beta}|\bar{\beta}^{n-2}(\bar{\beta}-\chi_1)(\bar{\beta}-\chi_2)|^2}{\mathbb{K}_n(\bar{\beta},\bar{\beta},\tilde{\mu})} - a_n.$$

By substituting the value of $\mathbb{K}_n(\bar{\beta},\bar{\beta},\tilde{\mu})$ from (2.15) into the previous equation and simplifying, we obtain the expression for a_n as given in (2.16).

This completes the proof. □

2.1. Para-orthogonal polynomials and Chain sequences

The connection between chain sequences and the continued fraction representation of the ratio of Gauss hypergeometric functions [3], that are pivotal in the study of orthogonal polynomials [10], is a well-known phenomenon. Furthermore, these hypergeometric functions serve as essential tools in diverse areas, including the inequalities among the Gauss hypergeometric functions, as discussed in [25]. Chain sequences can also be related to the complete monotonicity of sequences that are ratio of various special functions. For example, the inequalities involving ratio of gamma function and the corresponding logarithmic complete monotonicity results can be found in [16]. Subsequently, we proceed to represent the positive chain sequence in relation to the Verblunsky coefficients $\tilde{\alpha}_n$ associated with the measure $\tilde{\mu}$. To achieve this, it is essential to introduce the notion of para-orthogonal polynomials on the unit circle (POPUC). The POPUC associated with $\tilde{\Phi}_n$ is given by

$$\tilde{\Phi}_n^p(z;\zeta) := \tilde{\Phi}_n(z) - \frac{\tilde{\Phi}_n(\zeta)}{\tilde{\Phi}_n^*(\zeta)}\tilde{\Phi}_n^*(z) \quad \text{for } \zeta \in \partial\mathbb{D}. \tag{2.19}$$

The CD kernel can be expressed in the framework of the POPUC and this equivalence can be articulated as follows:

$$\mathbb{K}_n(z,w,\tilde{\mu}) = \frac{\overline{\tilde{\Phi}_{n+1}(\zeta)}\kappa_{n+1}^2}{\bar{w}(z-w)}\tilde{\Phi}_{n+1}^p(z;\zeta).$$

The relationship between the POPUC and the CD kernel is extensively explored in [21]. This connection is instrumental in deriving further insights, as exemplified in [9]. Notably, this linkage plays a crucial role in understanding the distribution of zeros of para-orthogonal polynomials, as demonstrated in [33, section 2.14]. For the historical background of the POPUC, we refer to [19,20].

By utilizing (2.19), we consider the sequence $\{\mathcal{L}_n(z;\zeta)\}_{n\geqslant 0}$ of monic polynomials with deg $\mathcal{L}_n = n$ in the variable z defined as

$$\mathcal{L}_n(z;\zeta) = \frac{1}{1+\frac{\tilde{\Phi}_{n+1}(\zeta)}{\tilde{\Phi}_{n+1}^*(\zeta)}\tilde{\alpha}_{n+1}}\frac{\tilde{\Phi}_{n+1}^p(z;\zeta)}{z-\zeta}, \tag{2.20}$$

where $\tilde{\alpha}_n$'s are Verblunsky coefficients associated with measure $\tilde{\mu}$. An equivalent formulation of (2.20) is given by

$$\mathcal{L}_n(z;\zeta) = \frac{z\tilde{\Phi}_n(z) - \zeta\frac{\tilde{\Phi}_n(\zeta)}{\tilde{\Phi}_n^*(\zeta)}\tilde{\Phi}_n^*(z)}{z-\zeta}. \tag{2.21}$$

Now we have a sequence of polynomials $\{\mathcal{R}_n\}_{n\geqslant 0}$, which is defined by

$$\mathcal{R}_n(z) = \mathcal{T}_{n-1}\mathcal{L}_n(z;1), \tag{2.22}$$

where

$$\mathcal{T}_{n-1} = \frac{\prod_{j=0}^{n-1}\left(1 - \frac{\tilde{\Phi}_n(1)}{\tilde{\Phi}_n^*(1)}\tilde{\alpha}_j\right)}{\prod_{j=0}^{n-1}\left(1 - \mathrm{Re}\left(\frac{\tilde{\Phi}_n(1)}{\tilde{\Phi}_n^*(1)}\tilde{\alpha}_j\right)\right)}.$$

Thus, as established in [11, Theorem 2.2], the sequence of polynomials (modified CD kernel) $\{\mathcal{R}_n\}_{n\geqslant 0}$ satisfies the three-term recurrence relation. Notably, one of the recurrence coefficients in this sequence forms a positive chain sequence. More specifically, the recursive expression for $\{\mathcal{R}_n\}_{n\geqslant 0}$ is given by

$$\mathcal{R}_{n+1}(z) = [(1+i\mathfrak{t}_{n+1})z + (1-i\mathfrak{t}_{n+1})]\mathcal{R}_n(z) - 4l_{n+1}z\mathcal{R}_{n-1}(z), n \geqslant 0, \tag{2.23}$$

with initial conditions $\mathcal{R}_{-1}(z) = 0$ and $\mathcal{R}_0(z) = 1$ (see also [31]). A similar recurrence relation is found in [14, equation 3.3], which is then simplified to a more concise form in [14, equation 3.7] using normalization. Significantly, the sequences $\{\mathfrak{t}_n\}_{n\geqslant 1}$ represent real parameters, while $\{l_n\}_{n\geqslant 0}$ constitutes a positive chain sequence. These parameters are determined by the following expressions:

$$\mathfrak{t}_n = \begin{cases} -\frac{\mathrm{Im}(\alpha_0)}{1-\mathrm{Re}(\alpha_0)} & \text{for } n=1, \\ -\frac{\mathrm{Im}\left(\frac{1-\bar{\alpha}_0-a_1}{1-\alpha-\bar{a}_1}(\alpha_1-\bar{a}_2\alpha_0)\right)}{1-\mathrm{Re}\left(\frac{1-\bar{\alpha}_0-a_1}{1-\alpha_0-\bar{a}_1}(\alpha_1-\bar{a}_2\alpha_0)\right)} & \text{for } n=2, \\ 0 & \text{for } n\geqslant 3, \end{cases}$$

and

$$l_{n+1} = \begin{cases} \frac{1}{4}\frac{\left(1-|\alpha_0-\bar{a}_1|^2\right)\left|1-\frac{1-\bar{\alpha}_0-a_1}{1-\alpha-\bar{a}_1}(\alpha_1-\bar{a}_2\alpha_0)\right|^2}{(1-\mathrm{Re}(\alpha_0-\bar{a}_1))\left[1-\mathrm{Re}\left(\frac{1-\bar{\alpha}_0-a_1}{1-\alpha_0-\bar{a}_1}(\alpha_1-\bar{a}_2\alpha_0)\right)\right]} & \text{for } n=1, \\ \frac{1}{4}\frac{1-|\alpha_1-\bar{a}_2\alpha_0|^2}{\left[1-\mathrm{Re}\left(\frac{1-\bar{\alpha}_0-a_1}{1-\alpha_0-\bar{a}_1}(\alpha_1-\bar{a}_2\alpha_0)\right)\right]} & \text{for } n=2, \\ \frac{1}{4} & \text{for } n\geqslant 3. \end{cases} \tag{2.24}$$

The positive chain sequence $\{l_n\}_{n\geqslant 1}$ can be expressed as $l_n = (1-g_{n-1})g_n$ for $n \geqslant 1$, wherein the sequence $\{g_n\}_{n\geqslant 0}$ is referred to as the parameter sequence associated with l_n [2]. This parameter sequence is given by

$$g_n = \begin{cases} \frac{1}{2}\frac{|1-\alpha_0|^2}{1-\mathrm{Re}(\alpha_0)} & \text{for } n=0, \\ \frac{1}{2}\frac{\left|1-\frac{1-\bar{\alpha}_0-a_1}{1-\alpha-\bar{a}_1}(\alpha_1-\bar{a}_2\alpha_0)\right|^2}{\left[1-\mathrm{Re}\left(\frac{1-\bar{\alpha}_0-a_1}{1-\alpha_0-\bar{a}_1}(\alpha_1-\bar{a}_2\alpha_0)\right)\right]} & \text{for } n=1, \\ \frac{1}{2} & \text{for } n\geqslant 2. \end{cases}$$

In addition, for $n \geqslant 2$ and $\zeta = 1$, we can write (2.21) as

$$\mathcal{L}_n(z;1) = \frac{z^{n-1}\tilde{\Phi}_2(z) - \frac{\tilde{\Phi}_2(1)}{\tilde{\Phi}_2^*(1)}\tilde{\Phi}_2^*(z)}{z-1}.$$

For $|z| < 1$, we have

$$\tilde{\Phi}_2^*(z) = \lim_{n\to\infty} \frac{(1-z)\tilde{\Phi}_2^*(1)}{\tilde{\Phi}_2(1)} \mathcal{L}_n(z;1).$$

Thus,

$$\lim_{n\to\infty} \mathcal{R}_n(z) = \frac{1-\alpha_0}{1-\mathrm{Re}(\alpha_0)} \frac{\left(1 - \frac{1-\bar{\alpha}_0-a_1}{1-\alpha-\bar{a}_1}(\alpha_1 - \bar{a}_2\alpha_0)\right)}{\left[1-\mathrm{Re}\left(\frac{1-\bar{\alpha}_0-a_1}{1-\alpha_0-\bar{a}_1}(\alpha_1-\bar{a}_2\alpha_0)\right)\right]} \frac{\tilde{\Phi}_2(1)}{(1-z)\tilde{\Phi}_2^*(1)} \tilde{\Phi}_2^*(z).$$

Note that for $\mathfrak{t}_n = 0$ and $l_n = \frac{1}{4}$, $n \geqslant 1$, (2.23) reads

$$\mathcal{R}_{n+1}(z) = (z+1)\mathcal{R}_n(z) - z\mathcal{R}_{n-1}(z). \tag{2.25}$$

The polynomial corresponding to this recurrence relation is given by

$$\mathcal{R}_n(z) = \frac{z^{n+1}-1}{z-1}. \tag{2.26}$$

The monic Christoffel polynomial of degree n obtained by the Christoffel transformation of normalized Lebesgue measure at point $\tilde{\gamma} = 1$ can be written in terms of the derivative of $\mathcal{R}_n(z)$ as:

$$\Phi_n(z;1) = \frac{1}{n+1}\frac{d}{dz}(z\mathcal{R}_n(z)) = \frac{(n+1)z^n + nz^{n-1} + \ldots + 2z + 1}{n+1}. \tag{2.27}$$

Additionally, the orthogonal polynomial corresponding to the Lebesgue measure on the unit circle can be expressed in terms of $\mathcal{R}_n(z)$ as follows:

$$\Phi_n(z) = z^n = \mathcal{R}_n(z) - \mathcal{R}_{n-1}(z).$$

The corresponding POPUC is given by $\Phi_n^p(z;\zeta) = z^n - \zeta^n \quad \text{for} \quad \zeta \in \partial\mathbb{D}$.

The polynomial $\Phi_n^p(z;\zeta)$, which has degree n, is orthogonal to the set $\{z, z^2, \ldots, z^{n-1}\}$ with respect to the Lebesgue measure, and all the zeros of $\Phi_n^p(z;\zeta)$ lie on the unit circle. Specifically, for $\zeta = 1$, we have

$$\Phi_n^p(z;1) = z^n - 1. \tag{2.28}$$

In Case 3, we will demonstrate the zeros of the polynomial resulting from the linear combination of two consecutive elements in (2.28).

More generally, the linear combination of two consecutive elements of POPUC, as defined in (2.19), is expressed as

$$\Phi_n^p(z;\zeta;\tilde{\gamma}_n) = \tilde{\Phi}_n^p(z;\zeta) + \tilde{\gamma}_n\tilde{\Phi}_{n-1}^p(z;\zeta). \tag{2.29}$$

An equivalent form of (2.29) can be derived by substituting the expression for $\tilde{\Phi}_n^p(z;\zeta)$ mentioned in (2.19).

$$\begin{aligned}\Phi_n^p(z;\zeta;\tilde{\gamma}_n) &= \tilde{\Phi}_n(z) - \frac{\tilde{\Phi}_n(\zeta)}{\tilde{\Phi}_n^*(\zeta)}\tilde{\Phi}_n^*(z) + \tilde{\gamma}_n\tilde{\Phi}_{n-1}(z) - \tilde{\gamma}_n\frac{\tilde{\Phi}_{n-1}(\zeta)}{\tilde{\Phi}_{n-1}^*(\zeta)}\tilde{\Phi}_{n-1}^*(z) \\ &= \tilde{\Phi}_n(z) + \tilde{\gamma}_n\tilde{\Phi}_{n-1}(z) - \frac{\tilde{\Phi}_n(\zeta) + \tilde{\gamma}_n\tilde{\Phi}_{n-1}(\zeta)}{\tilde{\Phi}_2^*(\zeta)}\tilde{\Phi}_2^*(z)\end{aligned} \tag{2.30}$$

The polynomials defined in 2.29 are referred to as "combined POPUC". It may be noted that, we are not calling (2.29) as quasi POPUC, because in the literature [7], this terminology is used for another sequence of polynomials. Interestingly, our terminology of POPUC coincides with the quasi POPUC of order one given in [7], wherein higher orders of quasi POPUC and the properties of their zeros are discussed.

Further simplifying, we write $\Phi_n^p(z;\zeta;\tilde{\gamma}_n)$ defined in (2.30) as:

$$\begin{aligned}\Phi_n^p(z;\zeta;\tilde{\gamma}_n) = z^n + (\tilde{\gamma}_n - \chi_1 - \chi_2)z^{n-1} + (\chi_1\chi_2 - \tilde{\gamma}_n\chi_1 - \tilde{\gamma}_n\chi_2)z^{n-2} + \chi_1\chi_2\tilde{\gamma}_n z^{n-3} \\ - \overline{\chi_1\chi_2}A_n z^2 + (\overline{\chi_1} + \overline{\chi_2})A_n - A_n,\end{aligned} \tag{2.31}$$

where

$$A_n = (\tilde{\gamma}_n + \zeta)\frac{\zeta^{n-3}(\zeta - \chi_1)(\zeta - \chi_2)}{(1 - \zeta\overline{\chi_1})(1 - \zeta\overline{\chi_2})} \implies |A_n| = |\tilde{\gamma}_n + \zeta|.$$

Next, we analyze the location of zeros of the polynomial $\Phi_n^p(z;\zeta;\tilde{\gamma}_n)$ defined in (2.31).

It is evident from the Cauchy theorem [29, page 247] that all the zeros of $\Phi_n^p(z;\zeta;\tilde{\gamma}_n)$ lie inside the disc $|z| < 1 + M$, where

$$\begin{aligned}M = \max\{&|\tilde{\gamma}_n + \zeta|, |\tilde{\gamma}_n - \chi_1 - \chi_2|, |\chi_1\chi_2 - \tilde{\gamma}_n(\chi_1 + \chi_2)|, |\chi_1\chi_2\tilde{\gamma}_n|, |\chi_1\chi_2||\tilde{\gamma}_n + \zeta|, \\ &|(\chi_1 + \chi_2)\tilde{\gamma}_n + \zeta|\}.\end{aligned}$$

We can also determine the bounds for the sum of the squares of the zeros of the polynomial. To achieve this, the companion matrix corresponding to the polynomial (2.31) is expressed as:

$$C_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix},$$

where the coefficients a_n are defined as follows: $a_0 = -A_n$, $a_1 = (\overline{\chi_1} + \overline{\chi_2})A_n$, $a_2 = -\overline{\chi_1\chi_2}A_n$, $a_{n-3} = \chi_1\chi_2\tilde{\gamma}_n$, $a_{n-2} = \chi_1\chi_2 - \tilde{\gamma}_n\chi_1 - \tilde{\gamma}_n\chi_2$ and $a_{n-1} = \tilde{\gamma}_n - \chi_1 - \chi_2$. All other coefficients are zero. Then, the determinantal expression for the polynomial (2.31) can be written as

$$\Phi_n^p(z;\zeta;\tilde{\gamma}_n) = \det(zI_n - C_n), \tag{2.32}$$

where I_n is the $n \times n$ identity matrix. It is clear that $z_1 = \zeta \in \partial\mathbb{D}$ is one of the zeros of the polynomial (2.31). Let the remaining zeros of $\Phi_n^p(z;\zeta;\tilde{\gamma}_n)$ be denoted as $z_2, z_3, \cdots, z_n$. From the Schur inequality [29, page 278], it follows that

$$\begin{aligned}\sum_{j=2}^{n} |z_j|^2 \leqslant (n-2) + |\tilde{\gamma}_n + \zeta|^2 + |\chi_1 + \chi_2|^2|\tilde{\gamma}_n + \zeta|^2 + |\chi_1\chi_2|^2|\tilde{\gamma}_n + \zeta|^2 + |\chi_1\chi_2\tilde{\gamma}_n|^2 \\ + |\chi_1\chi_2 - \tilde{\gamma}_n\chi_1 - \tilde{\gamma}_n\chi_2|^2 + |\tilde{\gamma}_n - \chi_1 - \chi_2|^2.\end{aligned}$$

Further, we impose conditions on the parameter $\tilde{\gamma}_n$ to analyze the zeros of the polynomial defined in (2.31), which we discuss in three cases.

Case 1. If we take $\tilde{\gamma}_n = -\zeta$, then $A_n = 0$ and (2.31) reduces to

$$\Phi_n^p(z;\zeta;-\zeta) = z^n - (\zeta+\chi_1+\chi_2)z^{n-1} + (\chi_1\chi_2+\zeta\chi_1+\zeta\chi_2)z^{n-2} - \chi_1\chi_2\zeta z^{n-3}. \tag{2.33}$$

By simplifying this equation, we obtain

$$\Phi_n^p(z;\zeta;-\zeta) = z^{n-3}(z-\chi_1)(z-\chi_2)(z-\zeta). \tag{2.34}$$

Thus, for the choice $\tilde{\gamma}_n = -\zeta$, exactly $n-1$ zeros of the polynomial $\Phi_n^p(z;\zeta;\tilde{\gamma}_n)$ lie inside the unit disk, while one zero lies on the unit circle.

Case 2. If we take $\chi_1 = 0$ and $\tilde{\gamma}_n = -\frac{1}{\overline{\chi}_2}$, then (2.31) reduces to

$$\Phi_n^p(z;\zeta;-\overline{\chi}_2^{-1}) = \frac{z\overline{\chi}_2 - 1}{\overline{\chi}_2}(z^{n-1} - \chi_2 z^{n-2} - (\zeta-\chi_2)\zeta^{n-2}). \tag{2.35}$$

This expression indicates that at least one zero lies outside the unit disc, specifically $z = \overline{\chi}_2^{-1}$, and one zero lie on the unit circle, namely $z = \zeta$. For the remaining zeros, we observe that no specific pattern emerges. In other words, $n-2$ zeros may either lie inside or outside the unit disk. It is noted that as the value of χ_2 approaches to zero, the largest zero of $\Phi_n^p(z;\zeta;-\overline{\chi}_2^{-1})$ for each $n \geqslant 2$ is $z = \overline{\chi}_2^{-1}$.

Zeros of $z^{n-1} - \chi_2 z^{n-2} - (\zeta-\chi_2)\zeta^{n-2}$		
$n=7$, $\chi_2 = 0.1-0.1i$ $\zeta = -e^{\frac{i\pi}{4}}$	$n=8$, $\chi_2 = 0.5$ $\zeta = e^{\frac{i\pi}{10}}$	$n=9$, $\chi_2 = 0.53+0.25i$ $\zeta = -e^{\frac{i\pi}{8}}$
$-0.9902+0.2546i$	$-0.8551+0.0848i$	$-0.9238-0.3826i$
$-0.7071-0.7071i$	$-0.5762-0.6521i$	$-0.9199+0.4213i$
$-0.2999+0.9782i$	$-0.4408+0.7572i$	$-0.3513-0.9461i$
$0.2664-0.9449i$	$0.1888-0.8929i$	$-0.3376+0.9921i$
$0.6738+0.7403i$	$0.3593+0.8538i$	$0.4622-0.9342i$
$0.9570-0.2211i$	$0.8731-0.4598i$	$0.4881+0.9998i$
–	$0.9510+0.3090i$	$1.0469-0.3508i$
–	–	$1.0654+0.4506i$

Table 1: *Zeros of* $\Phi_n^p(z;\zeta;-\overline{\chi}_2^{-1})$

For $n=8$, $\chi_2 = 0.5$, and $\zeta = e^{\frac{i\pi}{10}}$, six zeros of the polynomial defined in (2.31) lie inside the unit disk, as shown by the magenta dots in Figure 1 and given in Table 1. On the other hand, for $n=9$, $\chi_2 = 0.53+0.25i$, and $\zeta = -e^{\frac{i\pi}{8}}$, seven zeros, represented by green dots in Figure 1, lie outside the unit disk. Furthermore, for $n=7$, $\chi_2 = 0.1-0.1i$, and $\zeta = -e^{\frac{i\pi}{4}}$, the zeros represented as red dots are distributed both inside and outside the unit disk.

Case 3. If we take $\chi_1 = \chi_2 = 0$, then (2.31) reduces to

$$\Phi_n^p(z;1;\tilde{\gamma}_n) = z^n + \tilde{\gamma}_n z^{n-1} - \tilde{\gamma}_n - 1. \tag{2.36}$$

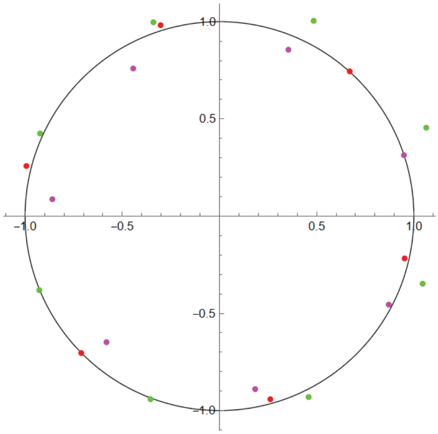

Figure 1: *Zeros of* $z^{n-1} - \chi_2 z^{n-2} - (\zeta - \chi_2)\zeta^{n-2}$, $n = 8$, $\chi_2 = 0.5$ $\zeta = e^{\frac{i\pi}{10}}$ *(magenta dots),* $n = 9$, $\chi_2 = 0.53 + 0.25i$ $\zeta = -e^{\frac{i\pi}{8}}$ *(green dots), and* $n = 7$, $\chi_2 = 0.1 - 0.1i$, $\zeta = -e^{\frac{i\pi}{4}}$ *(red dots).*

The polynomial defined in (2.36) is orthogonal to the set $\{z, z^2, \ldots, z^{n-2}\}$ with respect to the Lebesgue measure.

Zeros of $\Phi_n^p(z;1;\tilde{\gamma})$			
$n=6, \tilde{\gamma}=-0.9$	$n=7, \tilde{\gamma}=-0.2$	$n=6, \tilde{\gamma}=-2$	$n=7, \tilde{\gamma}=-9.1$
-0.583128	$-0.846258-0.419346i$	$-0.678351-0.458536i$	-0.964456
$-0.237066-0.553379i$	$-0.846258+0.419346i$	$-0.678351+0.458536i$	$-0.49786-0.834243i$
$-0.237066+0.553379i$	$-0.187738-0.942i$	$0.195377-0.848854i$	$-0.49786+0.834243i$
$0.47863-0.494043i$	$-0.187738+0.942i$	$0.195377+0.848854i$	$0.480095-0.864496i$
$0.47863+0.494043i$	$0.633997-0.755073i$	1	$0.480095+0.864496i$
1	$0.633997+0.755073i$	1.96595	1
–	1	–	9.0999

Table 2: *Zeros of* $\Phi_n^p(z;1;\tilde{\gamma})$

Interestingly, no longer, all the zeros of $\Phi_n^p(z;1;\tilde{\gamma}_n)$ lie on the unit circle. Particularly, when $\tilde{\gamma}_n = \tilde{\gamma} \in (-1,0)$, (2.36) becomes

$$\Phi_n^p(z;1;\tilde{\gamma}) := z^n + \tilde{\gamma} z^{n-1} - \tilde{\gamma} - 1. \tag{2.37}$$

It is evident that the zeros of (2.28) lie on the unit circle. However, when we perturb the polynomial to obtain (2.36), with one zero at $z = 1$ and the others depending on the parameters $\tilde{\gamma}_n$, interesting patterns emerge. For instance, setting $\tilde{\gamma}_n = \tilde{\gamma} \in (-1,0)$, the zeros of (2.37), except for $z = 1$, reside within the unit disk. This behavior is illustrated for $n = 6$, $\tilde{\gamma} = -0.9$ and $n = 7$, $\tilde{\gamma} = -0.2$ in the Table 2 and figures 2 and 3.

On the other hand, if $\tilde{\gamma} < -1$, then at most one zero of (2.36) extends beyond the unit disk, as detailed in the Table 2. Additionally, as $\tilde{\gamma}$ approaches -1 from within $(-1,0)$, the zeros of (2.37) tend towards the origin, while as $\tilde{\gamma}$ approaches 0, they move closer to the unit circle. The rational modification of (2.28) given by (2.26) exhibits a distinct behavior, with all its zeros lying on the boundary of the unit disk. Furthermore, figures 2 and 3 depict an intriguing alternation in the placement of zeros between (2.26) and (2.28) along the unit circle.

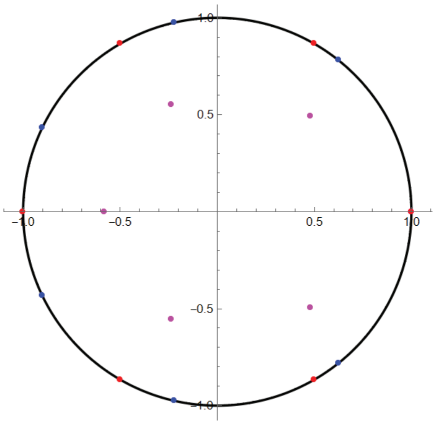

Figure 2: Zeros of $\Phi_6^p(z;1)$ (Red), $\mathcal{R}_6(z)$ (Blue), $\Phi_6^p(z;1;-0.9)$ (2.37) (Magenta)

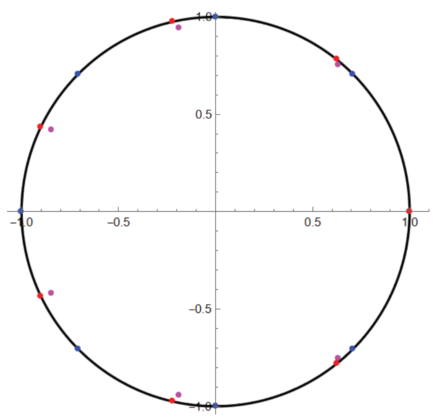

Figure 3: Zeros of $\Phi_7^p(z;1)$ (Red), $\mathcal{R}_7(z)$ (Blue), $\Phi_7^p(z;1;-0.2)$ (2.37) (Magenta)

3. Lubinsky type inequalities involving μ and $\tilde{\mu}$

In the previous section, it has been observed that the zeros of the linear combination of POPUC and quasi-orthogonal polynomials may lie outside the support of the measure. This leads to the question of examining the relation between the measure μ in BGM class and its corresponding measure $\tilde{\mu}$ involving the quasi orthogonal polynomials. In this section, we determine several inequalities involving μ and $\tilde{\mu}$. In particular, we are interesting in finding an inequality similar to Lubinsky inequality [24], which require one of the following norm inequalities.

3.1. Norm inequalities

Here, we discuss some norm inequalities involving $\tilde{\Phi}_n$ and Φ_n with respect to different measures.

PROPOSITION 4. *Let $\{\Phi_n\}_{n\geqslant 0}$ be a sequence of monic OPUC with respect to the measure μ. Let $\tilde{\Phi}_n(z) = \Phi_n(z) - a_n\Phi_{n-1}(z)$, $n \geqslant 1$ be a polynomial of degree exactly equal to n. Then*

$$\|\tilde{\Phi}_n\|_{\mu}^2 \leqslant (1+|a_n|^2)m_o(\mu),\ n \geqslant 1\ , \tag{3.1}$$

where $m_o(\mu) = \int_0^{2\pi} d\mu(e^{i\theta})$.

Proof. Let $\tilde{\Phi}_n(z) = \Phi_n(z) - a_n\Phi_{n-1}(z)$ be a polynomial of degree n. Using the minimization property of Φ_n with respect to μ, we get

$$\begin{aligned}\|\tilde{\Phi}_n\|_\mu^2 &= \int |\tilde{\Phi}_n(z)|^2 d\mu(z) = \int (\Phi_n(z) - a_n\Phi_{n-1}(z))\left(\overline{\Phi_n(z)} - \overline{a_n}\overline{\Phi_{n-1}(z)}\right) d\mu(z)\\ &= \|\Phi_n\|_\mu^2 + |a_n|^2\|\Phi_{n-1}\|_\mu^2 \leqslant \|z^n\|_\mu^2 + |a_n|^2\|z^{n-1}\|_\mu^2\\ &= (1+|a_n|^2)m_o(\mu).\end{aligned}$$

This completes proof. $\square$

Further, if we assume that μ is in BGM class, then it is possible to achieve sharp bounds that remain independent of a_n.

THEOREM 3.1. *Let* μ *be in BGM class. Suppose* $\{\Phi_n\}_{n\geqslant 0}$ *and* $\{\tilde{\Phi}_n\}_{n\geqslant 0}$ *are sequences of monic OPUC with respect to* μ *and* $\tilde{\mu}$*, respectively. Then, for* $n \geqslant 2$*, the following holds:*

1. $\|\tilde{\Phi}_n\|_{\tilde{\mu}}^2 = \|\tilde{\Phi}_n^*\|_{\tilde{\mu}}^2 = 2\pi K$.
2. $\|\tilde{\Phi}_n\|_{\mu}^2 = 2\pi K(1+|\beta|^2)$.
3. $2\pi K \leqslant \|\Phi_n\|_{\tilde{\mu}}^2 \leqslant 2\pi K(1+|\beta|^2) dist(\overline{\beta}, \operatorname{supp}\mu)^{-2}$.

Proof. If μ is in BGM class, then the polynomial $\tilde{\Phi}_n$ is given by $\tilde{\Phi}_n(z) = z^{n-2}(z-\chi_1)(z-\chi_2)$.

1. We have

$$\begin{aligned}\|\tilde{\Phi}_n\|_{\tilde{\mu}}^2 &= \int_{\partial\mathbb{D}} |z^{n-2}(z-\chi_1)(z-\chi_2)|^2 K\frac{1}{|z-\chi_1|^2|z-\chi_2|^2}|dz|\\ &= K\int_{\partial\mathbb{D}} |dz|\\ &= 2\pi K.\end{aligned}$$

Since $\|\tilde{\Phi}_n\|_{\tilde{\mu}}^2 = \|\tilde{\Phi}_n^*\|_{\tilde{\mu}}^2$, it follows $\|\tilde{\Phi}_n^*\|_{\tilde{\mu}}^2 = 2\pi K$.

2. We can write

$$\begin{aligned}\|\tilde{\Phi}_n\|_{\mu}^2 &= \int_{\partial\mathbb{D}} |z^{n-2}(z-\chi_1)(z-\chi_2)|^2 K\frac{|z-\overline{\beta}|^2}{|z-\chi_1|^2|z-\chi_2|^2}|dz|\\ &= K\int_{\partial\mathbb{D}} |z-\overline{\beta}|^2|dz|\\ &= K\int_0^{2\pi}(1-\overline{\beta}e^{-i\theta} - \beta e^{i\theta} + |\beta|^2)d\theta\\ &= 2\pi K(1+|\beta|^2).\end{aligned}$$

3. Using the minimization property of $\tilde{\Phi}_n$ with respect to $\tilde{\mu}$, we obtain

$$\|\Phi_n\|_{\tilde{\mu}}^2 \geqslant \|\tilde{\Phi}_n\|_{\tilde{\mu}}^2 = 2\pi K. \tag{3.2}$$

On the other hand

$$\begin{aligned}\|\Phi_n\|_{\tilde{\mu}}^2 &= \int_{\partial\mathbb{D}} |\Phi_n(z)|^2 d\tilde{\mu}(z)\\ &= \int_{\partial\mathbb{D}} \frac{|\Phi_n(z)|^2}{|z-\bar{\beta}|^2} d\mu(z)\\ &\leqslant dist(\bar{\beta}, \operatorname{supp}\mu)^{-2} \int_{\partial\mathbb{D}} |\Phi_n(z)|^2 d\mu(z)\\ &\leqslant dist(\bar{\beta}, \operatorname{supp}\mu)^{-2} \int_{\partial\mathbb{D}} |\tilde{\Phi}_n(z)|^2 d\mu(z).\end{aligned}$$

The last inequality was obtained using the minimization property of Φ_n with respect to μ. Now, using item (2), we get:

$$\|\Phi_n\|_{\tilde{\mu}}^2 \leqslant dist(\bar{\beta}, \operatorname{supp}\mu)^{-2} 2\pi K(1+|\beta|^2). \tag{3.3}$$

This completes the proof. □

Next, we obtain the measure of the unit circle with respect to $\tilde{\mu}$ and μ using the Cauchy integral formula.

PROPOSITION 5. *Let $\tilde{\mu}$ and μ be the measures given in (2.10) and (2.9), respectively. Then we have the following*

$$\int_{\partial\mathbb{D}} d\tilde{\mu}(z) = 2\pi K \frac{(1-|\chi_1|^2|\chi_2|^2)}{(1-|\chi_1|^2)(1-|\chi_2|^2)|1-\chi_1\overline{\chi_2}|^2}, \tag{3.4}$$

$$\int_{\partial\mathbb{D}} d\mu(z) = 2\pi K \frac{(1-|\chi_1|^2)(1+|\beta|^2 - 2\mathrm{Re}(\beta\chi_2)) + (1-|\chi_2|^2)\left((1+|\beta|^2)|\chi_1|^2 - 2\mathrm{Re}(\beta\chi_1)\right)}{(1-|\chi_1|^2)(1-|\chi_2|^2)|1-\chi_1\overline{\chi_2}|^2}. \tag{3.5}$$

Proof. For the given measures $\tilde{\mu}$ and μ, we write

$$\begin{aligned}\int_{\partial\mathbb{D}} d\tilde{\mu}(z) &= \int_{\partial\mathbb{D}} K \frac{1}{|z-\chi_1|^2|z-\chi_2|^2}|dz|\\ &= \frac{K}{i}\int_{\partial\mathbb{D}} \frac{z}{(z-\chi_1)(z-\chi_2)(1-z\overline{\chi_1})(1-z\overline{\chi_2})} dz.\end{aligned}$$

Since χ_1 and χ_2 lie inside the unit disk, by the Cauchy integral formula, it follows that

$$\int_{\partial\mathbb{D}} d\tilde{\mu}(z) = \frac{2\pi K}{\chi_1-\chi_2}\left[\frac{\chi_1}{(1-|\chi_1|^2)(1-\chi_1\overline{\chi_2})} - \frac{\chi_2}{(1-|\chi_2|^2)(1-\chi_2\overline{\chi_2})}\right].$$

By simplifying the right-hand side of the above equation, the factor $\chi_1-\chi_2$ will cancel out, which proves (3.4).

On the other hand, we write

$$\begin{aligned}\int_{\partial\mathbb{D}} d\mu(z) &= \int_{\partial\mathbb{D}} K\frac{|z-\bar{\beta}|^2}{|z-\chi_1|^2|z-\chi_2|^2}|dz| \\ &= \frac{K}{i}\int_{\partial\mathbb{D}} \frac{(z-\bar{\beta})(1-z\beta)}{(z-\chi_1)(z-\chi_2)(1-z\overline{\chi_1})(1-z\overline{\chi_2})}dz \\ &= \frac{2\pi K}{(\chi_1-\chi_2)}\left[\frac{(\chi_1-\bar{\beta})(1-\chi_1\beta)}{(1-|\chi_1|^2)(1-\chi_1\overline{\chi_2})} - \frac{(\chi_2-\bar{\beta})(1-\chi_2\beta)}{(1-|\chi_2|^2)(1-\chi_2\overline{\chi_1})}\right].\end{aligned}$$

The final equality is obtained by solving the complex integral using the Cauchy integral formula. Upon simplifying the right-hand side of the equation, the factor $\chi_1-\chi_2$ cancels out, resulting in a compact form

$$\int_{\partial\mathbb{D}} d\mu(z) = 2\pi K\frac{(1+|\beta|^2)(1-|\chi_1|^2|\chi_2|^2)-2(1-|\chi_1|^2)\mathrm{Re}\,(\beta\chi_2)-2(1-|\chi_2|^2)\mathrm{Re}\,(\beta\chi_1)}{(1-|\chi_1|^2)(1-|\chi_2|^2)|1-\chi_1\overline{\chi_2}|^2}.$$

By writing $1-|\chi_1|^2|\chi_2|^2 = (1-|\chi_1|^2)+|\chi_1|^2(1-|\chi_2|^2)$, we can further simplify and obtain (3.5) and this completes the proof. □

3.2. Lubinsky type inequality without ordering of the measures

In [24], Lubinsky introduced an important tool, which Simon calls Lubinsky inequality [33], to prove the universality limits, which is essential for giving information about the zeros of para-orthogonal polynomials on the unit circle (POPUC). We have a well-developed theory in the literature [27, 32] to study the asymptotic behaviour of the Christoffel function, which makes the Lubinsky inequality "a powerful tool" since this allows us to control the off-diagonal CD kernel to diagonal CD kernel. If we drop the hypothesis of the Lubinsky inequality i.e. work with a general pair of measures, then the upper bound for L^2-norm of the CD kernel of μ_2 with respect to μ_1 will create a challenge. In our setting of Theorem 3.3, we drop the hypothesis of the Lubinsky inequality and get the control of the off-diagonal CD kernel from the diagonal CD kernels.

For clarity, we give the statement of the Lubinsky inequality. Its proof can be found in [34] (see also [24]).

THEOREM 3.2. *Let $\mu_1 \leqslant \mu_2$. Then for any complex numbers z and w, we have*

$$|\mathbb{K}_n(z,w,\mu_1)-\mathbb{K}_n(z,w,\mu_2)|^2 \leqslant \mathbb{K}_n(w,w,\mu_1)\left[\mathbb{K}_n(z,z,\mu_1)-\mathbb{K}_n(z,z,\mu_2)\right].$$

LEMMA 3.1. *Let μ be a positive Borel measure supported on the unit circle as defined in Definition 1.1 and $\mathbb{K}_n(z,s,\tilde{\mu})$ denote the CD kernel corresponding to the measure $\tilde{\mu}$ as defined in (1.2). Then*

$$\|\mathbb{K}_n(z,s,\tilde{\mu})\|_{\mu}^2 \leqslant 4\mathbb{K}_n(z,z,\tilde{\mu}). \tag{3.6}$$

Proof. For $\tilde{\mu}$ given in (2.10), we write

$$\|\mathbb{K}_n(z,s,\tilde{\mu})\|_{\mu}^2 = \int_{\partial\mathbb{D}} |\mathbb{K}_n(z,s,\tilde{\mu})|^2 K \frac{|s-\bar{\beta}|^2}{|s-\chi_1|^2|s-\chi_2|^2}|ds| \tag{3.7}$$
$$\leqslant 4\int_{\partial\mathbb{D}} |\mathbb{K}_n(z,s,\tilde{\mu})|^2 d\tilde{\mu}(s)$$
$$= \int_{\partial\mathbb{D}} \mathbb{K}_n(z,s,\tilde{\mu})\mathbb{K}_n(s,z,\tilde{\mu})d\tilde{\mu}(s).$$

By using the reproducing property of the CD kernel with respect to $\tilde{\mu}$, we get

$$\|\mathbb{K}_n(z,s,\tilde{\mu})\|_{\mu}^2 \leqslant 4\mathbb{K}_n(z,z,\tilde{\mu}).$$

This completes the proof. □

The above Lemma helps to achieve the following Lubinsky-type inequality.

THEOREM 3.3. *Let μ be a positive Borel measure supported on the unit circle as defined in Definition 1.1. Suppose $\mathbb{K}_n(z,w,\mu)$ and $\mathbb{K}_n(z,w,\tilde{\mu})$ denote the CD kernels corresponding to the measure μ and $\tilde{\mu}$, respectively, as given in (1.2). Then, for any complex numbers z and w, we have*

$$|\mathbb{K}_n(z,w,\mu) - \mathbb{K}_n(z,w,\tilde{\mu})|^2 \leqslant \mathbb{K}_n(w,w,\mu)\left[\mathbb{K}_n(z,z,\mu) + 2\mathbb{K}_n(z,z,\tilde{\mu})\right]. \tag{3.8}$$

Proof. By the reproducing property of CD kernel, we can write

$$\mathbb{K}_n(z,w,\mu) - \mathbb{K}_n(z,w,\tilde{\mu}) = \int (\mathbb{K}_n(z,s,\mu) - \mathbb{K}_n(z,s,\tilde{\mu}))\mathbb{K}_n(s,w,\mu)d\mu(s),$$

and using the Cauchy-Schwarz inequality, the above expression becomes

$$|\mathbb{K}_n(z,w,\mu) - \mathbb{K}_n(z,w,\tilde{\mu})|^2 \leqslant \int |\mathbb{K}_n(z,s,\mu) - \mathbb{K}_n(z,s,\tilde{\mu})|^2 d\mu(s) \int |\mathbb{K}_n(s,w,\mu)|^2 d\mu(s)$$
$$= \mathbb{K}_n(w,w,\mu)\left(\int |\mathbb{K}_n(z,s,\mu) - \mathbb{K}_n(z,s,\tilde{\mu})|^2 d\mu(s)\right). \tag{3.9}$$

Now, we denote the bracketed term in (3.9) as I then

$$I = \int \mathbb{K}_n(z,s,\mu)\mathbb{K}_n(s,z,\mu)d\mu(s) - \int \overline{\mathbb{K}_n(z,s,\tilde{\mu})}\mathbb{K}_n(z,s,\mu)d\mu(s)$$
$$- \int \mathbb{K}_n(z,s,\tilde{\mu})\mathbb{K}_n(s,z,\mu)d\mu(s) + \int \mathbb{K}_n(z,s,\tilde{\mu})\overline{\mathbb{K}_n(z,s,\tilde{\mu})}d\mu(s). \tag{3.10}$$

Substituting (3.6), in (3.10), we have

$$I \leqslant \mathbb{K}_n(z,z,\mu) - 2\mathbb{K}_n(z,z,\tilde{\mu}) + 4\mathbb{K}_n(z,z,\tilde{\mu}). \tag{3.11}$$

Hence, by using (3.11) in (3.9), we obtain (3.8) which completes the proof. □

REMARK 3.1. In one sense, Theorem 3.3 improves Theorem 3.2 since the ordering between the μ and $\tilde{\mu}$ ($\mu \lessgtr \tilde{\mu}$) is not required. On the other hand, we obtain the bound in terms of the diagonal kernel, as in the Lubinsky inequality, but the bound is not sharp. This non-sharpness is due to the fact that $\max|s-\overline{\beta}| = 2$, for $s \in \partial\mathbb{D}$ and $\beta \in \overline{\mathbb{D}}$, in (3.7). This maximum value is not sharp. We can obtain better optimum either for special values of β or considering the more particular measure.

3.3. Sub-reproducing property

We observe that $\mathbb{K}_n(z,w,\mu)$ denotes the kernel polynomials for orthonormal polynomials. It is important to note that normalization plays a crucial role in defining kernel polynomials, providing the reproducing property [34, eq 1.18] and CD formula. On the other hand, if we define the kernel polynomials for orthogonal polynomials, then it will not yield the reproducing property. Nevertheless, we can still inquire about some estimates related to the reproducing type property for the kernel polynomial of orthogonal polynomials. We define

$$\mathcal{K}_n(z,w,\mu) = \sum_{j=0}^{n} \overline{\Phi_j(w)}\Phi_j(z). \tag{3.12}$$

We refer to $\mathcal{K}_n(z,w,\mu)$ as kernel-type polynomials. In the subsequent result, we obtain one-sided inequality for the reproducing property, which is called sub-reproducing property.

PROPOSITION 6. (Sub-Reproducing property) *Let $\mathcal{K}_n(z,w,\mu)$ denote the polynomial of degree n in z, as defined in (3.12). If $p(z)$ is any polynomial of degree at most n such that* $\mathrm{Re}\,(\langle p,\Phi_j\rangle) \geqslant 0$ *for every $j = 0,1,\cdots,n$, then for any $w \in \mathbb{C}$ such that $\Phi_j(w) \geqslant 0$ for each $j = 0,1,\cdots,n$, we have*

$$\mathrm{Re}\left(\int \overline{p(z)}\mathcal{K}_n(z,w,\mu)d\mu(z)\right) \leqslant m_0(\mu)\mathrm{Re}\,(p(w)). \tag{3.13}$$

If μ is a probability measure on the unit circle, then

$$\mathrm{Re}\left(\int \overline{p(z)}\mathcal{K}_n(z,w,\mu)d\mu(z)\right) \leqslant \mathrm{Re}\,(p(w)). \tag{3.14}$$

Proof. Let $p(z) = \sum_{j=0}^{n} \beta_j\Phi_j(z)$. For any $w \in \mathbb{C}$ such that $\Phi_j(w) \geqslant 0$ for each $j = 0,1,\cdots,n$, we can write

$$\begin{aligned}\mathrm{Re}\left(\int \overline{p(z)}\mathcal{K}_n(z,w,\mu)d\mu(z)\right) &= \mathrm{Re}\left(\int \sum_{j=0}^{n} \overline{\beta_j\Phi_j(z)} \sum_{i=0}^{n} \overline{\Phi_i(w)}\Phi_i(z)d\mu(z)\right)\\ &= \sum_{j=0}^{n} \mathrm{Re}\,(\overline{\beta_j\Phi_j(w)})\|\Phi_j\|_\mu^2\\ &\leqslant m_0(\mu)\mathrm{Re}\,(p(w)),\end{aligned}$$

where the inequality follows from the minimization property of Φ_n with respect to μ. $\square$

COROLLARY 3.1. *Let $\mathcal{K}_n(z,\eta,\mu)$ denote the polynomial of degree n in η, as defined in (3.12). Then for any $z \in \mathbb{C}$ and $m_o(\mu) = \int_0^{2\pi} d\mu(e^{i\theta})$, we have*

$$\int \mathcal{K}_n(z,\eta,\mu)\mathcal{K}_n(\eta,z,\mu)d\mu(\eta) \leqslant m_0(\mu)\mathcal{K}_n(z,z,\mu). \tag{3.15}$$

Proof. For any $z \in \mathbb{C}$, we write

$$\int \mathcal{K}_n(z,\eta,\mu)\mathcal{K}_n(\eta,z,\mu)d\mu(\eta) = \sum_{j=0}^{n} \overline{\Phi_j(z)}\Phi_j(z)\|\Phi_j\|_\mu^2 \leqslant m_0(\mu)\mathcal{K}_n(z,z,\mu). \quad \square$$

In the next theorem, we obtain estimates for the absolute difference of diagonal elements of the kernel-type polynomials with respect to the measures μ and $\tilde{\mu}$.

THEOREM 3.4. *Let $z \in \overline{\mathbb{D}}$ and $\{\alpha_j\}_{j\geqslant 0}$ be representing the Verblunsky coefficients for the measure μ, we have*

$$|\mathcal{K}_n(z,z,\tilde{\mu}) - \mathcal{K}_n(z,z,\mu)| \leqslant \sum_{j=0}^{n}\left(\left(|\alpha_{j-1}|+1\right)^2 + 2|a_j|^2\right)\exp\left(2\sum_{k=0}^{j-2}|\alpha_k|\right).$$

Moreover, if $\alpha_n \neq 0$ for $n \geqslant 1$, then

$$|\mathcal{K}_n(z,z,\tilde{\mu}) - \mathcal{K}_n(z,z,\mu)| \leqslant M + 6\sum_{j=3}^{n}\frac{1}{|\alpha_{j-2}|^2}e^{2j-2}, \tag{3.16}$$

where $M = 2e^4(6+\sum_{j=0}^{2}|a_j|^2)$ and a_j's given in (1.4).

Proof. We can write the reproducing kernel-type polynomial with respect to the measure $\tilde{\mu}$ as

$$\begin{aligned}
\mathcal{K}_n(z,w,\tilde{\mu}) &= \sum_{j=0}^{n}(\Phi_j(z) - a_j\Phi_{j-1}(z))(\overline{\Phi_j(w)} - \overline{a_j}\overline{\Phi_{j-1}(w)}) \\
&= \sum_{j=0}^{n}\Phi_j(z)\overline{\Phi_j(w)} - \sum_{j=0}^{n}a_j\Phi_{j-1}(z)\overline{\Phi_j(w)} - \sum_{j=0}^{n}\overline{a_j}\Phi_j(z)\overline{\Phi_{j-1}(w)} \\
&\quad + \sum_{j=0}^{n}|a_j|^2\Phi_{j-1}(z)\overline{\Phi_{j-1}(w)},
\end{aligned} \tag{3.17}$$

which implies

$$
\begin{aligned}
&\mathcal{K}_n(z,w,\tilde{\mu}) - \mathcal{K}_n(z,w,\mu) \\
&= \sum_{j=0}^{n} |a_j|^2 \Phi_{j-1}(z)\overline{\Phi_{j-1}(w)} - \sum_{j=0}^{n} a_j \Phi_{j-1}(z)\overline{\Phi_j(w)} \\
&\quad + \sum_{j=0}^{n} \overline{a_j}\,\overline{\Phi_{j-1}(z)}\Phi_j(w) - \sum_{j=0}^{n} \overline{a_j}\,\overline{\Phi_{j-1}(z)}\Phi_j(w) - \sum_{j=0}^{n} \overline{a_j}\Phi_j(z)\overline{\Phi_{j-1}(w)} \\
&= \sum_{j=0}^{n} |a_j|^2 \Phi_{j-1}(z)\overline{\Phi_{j-1}(w)} - \sum_{j=0}^{n} 2\mathrm{Re}\left(a_j \Phi_{j-1}(z)\overline{\Phi_j(w)}\right) + \sum_{j=0}^{n} \overline{a_j}\,\overline{\Phi_{j-1}(z)}\Phi_j(w) \\
&\quad - \sum_{j=0}^{n} \overline{a_j}\Phi_j(z)\overline{\Phi_{j-1}(w)}.
\end{aligned}
$$

For $z = w$, we have

$$
\mathcal{K}_n(z,z,\tilde{\mu}) - \mathcal{K}_n(z,z,\mu) = \sum_{j=0}^{n} |a_j|^2 \Phi_{j-1}(z)\overline{\Phi_{j-1}(z)} - \sum_{j=0}^{n} 2\mathrm{Re}\left(a_j \Phi_{j-1}(z)\overline{\Phi_j(z)}\right).
$$

Using the triangle inequality, we get

$$
|\mathcal{K}_n(z,z,\tilde{\mu}) - \mathcal{K}_n(z,z,\mu)| \leqslant \sum_{j=0}^{n} |a_j|^2 |\Phi_{j-1}(z)|^2 + \sum_{j=0}^{n} 2\left|\mathrm{Re}\left(a_j \Phi_{j-1}(z)\overline{\Phi_j(z)}\right)\right|.
$$

We know that $2|\mathrm{Re}\,(zw)| \leqslant |z|^2 + |w|^2$. Hence,

$$
|\mathcal{K}_n(z,z,\tilde{\mu}) - \mathcal{K}_n(z,z,\mu)| \leqslant \sum_{j=0}^{n} |\Phi_j(z)|^2 + 2\sum_{j=0}^{n} |a_j|^2 |\Phi_{j-1}(z)|^2.
$$

For $z \in \partial\mathbb{D}$, using the inequality [32, equation 1.5.19] $|\Phi_{n+1}(z)| \leqslant (1+|\alpha_n|)|\Phi_n(z)|$, we can write

$$
|\mathcal{K}_n(z,z,\tilde{\mu}) - \mathcal{K}_n(z,z,\mu)| \leqslant \sum_{j=0}^{n} \left((|\alpha_{j-1}|+1)^2 + 2|a_j|^2 \right) |\Phi_{j-1}(z)|^2,
$$

from which again using the inequality [32, equation 1.5.17], we obtain the desired result

$$
|\mathcal{K}_n(z,z,\tilde{\mu}) - \mathcal{K}_n(z,z,\mu)| \leqslant \sum_{j=0}^{n} \left((|\alpha_{j-1}|+1)^2 + 2|a_j|^2 \right) \exp\left(2\sum_{k=0}^{j-2} |\alpha_k| \right).
$$

If $\alpha_n \neq 0$ for $n \geqslant 1$, then using $a_{n+1} = \frac{\overline{\alpha_n}}{\overline{\alpha_{n-1}}}$ [5, Theorem 4], we have

$$
\begin{aligned}
|\mathcal{K}_n(z,z,\tilde{\mu}) - \mathcal{K}_n(z,z,\mu)| &\leqslant \sum_{j=0}^{2} \left((|\alpha_{j-1}|+1)^2 + 2|a_j|^2 \right) \exp\left(2\sum_{k=0}^{j-2} |\alpha_k| \right) \\
&\quad + \sum_{j=3}^{n} \left((|\alpha_{j-1}|+1)^2 + 2\frac{|\alpha_{j-1}|^2}{|\alpha_{j-2}|^2} \right) \exp\left(2\sum_{k=0}^{j-2} |\alpha_k| \right) \\
&\leqslant M + 6\sum_{j=3}^{n} \frac{1}{|\alpha_{j-2}|^2} e^{2j-2},
\end{aligned}
$$

where $M = 2e^4(6 + \sum_{j=0}^{2} |a_j|^2)$. $\square$

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