Journal of
Mathematical
Inequalities
Volume 19, Number 4 (2025), 1043–1058 doi:10.7153/jmi-2025-19-67

MUCKENHOUPT WEIGHTS ASSOCIATED WITH A CLASS OF HOMOGENEOUS TREES

LUOBIN LIU AND JIANG ZHOU*

(*Communicated by M. Krnić*)

Abstract. In this paper, the authors introduce the Muckenhoupt weights on a class of homogeneous trees, study some important properties of Muckenhoupt weights, and establish an equivalence of Muckenhoupt weights on the trees. As applications, the characterizations of the maximal operator M associated with admissible trapezoids on weighted Lebesgue spaces are obtained.

1. Introduction

The origin of trees can be traced back to the work of Euler, who laid the foundation of graph theory in the study of the Königsberg bridge problem in 1736. While the research conducted during that period primarily focused on general graphs rather than trees, the significance of this work for advancing tree theory should not be overlooked. In 1857, Cayley [4] explicitly introduced the concept of trees while calculating the isomers of saturated hydrocarbons. In the latter half of the 19th century, the application of trees in circuit theory and network analysis developed and became one of the essential tools for circuit analysis, see for instance [7, 8, 17]. In 1972, Cartier [5] first study of problems on harmonic analysis in the framework of trees, laying the foundation for the theory of harmonic analysis on trees. For work on tree-based harmonic analysis theory since the 1980s, see [11, 13, 14].

For a vertex set $\mathscr{V}$ on an infinite homogeneous tree T, equipped with the natural distance d and a flow measure μ, the metric measure space $(\mathscr{V},d,\mu)$ does not satisfy the doubling condition and exhibits exponential growth. Consequently, the classical Calderón-Zygmund decomposition theory is not applicable in this setting. In 2003, Hebisch and Steger [9] established an abstract Calderón-Zygmund decomposition technology applicable to $(\mathscr{V},d,\mu)$ and obtained the weak $(1,1)$ boundedness of the maximal operator associated with admissible trapezoids R. In 2020, using the technology in [9], Arditti, Tabacco and Vallarino [2] introduced the atomic Hardy spaces $H^1(\mu)$ on the tree T, studied its properties and obtained the boundedness of maximal operators associared with trapezoids on $(\mathscr{V},d,\mu)$. In 2021, Arditti, Tabacco, and Vallarino [3]

Mathematics subject classification (2020): 42B25, 05C05, 42B35.
Keywords and phrases: Homogeneous trees, Lebesgue spaces, exponential growth, Muckenhoupt weight, maximal operator.
* Corresponding author.

introduced the BMO(μ) spaces on the tree T, further developing the harmonic analysis theory on trees. However, we find a lack of research on weighted theory for such tree. Therefore, this paper aims to establish a Muckenhoupt weight theory for the tree T.

In classical harmonic analysis, the research on weighted theory began with Muckenhoupt's work [12]. He proved that for $1 < p < \infty$, the Hardy-Littlewood maximal operator is bounded on $L^p_\omega(\mathbb{R}^n)$ if and only if the weight ω satisfies the following condition:

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$, and $|\cdot|$ denotes the Lebesgue measure of a set. The weight ω that satisfies the above inequality is later referred to as an A_p weight or a Muckenhoupt A_p weight. In 1974, Coifman and Fefferman [6] further simplified the proof of the boundedness of the Hardy-Littlewood maximal operator with weighted norm and obtained some properties of the A_p weights, especially when $\omega \in A_p$, the reverse Hölder inequality holds.

Due to the significant role in establishing various weighted inequalities and analyzing the properties of function spaces, discrete Muckenhoupt weights have been studied by many scholars. In 2021, Saker et al. [15] proved some fundamental properties of the discrete Muckenhoupt weights. In the same year, Saker and Agarwal [16] established the discrete Rubio de Francia extrapolation theorem.

Inspired by the aforementioned works, we establish the Muckenhoupt weight theory on this tree T. In Section 2, we present some definitions about infinite homogeneous trees, including the Muckenhoupt classes $\tilde{A}_p$ and A_p, as well as the definition of the reverse Hölder classes RH_r. In Section 3, some properties of $\tilde{A}_p$ and A_p weights are proven. Additionally, we establish an equivalence between $\tilde{A}_p$ weights associated with levels and $\mathscr{A}_p(\mathbb{Z})$. Through this equivalence, we demonstrate that some of the weights associated with levels belong to $\tilde{A}_p$. In section 4, certain conditions for the weighted weak and strong boundedness of the maximal operator are obtained.

In this paper, let $\omega(x) \geqslant 0$ be a weight, and suppose $E \subset \mathscr{V}$, we define $\omega(E) = \sum_{x\in E} \omega(x) q^{l(x)}$. The letter C denotes a constant not necessarily the same at each occurrence, p' is the conjugate exponent to p, i.e. $1/p' + 1/p = 1$, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ represent the floor and ceiling functions, respectively, $\mathbb{Z}$ denotes the set of all integers and $\mathbb{N} = \{0, 1, 2, \cdots\}$.

2. Preliminaries

2.1. Homogeneous tree

DEFINITION 1. [9] An infinite homogeneous tree of order $q+1$ is a graph $T = (\mathscr{V}, \mathscr{E})$ that satisfies the following conditions:

(i) T is connected and acyclic;

(ii) Each vertex in $\mathscr{V}$ has exactly $q+1$ neighbors,

where $\mathscr{V}$ is the set of vertices, and $\mathscr{E}$ is the set of edges. The distance $d(x,y)$ between two vertices x and y in T is the length of the shortest path between x and y.

DEFINITION 2. [1] A doubly-infinite geodesic g in the infinite homogeneous tree $T=(\mathscr{V},\mathscr{E})$ is a subset of $\mathscr{V}$ that satisfies the following conditions:

(i) For each element $x\in g$, there are exactly two neighbours of x in g;

(ii) For arbitrary two vertices $x,y\in g$, the shortest path between x and y is contained in g.

DEFINITION 3. [9] Let $T=(\mathscr{V},\mathscr{E})$ be an infinite homogeneous tree and $N:g\to\mathbb{Z}$ is a mapping on the double-infinite geodesic g such that

$$N(x)-N(y)=d(x,y), \quad \forall x,y\in g.$$

Choosing an origin $o\in g$ such that $N(o)=0$, and determining an orientation for g, one can obtain a numberation of the vertices on g.

Then, define level function $l:\mathscr{V}\to\mathbb{Z}$ as:

$$l(x)=N(x')-d(x,x'),$$

where x' is the only vertex in g such that $d(x,x')=\min\{d(x,z):z\in g\}$. For $x,y\in\mathscr{V}$, x lies below y or y lies above x, if $l(y)-l(x)=d(x,y)$.

DEFINITION 4. [9] Let $T=(\mathscr{V},\mathscr{E})$ be an infinite homogeneous tree of order $q+1$, μ is a measure on $\mathscr{V}$ such that for any non-negative function $f:\mathscr{V}\to\mathbb{R}$,

$$\int_{\mathscr{V}} f d\mu=\sum_{x\in\mathscr{V}} f(x)q^{l(x)}.$$

DEFINITION 5. [2] Let $T=(\mathscr{V},\mathscr{E})$ be an infinite homogeneous tree and equipped with the natural distance d. Given $x'\in\mathscr{V}$ and $r\geqslant 1$. The sphere $S_r(x'):=\{x\in\mathscr{V}: d(x,x')=r\}$ and the closed ball $B_r(x'):=\{x\in\mathscr{V}:d(x,x')\leqslant r\}$.

2.2. Admissible trapezoids and Calderón-Zygmund sets

In [2], Arditti, Tabacco and Vallarino proved that the measure μ is of exponential growth, thus the space $(\mathscr{V},d,\mu)$ does not satisfy the doubling condition. Therefore, they used two special geometric structures to study the related theory of $(\mathscr{V},d,\mu)$, namely admissible trapezoids and Calderón-Zygmund sets.

DEFINITION 6. [1] A subset R of $\mathscr{V}$ is called an admissible trapezoid if it satisfies at least one of the following conditions:

(i) $R=\{x_R\}$ with $x_R\in\mathscr{V}$, that is R consists of a single vertex;

(ii) There exists $x_R\in\mathscr{V}$ and $h(R)\in\mathbb{Z}^+$ such that

$$R=\{x\in\mathscr{V}: x \text{ lies below } x_R, h(R)\leqslant l(x_R)-l(x)<2h(R)\},$$

where $h(R)$ is called the height of R and $h(R)=1$ in the first case and $h(R)=h\in\mathbb{Z}^+$ in the second case. The vertex x_R is called the root node of the admissible trapezoid R. In addition, the quantity $w(R)=q^{l(x_R)}$ is called the width of R. According to the structure of the tree T, it is easy to obtain $\mu(R)=h(R)w(R)$.

DEFINITION 7. [1] Let R be an admissible trapezoid, its envelope $\tilde{R}$ is defined as follows:

(i) If R consists of a single vertex, then $\tilde{R}=R$;

(ii) Other situations,

$$\tilde{R}=\left\{x\in\mathscr{V}: x \text{ lies below } x_R, \frac{h(R)}{2}\leqslant l(x_R)-l(x)<4h(R)\right\}.$$

The envelope of an admissible trapezoid is also called a Calderón-Zygmund set.

LEMMA 1. [2] *Let R be an admissible trapezoid, then $\mu(\tilde{R})\leqslant 4\mu(R)$.*

2.3. Muckenhoupt weights and the reverse Hölder inequality

Next, we introduce the definition of Muckenhoupt weights and the reverse Hölder class on the space $(\mathscr{V},d,\mu)$. We first recall the definitions of weighted Lebesgue and weak Lebesgue spaces.

DEFINITION 8. Let $0<p<\infty$, the weighted Lebesgue space $L^p_\omega(\mathscr{V})$ is defined as follows:

$$L^p_\omega(\mathscr{V})=\{f:\|f\|_{L^p_\omega}<\infty\},$$

where $\|f\|_{L^p_\omega}:=\left(\sum_{x\in\mathscr{V}}|f(x)|^p\omega(x)q^{l(x)}\right)^{\frac{1}{p}}$.

The weighted weak Lebesgue space WL^p_ω is defined as follows:

$$WL^p_\omega(\mathscr{V})=\{f:\|f\|_{WL^p_\omega}<\infty\},$$

where $\|f\|_{WL^p_\omega}:=\sup_{\lambda>0}\lambda\,\omega(\{x\in\mathscr{V}:f(x)>\lambda\})^{\frac{1}{p}}$.

For $p=\infty$, $\|f\|_{L^\infty_\omega}:=\|f\|_{L^\infty}$ and $L^\infty_\omega(\mathscr{V})=L^\infty(\mathscr{V})$.

DEFINITION 9. ($\tilde{A}_p$ and A_p weights) Denote $\tilde{\mathscr{R}}$ as the set of all Calderón-Zygmund sets. Let ω be a weight, $1\leqslant p<\infty$, if ω satisfies the following condition:

$$\sup_{\tilde{R}\in\tilde{\mathscr{R}}}\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)q^{l(x)}\right)\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^{-\frac{1}{p-1}}q^{l(x)}\right)^{p-1}<\infty, \tag{1}$$

then ω is called an $\tilde{A}_p$ weight, denoted as $\omega\in\tilde{A}_p$.

When $p=1$, $\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^{-\frac{1}{p-1}}q^{l(x)}\right)^{p-1}$ is understood as $\left(\inf_{x\in\tilde{R}}\omega(x)\right)^{-1}$.

Denote $\mathscr{R}$ as the set of all admissible trapezoids. Observe that, for any admissible trapezoid R, it follows from Lemma 1 that $\mu(\tilde{R}) \leqslant 4\mu(R)$. Therefore, if $\omega \in \tilde{A}_p$, then ω also satisfies the following condition:

$$\sup_{R\in\mathscr{R}} \left(\frac{1}{\mu(R)}\sum_{x\in R}\omega(x)q^{l(x)}\right)\left(\frac{1}{\mu(R)}\sum_{x\in R}\omega(x)^{-\frac{1}{p-1}}q^{l(x)}\right)^{p-1} < \infty, \tag{2}$$

ω that satisfies inequality (2) is called an A_p weight, denoted as $\omega \in A_p$.

DEFINITION 10. If there exist $1 < r < \infty$ and $C > 0$ such that, for all Calderón-Zygmund sets $\tilde{R}$,

$$\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^r q^{l(x)}\right)^{\frac{1}{r}} \leqslant \frac{C}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)q^{l(x)}, \tag{3}$$

then ω belongs to reverse Hölder class, denoted as $\omega \in RH_r$.

3. Properties of $\tilde{A}_p$ weights

In this section, we discuss the main results of the article concerning $\tilde{A}_p$ weights on the infinite homongeneous tree T. These results include the equivalent propositions of Muckenhoupt weights, the relationship between $\tilde{A}_p$ weights class and BMO spaces, the equivalence between $\tilde{A}_p$ weights in relation to level l and $\mathscr{A}_p(\mathbb{Z})$ and other properties.

First, we present several key properties of $\tilde{A}_p$. As the proofs are similar to those for the classical Euclidean space, we will omit them.

THEOREM 1. (i) *For* $1 \leqslant p < \infty$, $\omega \in \tilde{A}_p$ *if and only if*

$$\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}|f(x)|q^{l(x)} \leqslant C\left(\frac{\sum_{x\in\tilde{R}}|f(x)|^p\omega(x)q^{l(x)}}{\sum_{x\in\tilde{R}}\omega(x)q^{l(x)}}\right)^{\frac{1}{p}}, \quad \forall f \in L^p_\omega(\mathscr{V}),$$

for all Calderón-Zygmund set $\tilde{R} \subset \mathscr{V}$;

(ii) *If* $1 \leqslant p < s < \infty$, *then* $\tilde{A}_p \subset \tilde{A}_s$;

(iii) *If* $p > 1$, *then* $\omega \in \tilde{A}_p$ *if and only if* $\omega^{1-p'} \in \tilde{A}_{p'}$;

(iv) *For* $p > 1$, *if* $\omega \in \tilde{A}_p$, *then for any* $0 < \varepsilon < 1$, $\omega^\varepsilon \in \tilde{A}_p$;

(v) *If* $\omega_1, \omega_2 \in \tilde{A}_1$, *then* $\omega_1\omega_2^{1-p} \in \tilde{A}_p$.

REMARK 1. If we replace $\tilde{R}$ with R, the results of Theorem 1 still hold.

Next, we will demonstrate the relationship between $\tilde{A}_p$ weights and BMO spaces. In [3], Arditti, Tabacco and Vallarino introduced BMO spaces on the trees T. Let $t \in [1,\infty)$. The space $\mathrm{BMO}_t(\mu)$ is defined as follows:

$$\mathrm{BMO}_t(\mu) = \left\{f : \|f\|_{\mathrm{BMO}_t} = \sup_{\tilde{R}}\left(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}}|f - f_{\tilde{R}}|^t d\mu\right)^{1/t} < \infty\right\}.$$

where $f_{\tilde{R}} = \frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} f d\mu$.

LEMMA 2. (Jensen's inequality) *Let E be a measurable set, $f\phi \in L(E)$, $\phi \in L(E)$, $\phi \geqslant 0$ and $\int_E \phi(x)dx > 0$, if g is a convex function, then*

$$g\left(\frac{\int_E f\phi}{\int_E \phi}\right) \leqslant \frac{\int_E g(f)\phi}{\int_E \phi}.$$

THEOREM 2. *For $p > 1$, if $\omega \in \tilde{A}_p$, then $\phi(x) := \ln\omega(x) \in \mathrm{BMO}_1(\mu)$.*

Proof. Since $\omega \in \tilde{A}_p$ and $p > 1$, there exists a constant $C > 0$ such that for all $\tilde{R}$

$$\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} \omega(x)q^{l(x)}\right)\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} \omega(x)^{-\frac{1}{p-1}}q^{l(x)}\right)^{p-1} \leqslant C.$$

Let $\phi(x) = \ln\omega(x)$, then $\omega(x) = e^{\phi(x)}$, we have

$$\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} e^{\phi(x)}q^{l(x)}\right)\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} e^{-\frac{\phi(x)}{p-1}}q^{l(x)}\right)^{p-1} \leqslant C.$$

The above inequality is equivalent to

$$\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} e^{\phi(x)-\phi_{\tilde{R}}}q^{l(x)}\right)\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} e^{-\frac{\phi(x)-\phi_{\tilde{R}}}{p-1}}q^{l(x)}\right)^{p-1} \leqslant C. \tag{4}$$

If let $E = \tilde{R}$, $f(x) = \phi(x) - \phi_{\tilde{R}}$, $\phi(x) \equiv 1$ and $g(x) = e^x$, then from Lemma 2, it follows that

$$\exp\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} |\phi(x) - \phi_{\tilde{R}}|q^{l(x)}\right) \leqslant \frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} e^{\phi(x)-\phi_{\tilde{R}}}q^{l(x)}.$$

Since

$$\sum_{x\in\tilde{R}} |\phi(x) - \phi_{\tilde{R}}|q^{l(x)} = \sum_{x\in\tilde{R}} \phi(x)q^{l(x)} - \sum_{x\in\tilde{R}}\left(\frac{1}{\mu(\tilde{R})}\sum_{y\in\tilde{R}} \phi(y)q^{l(y)}\right)q^{l(x)} = 0,$$

then

$$1 \leqslant \frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} e^{\phi(x)-\phi_{\tilde{R}}}q^{l(x)}. \tag{5}$$

If let $E = \tilde{R}$, $f(x) = \phi(x) - \phi_{\tilde{R}}$, $\phi(x) \equiv 1$ and $g(x) = e^{-\frac{x}{p-1}}$, similarly to the above proof, we can obtain

$$1 \leqslant \frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}} e^{-\frac{\phi(x)-\phi_{\tilde{R}}}{p-1}}q^{l(x)}. \tag{6}$$

From (4), (5) and (6), we have

$$\sup_{\tilde{R}} \frac{1}{\mu(\tilde{R})} \sum_{x \in \tilde{R}} e^{\phi(x) - \phi_{\tilde{R}}} q^{l(x)} < \infty$$

and

$$\sup_{\tilde{R}} \frac{1}{\mu(\tilde{R})} \sum_{x \in \tilde{R}} e^{-\frac{\phi(x) - \phi_{\tilde{R}}}{p-1}} q^{l(x)} < \infty.$$

Since $x < e^x$, it is obvious that

$$\sup_{\tilde{R}} \frac{1}{\mu(\tilde{R})} \sum_{x \in \tilde{R}} |\phi(x) - \phi_{\tilde{R}}| q^{l(x)} < \infty.$$

This shows that $\phi(x) \in \mathrm{BMO}_1(\mu)$. □

Next, we will establish the equivalence between the $\tilde{A}_p$ weight associated with level and $\mathscr{A}_p(\mathbb{Z})$.

DEFINITION 11. [10] A discrete weight ω is said to belong to the *discrete Muckenhoupt class* $\mathscr{A}_1(\mathbb{Z})$ if

$$\|\omega\|_{\mathscr{A}_1(\mathbb{Z})} := \sup_{J \subseteq \mathbb{Z}} \frac{1}{|J|} \left(\frac{1}{\inf\limits_{k \in J} \omega(k)} \sum_{k \in J} \omega(k) \right) < \infty.$$

For $1 < p < \infty$, a discrete weight ω is said to belong to the *discrete Muckenhoupt class* $\mathscr{A}_p(\mathbb{Z})$ if

$$\|\omega\|_{\mathscr{A}_p(\mathbb{Z})} := \sup_{J \subseteq \mathbb{Z}} \left(\frac{1}{|J|} \sum_J \omega \right) \left(\frac{1}{|J|} \sum_J \omega^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

where $\|\omega\|_{\mathscr{A}_p(\mathbb{Z})}$ denotes the norm of weight ω and J is any bounded interval in $\mathbb{Z}$.

Define

$$\mathscr{A}(\mathbb{Z})_\infty := \bigcup_{1 \leqslant p < \infty} \mathscr{A}_p(\mathbb{Z}).$$

THEOREM 3. *Let* $\omega(x) = f(l(x)) > 0$, *then* $\omega \in \tilde{A}_p$ *if and only if* $f \in \mathscr{A}_p(\mathbb{Z})$.

Proof. For all $\tilde{R} \in \tilde{\mathscr{R}}$,

$$\begin{aligned}\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)q^{l(x)} &= \frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}f(l(x))q^{l(x)} \\ &= \frac{1}{\mu(\tilde{R})}\sum_{k=l(x_R)-4h+1}^{\lfloor l(x_R)-\frac{h}{2}\rfloor}\sum_{x\in\tilde{R},l(x)=k}f(l(x))q^{k} \\ &= \frac{1}{Lq^{l(x_R)}}\sum_{k=l(x_R)-4h+1}^{\lfloor l(x_R)-\frac{h}{2}\rfloor}f(k)q^{l(x_R)-k}q^{k} \\ &= \frac{1}{L}\sum_{k=l(x_R)-4h+1}^{\lfloor l(x_R)-\frac{h}{2}\rfloor}f(k),\end{aligned}$$

where L denotes the number of levels, that is $L = 4h(R) - \lceil \frac{h(R)}{2} \rceil$.

By a similar argument, we can also obtain

$$\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^{-\frac{1}{p-1}}q^{l(x)}\right)^{p-1} = \left(\frac{1}{L}\sum_{k=l(x_R)-4h+1}^{\lfloor l(x_R)-\frac{h}{2}\rfloor}f(k)^{-\frac{1}{p-1}})\right)^{p-1}.$$

Let J' is the collection of all simple points in $\mathbb{Z}$ or sets of the form $\mathbb{Z}\cap[l(x_R)-4h+1,\lfloor l(x_R)-\frac{h}{2}\rfloor]$.

Then (1) holds if and only if the following inequality is true

$$\sup_{J'\in\mathscr{J}'}\left(\frac{1}{|J'|}\sum_{k\in J'}f(k)\right)\left(\frac{1}{|J'|}\sum_{k\in J'}f(k)^{-\frac{1}{p-1}}\right)^{p-1} < \infty.$$

Next, we define a family of sets $\mathscr{J}$ as:

$$\mathscr{J} = \{[a,b]\cap\mathbb{Z} | a,b\in\mathbb{Z}, a\leqslant b\}.$$

According to the definitions of $\mathscr{J}'$ and $\mathscr{J}$, we have there exists $J_1' \in \mathscr{J}'$ such that

$$J_1' \subset J \subset 2J_1' =: J_2'.$$

On one hand, it is obvious that

$$\begin{aligned}&\sup_{J'\in\mathscr{J}'}\left(\frac{1}{|J'|}\sum_{k\in J'}f(k)\right)\left(\frac{1}{|J'|}\sum_{k\in J'}f(k)^{-\frac{1}{p-1}}\right)^{p-1} \\ &\leqslant \sup_{J\in\mathscr{J}}\left(\frac{1}{|J|}\sum_{k\in J}f(k)\right)\left(\frac{1}{|J|}\sum_{k\in J}f(k)^{-\frac{1}{p-1}}\right)^{p-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sup_{J\in\mathscr{J}}\left(\frac{1}{|J|}\sum_{k\in J}f(k)\right)\left(\frac{1}{|J|}\sum_{k\in J}f(k)^{-\frac{1}{p-1}}\right)^{p-1}\\
&\leqslant \sup_{J'\in\mathscr{J}'}\left(\frac{1}{|J_1'|}\sum_{k\in J_2}f(k)\right)\left(\frac{1}{|J_1'|}\sum_{k\in J_2}f(k)^{-\frac{1}{p-1}}\right)^{p-1}\\
&=2^p\sup_{J'\in\mathscr{J}'}\left(\frac{1}{|J_2'|}\sum_{k\in J_2'}f(k)\right)\left(\frac{1}{|J_2'|}\sum_{k\in J_2'}f(k)^{-\frac{1}{p-1}}\right)^{p-1}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\sup_{J'\in\mathscr{J}'}\left(\frac{1}{|J'|}\sum_{k\in J}f(k)\right)\left(\frac{1}{|J'|}\sum_{k\in J'}f(k)^{-\frac{1}{p-1}}\right)^{p-1}\\
&\approx \sup_{J\in\mathscr{J}}\left(\frac{1}{|J|}\sum_{k\in J}f(k)\right)\left(\frac{1}{|J|}\sum_{k\in J}f(k)^{-\frac{1}{p-1}}\right)^{p-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sup_{\tilde{R}}\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)q^{l(x)}\right)\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^{-\frac{1}{p-1}}q^{l(x)}\right)^{p-1}<\infty\\
\Longleftrightarrow\ &\sup_{J'\in\mathscr{J}'}\left(\frac{1}{|J'|}\sum_{k\in J'}f(k)\right)\left(\frac{1}{|J'|}\sum_{k\in J'}f(k)^{-\frac{1}{p-1}}\right)^{p-1}<\infty\\
\Longleftrightarrow\ &\sup_{J\in\mathscr{J}}\left(\frac{1}{|J|}\sum_{k\in J}f(k)\right)\left(\frac{1}{|J|}\sum_{k\in J}f(k)^{-\frac{1}{p-1}}\right)^{p-1}<\infty.
\end{aligned}$$

This shows that $\omega\in\tilde{A}_p$ if and only if $f\in\mathscr{A}_p(\mathbb{Z})$. □

REMARK 2. In [10], Hao, Li and Yang established the equivalence relationship between $\mathscr{A}_p(\mathbb{N})$ and $\mathscr{A}_p(\mathbb{Z})$. So, by Theorem 3, for f defined on $\mathbb{N}$ and $\omega=f(|l(x)|)$, we also get that $\omega\in\tilde{A}_p$ if and only if $f\in\mathscr{A}_p(\mathbb{N})$.

Using the above equivalence relationship, we find that the power functions related to level belong to the $\tilde{A}_p$ weight class.

THEOREM 4. *For $1<p<\infty$, let $-1<\alpha<p-1$ and $\omega(x)=(|l(x)|+1)^\alpha$, then $\omega\in\tilde{A}_p$.*

Proof. From Theorem 3 and Remark 2, it is sufficient to prove that, for $f(k) = (k+1)^{\alpha}$, $f \in \mathscr{A}_p(\mathbb{N})$. Let $J = [a, b-1] \cap \mathbb{N}$, $a \in \mathbb{N}$ and $b = 1, 2, \cdots$, then

$$\begin{aligned}
&\left(\frac{1}{|J|}\sum_J (k+1)^{\alpha}\right)\left(\frac{1}{|J|}\sum_J (k+1)^{-\frac{\alpha}{p-1}}\right)^{p-1} \\
&\approx \left(\frac{1}{b-a}\int_a^b (x+1)^{\alpha}\,dx\right)\left(\frac{1}{b-a}\int_a^b (x+1)^{-\frac{\alpha}{p-1}}\,dx\right)^{p-1} \\
&= \frac{1}{(b-a)^p}\left(\frac{(b+1)^{\alpha+1}-(a+1)^{\alpha+1}}{\alpha+1}\right)\left(\frac{(b+1)^{1-\frac{\alpha+1}{p-1}} - a^{1-\frac{\alpha}{p-1}}}{1-\frac{\alpha}{p-1}}\right)^{p-1} \\
&= \frac{(a+1)^p}{(b-a)^p}\left(\frac{\left(\frac{b+1}{a+1}\right)^{\alpha+1}-1}{\alpha+1}\right)\left(\frac{\left(\frac{b+1}{a+1}\right)^{1-\frac{\alpha}{p-1}}-1}{1-\frac{\alpha}{p-1}}\right)^{p-1}.
\end{aligned}$$

Case 1. If $1 \leqslant \frac{b+1}{a+1} < \frac{3}{2}$, then

$$\left(\frac{b+1}{a+1}\right)^{\alpha+1} - 1 \approx (\alpha+1)\left(\frac{b+1}{a+1}-1\right) = (\alpha+1)\left(\frac{b-a}{a+1}\right)$$

and

$$\left(\frac{b+1}{a+1}\right)^{1-\frac{\alpha}{p-1}} - 1 \approx \left(1-\frac{\alpha}{p-1}\right)\left(\frac{b+1}{a+1}-1\right) = \left(1-\frac{\alpha}{p-1}\right)\left(\frac{b-a}{a+1}\right).$$

Thus,

$$\frac{(a+1)^p}{(b-a)^p}\left(\frac{\left(\frac{b+1}{a+1}\right)^{\alpha+1}-1}{\alpha+1}\right)\left(\frac{\left(\frac{b+1}{a+1}\right)^{1-\frac{\alpha}{p-1}}-1}{1-\frac{\alpha}{p-1}}\right)^{p-1} \approx 1. \tag{7}$$

Case 2. If $\frac{b+1}{a+1} \geqslant \frac{3}{2}$, then

$$\begin{aligned}
&\frac{(a+1)^p}{(b-a)^p}\left(\frac{\left(\frac{b+1}{a+1}\right)^{\alpha+1}-1}{\alpha+1}\right)\left(\frac{\left(\frac{b+1}{a+1}\right)^{1-\frac{\alpha}{p-1}}-1}{1-\frac{\alpha}{p-1}}\right)^{p-1} \\
&\leqslant \left(\frac{a+1}{b-a}\right)^p\left(\frac{\left(\frac{b+1}{a+1}\right)^{\alpha+1}}{\alpha+1}\right)\left(\frac{\left(\frac{b+1}{a+1}\right)^{1-\frac{\alpha}{p-1}}}{1-\frac{\alpha}{p-1}}\right)^{p-1} \\
&= \left(\frac{b+1}{b-a}\right)^p \frac{1}{(\alpha+1)(1-\frac{\alpha}{p-1})} \\
&\leqslant 2^p \frac{1}{(\alpha+1)(1-\frac{\alpha}{p-1})}
\end{aligned} \tag{8}$$

In conclusion, from (7) and (8), it is follows that $f \in \mathscr{A}_p(\mathbb{N})$. Thus, $\omega(x) \in \tilde{A}_p$. $\square$

4. Weighted norm inequality for the maximal operator

As an application of Muckenhoupt weights, we will study the weighted estimates of the maximal operator, starting with its definition.

DEFINITION 12. [2] The maximal operator is defined as follows:

$$Mf(x) = \sup_{R\in\mathscr{R},\, R\ni x} \frac{1}{\mu(R)} \sum_{R} |f(v)|q^{l(v)}.$$

LEMMA 3. [1] *For any Calderón-Zygmund set $\tilde{R}$, there are three admissible trapezoids P_1,P_2,P_3 with $\mu(P_1)\leqslant\mu(P_2)\leqslant\mu(P_3)$ such that $\tilde{R}\subset P_1\cup P_2\cup P_3$ and $\mu(P_i)\leqslant 2q\mu(\tilde{R})$ for $i=1,2,3$.*

LEMMA 4. *If $\omega\in A_p$ with $p\geqslant 1$, then $\omega(\tilde{R})\leqslant C(8q)^p\omega(R)$.*

Proof. From Hölder's inequality, the definition of $\tilde{A}_p$ and Lemma 3, we have

$$\begin{aligned}
\frac{1}{4}\leqslant\frac{\mu(R)}{\mu(\tilde{R})} &= \frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\chi_R(x)q^{l(x)}\\
&= \frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\chi_R(x)\omega(x)^{\frac{1}{p}}\omega(x)^{-\frac{1}{p}}q^{l(x)}\\
&\leqslant \left(\frac{1}{\mu(\tilde{R})}\sum_{x\in R}\omega(x)q^{l(x)}\right)^{\frac{1}{p}}\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^{-\frac{p'}{p}}q^{l(x)}\right)^{\frac{1}{p'}}\\
&\leqslant \left(\frac{\omega(R)}{\mu(\tilde{R})}\right)^{\frac{1}{p}}\left(\frac{1}{\mu(\tilde{R})}\sum_{i=1}^{3}\sum_{x\in P_i}\omega(x)^{-\frac{p'}{p}}q^{l(x)}\right)^{\frac{1}{p'}}\\
&\leqslant \left(\frac{\omega(R)}{\mu(\tilde{R})}\right)^{\frac{1}{p}}\sum_{i=1}^{3}\left(\frac{2q}{\mu(P_i)}\sum_{x\in P_i}\omega(x)^{-\frac{p'}{p}}q^{l(x)}\right)^{\frac{1}{p'}}\\
&\leqslant \left(\frac{\omega(R)}{\mu(\tilde{R})}\right)^{\frac{1}{p}}\sum_{i=1}^{3}(2q)^{\frac{1}{p'}}\left(\frac{C}{\mu(P_i)}\sum_{x\in P_i}\omega(x)q^{l(x)}\right)^{-\frac{1}{p}}\\
&\leqslant \left(\frac{\omega(R)}{\mu(\tilde{R})}\right)^{\frac{1}{p}}(2q)^{\frac{1}{p'}}\left(C\sum_{i=1}^{3}\frac{\sum_{x\in P_i}\omega(x)q^{l(x)}}{\mu(P_i)}\right)^{-\frac{1}{p}}\\
&\leqslant \left(\frac{\omega(R)}{\mu(\tilde{R})}\right)^{\frac{1}{p}}(2q)^{\frac{1}{p'}}\left(C\frac{\sum_{i=1}^{3}\sum_{x\in P_i}\omega(x)q^{l(x)}}{\mu(P_3)}\right)^{-\frac{1}{p}}
\end{aligned}$$

$$\leqslant \left(\frac{\omega(R)}{\mu(\tilde{R})}\right)^{\frac{1}{p}} (2q)^{\frac{1}{p'}} \left(\frac{C\mu(P_3)}{\omega(\tilde{R})}\right)^{\frac{1}{p}}$$
$$\leqslant 2q \left(\frac{\omega(R)}{\mu(\tilde{R})}\right)^{\frac{1}{p}} \left(\frac{C\mu(\tilde{R})}{\omega(\tilde{R})}\right)^{\frac{1}{p}}.$$

Therefore, $\omega(\tilde{R}) \leqslant C(8q)^p \omega(R)$. □

THEOREM 5. *Let $1 \leqslant p < \infty$ and ω be a weight. M is bounded from L^p_ω to WL^p_w if and only if $\omega \in A_p$.*

Proof. When $p = 1$, given $x_0 \in R_1 \subset R_2$ and let $f = \chi_{R_1}$. If $\xi \in R_2$, then

$$Mf(x) = \sup_{R \in \mathscr{R}, R \ni x} \frac{1}{\mu(R)} \sum_{y \in R} |f(y)| q^{l(y)} \geqslant \frac{1}{\mu(R_2)} \sum_{y \in R_2} |f(y)| q^{l(y)} = \frac{\mu(R_1)}{\mu(R_2)}.$$

Therefore, $R_2 \subset \{x \in \mathscr{V} : Mf(x) \geqslant \frac{\mu(R_1)}{\mu(R_2)}\} =: \mathscr{M}$.

Furthermore,

$$\sum_{x \in R_2} \omega(x) q^{l(x)} \leqslant \sum_{x \in \mathscr{M}} \omega(x) q^{l(x)} = \omega(\mathscr{M}) \leqslant C \frac{\mu(R_2)}{\mu(R_1)} \sum_{x \in R_1} \omega(x) q^{l(x)}.$$

It shows that

$$\frac{1}{\mu(R_2)} \sum_{x \in R_2} \omega(x) q^{l(x)} \leqslant \frac{C}{\mu(R_1)} \sum_{x \in R_1} \omega(x) q^{l(x)}.$$

If $R_1 = \{x\}$, then for any admissible trapezoid R_2 containing x, we have

$$\frac{1}{\mu(R_2)} \sum_{x \in R_2} \omega(x) q^{l(x)} \leqslant C\omega(x).$$

Taking the supremum on the left side and the infimum on right side, we obtain $\omega \in A_1$.

When $p > 1$, since $\omega = 0$ obviously does not belong to A_p, for any fixed admissible trapezoid R, we may assume $\omega(x) > 0$.

Let $f(x) = \omega(x)^{-\frac{1}{p-1}} \chi_R(x)$ and $\lambda = \frac{1}{\mu(R)} \sum_{x \in R} f(x) q^{l(x)}$, then

$$\begin{aligned} \omega(\{x \in \mathscr{V} : Mf(x) > \lambda\}) &\leqslant \frac{C}{\lambda^p} \sum_{x \in R} \omega(x)^{-\frac{p}{p-1}} \omega(x) q^{l(x)} \\ &= \frac{C\mu(R)^p}{\left(\sum_{x \in R} \omega(x)^{-\frac{1}{p-1}} q^{l(x)}\right)^p} \sum_{x \in R} \omega(x)^{-\frac{1}{p-1}} q^{l(x)} \\ &= C\mu(R)^p \left(\sum_{x \in R} \omega(x)^{-\frac{1}{p-1}} q^{l(x)}\right)^{1-p}. \end{aligned}$$

From the definition of λ, it is obvious that $R \subset \{x \in \mathscr{V} : Mf(x) > \lambda\}$. Thus,

$$\omega(R) \leqslant \omega(\{x \in \mathscr{V} : Mf(x) > \lambda\}) \leqslant C\mu(R)^p \left(\sum_{x \in R} \omega(x)^{-\frac{1}{p-1}} q^{l(x)} \right)^{1-p}.$$

It implies that

$$\left(\frac{1}{\mu(R)} \sum_{x \in R} \omega(x) q^{l(x)} \right) \left(\frac{1}{\mu(R)} \sum_{x \in R} \omega(x)^{-\frac{1}{p-1}} q^{l(x)} \right)^{p-1} \leqslant C.$$

Therefore, $\omega \in A_p$.

Next, we will prove that the following inequality holds:

$$\lambda \omega(\{x \in \mathscr{V} : Mf(x) > \lambda\})^{\frac{1}{p}} \leqslant C\|f\|_{L^p_\omega}, \quad \forall \lambda > 0. \tag{9}$$

Let's assume $f \in L^1(\mathscr{V}, \mu)$. Otherwise, replace $f\chi_{B_r(o)}$ with f. Indeed, if $f \in L^1(\mathscr{V}, \mu)$, (9) is valid. Therefore, for all $f \in L^p_\omega(\mathscr{V})$, we set $g_r(x) = f(x)\chi_{B_r(o)}$, then

$$\begin{aligned} \sum_{x \in \mathscr{V}} |g_r(x)| q^{l(x)} &= \sum_{x \in B_r(o)} |f(x)| q^l(x) \\ &\leqslant \left(\sum_{x \in B_r(o)} |f(x)|^p \omega(x) q^{l(x)} \right) \left(\sum_{x \in B_r(o)} \omega(x)^{-\frac{p'}{p}} q^{l(x)} \right)^{\frac{1}{p'}} < \infty. \end{aligned}$$

This shows that $g_r(x) \in L^1(\mathscr{V})$. Thus, for all $r > 0$, we have

$$\omega(\{x : Mg_r(x) > \lambda\}) \leqslant C \left(\frac{\|g_r\|_{L^p_\omega}}{\lambda} \right)^p.$$

On the one hand, since $g_r \nearrow f$, from Levi's Lemma, it follows that

$$\|g_r\|_{L^p_\omega} \to \|f\|_{L^p_\omega},\ r \to \infty.$$

On the other hand, $Mf(x) > \lambda$ if and only if there exists an admissible trapezoid R_0 such that

$$\frac{1}{\mu(R_0)} \sum_{y \in R_0} |f(y)| q^{l(y)} > \lambda,$$

and for sufficiently large r, it is obvious that $R_0 \subset B_r(0)$ and then $Mg_r(x) > \lambda$. Thus,

$$\{Mf(x) > \lambda\} \subset \bigcup_{r=0}^{\infty} \{x : Mg_r(x) > \lambda\}.$$

Furthermore, we have

$$\omega(\{x : Mf(x) > \lambda\}) \leqslant \omega\left(\lim_{r \to \infty} \{x : Mg_r(x) > \lambda\} \right) = \lim_{r \to \infty} \omega(\{x : Mg_r(x) > \lambda\}).$$

Clearly,

$$\omega(\{x : Mg_r(x) > \lambda\}) \leqslant \omega(\{x : Mf(x) > \lambda\}).$$

Therefore,

$$\lim_{r\to\infty} \omega\left(\{x : Mg_r(x) > \lambda\}\right) = \omega\left(\{x : Mf(x) > \lambda\}\right).$$

From the above, it is sufficient to prove the case for $f \in L^1(\mathscr{V}, \mu)$. Define S_0 as the family of all admissible trapezoids R such that

$$\sum_{x\in R} |f(x)|q^{l(x)} \geqslant \lambda\mu(R).$$

By employing the methods described in [2, p. 29], we can find a countable set of admissible trapezoids R_j, such that

$$\{x : Mf(x) > \lambda\} \subset \bigcup_j \tilde{R}_j.$$

From Lemma 4 and Theorem 1, we obtain

$$\begin{aligned}
\omega\left(\{x\in\mathbb{R}^n : Mf(x) > \lambda\}\right) &\leqslant \sum_j \omega(\tilde{R}_j) \\
&\leqslant C(8q)^p \sum_j \omega(R_j) \\
&\leqslant C(8q)^p \sum_j \left(\frac{\mu(R_j)}{\sum\limits_{x\in R_j} |f(x)|q^{l(x)}}\right)^p \sum_{x\in R_j} |f(x)|^p \omega(x) q^{l(x)} \\
&\leqslant C(8q)^p \sum_j \left(\frac{1}{\lambda}\right)^p \sum_{x\in R_j} |f(x)|^p \omega(x) q^{l(x)} \\
&\leqslant C(8q)^p \left(\frac{1}{\lambda}\right)^p \|f\|^p_{L^p_w}.
\end{aligned}$$

The proof of Theorem 5 is complete. □

In Theorem 5, we have obtained the necessary and sufficient condition for the weighted weak (p,p) boundedness of the maximal operator. However, due to the lack of corresponding techniques as in $\mathbb{R}^n$, we are unable to prove that $\omega \in RH_r$ when $\omega \in \tilde{A}_p$ and thus, the necessary condition for the (p,p) boundedness of the maximal operator for $p > 1$. Nevertheless, we still provided a sufficient condition for the (p,p) boundedness of the maximal operator. We begin with a lemma.

LEMMA 5. *If $\omega \in \tilde{A}_p$ with $p > 1$, and there exists $r > 1$ such that $\omega^{1-p'} \in RH_r$, then $\omega \in \tilde{A}_s$ for some $1 < s < p$.*

Proof. According to the definition of RH_r, we have

$$\left(\frac{1}{\mu(\tilde{R})} \sum_{x\in\tilde{R}} \omega(x)^{(1-p')r} q^{l(x)}\right)^{\frac{1}{r}} \leqslant \frac{C}{\mu(\tilde{R})} \sum_{x\in\tilde{R}} \omega(x)^{1-p'} q^{l(x)}.$$

Let $s' = 1 + r(p' - 1)$, it is easy to see that $1 < s < p$. Thus,

$$\begin{aligned}
&\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)q^{l(x)}\right)\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^{1-s'}q^{l(x)}\right)^{\frac{1}{s'-1}}\\
&=\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)q^{l(x)}\right)\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^{(1-s')r}q^{l(x)}\right)^{\frac{1}{r(s'-1)}}\\
&\leqslant C\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)q^{l(x)}\right)\left(\frac{1}{\mu(\tilde{R})}\sum_{x\in\tilde{R}}\omega(x)^{1-s'}q^{l(x)}\right)^{\frac{1}{s'-1}}\\
&\leqslant C. \quad \square
\end{aligned}$$

THEOREM 6. *If* $\omega \in \tilde{A}_p$ *with* $p > 1$*, and there exists* $r > 1$ *such that* $\omega(x)^{1-p'} \in RH_r$*, then* $\|Mf\|_{L^p_\omega} \leqslant C\|f\|_{L^p_\omega}$.

Proof. According to Lemma 5, we have $\omega \in \tilde{A}_s$ for some $1 < s < p$. Thus, from Theorem 5, it follows that $\|Mf\|_{WL^s_\omega} \leqslant C\|f\|_{L^s_\omega}$. Since $\|\cdot\|_{L^\infty_\omega} = \|\cdot\|_{L^\infty}$, then M is bounded from L^∞_ω to L^∞_ω. By Marcinkiewicz's interpolation theorem, we obtain $\|Mf\|_{L^p_\omega} \leqslant C\|f\|_{L^p_\omega}$. The proof of Theorem 6 is complete. $\square$

In Section 3, we have established the equivalence between $\tilde{A}_p$ weights related to level and $\mathscr{A}_p$ weights on the integer set $\mathbb{Z}$. Therefore, for weights related to level, the necessary and sufficient condition for the weighted boundedness of the maximal operator is $\omega \in \tilde{A}_p$.

Whether there exists a relationship between $\tilde{A}_p$, A_p and RH_r, similar to that in $\mathbb{R}^n$, can be considered an open question.

Acknowledgements. The authors thank the referees for their careful reading and helpful comments which indeed improved the presentation of this article.

Funding information. The research was supported by National Natural Science Foundation of China (Grant No. 12461021).

Authors contributions. All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest. Authors state no conflict of interest.

REFERENCES

[1] L. ARDITTI, *Analysis on weighted homogeneous trees*, Master's thesis (2018).
[2] L. ARDITTI, A. TABACCO AND M. VALLARINO, *Hardy spaces on weighted homogeneous trees*, in *Advances in microlocal and time-frequency analysis*, 21–39, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham.
[3] L. ARDITTI, A. TABACCO AND M. VALLARINO, *BMO spaces on weighted homogeneous trees*, J. Geom. Anal. **31** (2021), no. 9, 8832–8849.
[4] A. CAYLEY, *On the theory of analytic forms called trees*, Collected Mathematical Papers of Arthur Cayley, Cambridge Univ. Press, **3** (1890) 242–246.
[5] P. CARTIER, *Harmonic analysis on trees*, in *Harmonic analysis on homogeneous spaces* (Proc. Sympos. Pure Math., vol. XXVI, Williams Coll., Williamstown, Mass., 1972), pp. 419–424, Proc. Sympos. Pure Math., vol. XXVI, Amer. Math. Soc., Providence, RI.
[6] R. R. COIFMAN AND C. L. FEFFERMAN, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
[7] W.-K. CHEN, *Applied graph theory*, second revised edition, North-Holland Series in Applied Mathematics and Mechanics, vol. 13, North-Holland, Amsterdam-New York-Oxford, 1976.
[8] L. R. FOULDS, *Graph theory applications*, Universitext, Springer, New York, 1992.
[9] W. HEBISCH AND T. STEGER, *Multipliers and singular integrals on exponential growth groups*, Math. Z. **245** (2003), no. 1, 37–61.
[10] X. HAO, B. D. LI AND S. YANG, *The Hardy-Littlewood maximal operator on discrete weighted Morrey spaces*, Acta Math. Hungar. **172** (2024), no. 2, 445–469.
[11] A. KORÁNYI AND M. A. PICARDELLO, *Boundary behaviour of eigenfunctions of the Laplace operator on trees*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **13** (1986), no. 3, 389–399.
[12] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
[13] A. G. SETTI, *L^p and operator norm estimates for the complex time heat operator on homogeneous trees*, Trans. Amer. Math. Soc. **350** (1998), no. 2, 743–768.
[14] A. V. SOBOLEV AND M. Z. SOLOMYAK, *Schrödinger operators on homogeneous metric trees: spectrum in gaps*, Rev. Math. Phys. **14** (2002), no. 5, 421–467.
[15] S. H. SAKER et al., *Some basic properties and fundamental relations for discrete Muckenhoupt and Gehring classes*, Adv. Difference Equ. **2021** (2021), no. 8, 1–22.
[16] S. H. SAKER AND R. P. AGARWAL, *Theory of discrete Muckenhoupt weights and discrete Rubio de Francia extrapolation theorems*, Appl. Anal. Discrete Math. **15** (2021), no. 2, 295–316.
[17] D. B. WEST, *Introduction to graph theory*, Prentice Hall, Upper Saddle River, NJ, 1996.

(Received September 28, 2024)

Luobin Liu
College of Mathematics and System Sciences
Xinjiang University
Urumqi 830017
e-mail: 107552300589@stu.xju.edu.cn

Jiang Zhou
College of Mathematics and System Sciences
Xinjiang University
Urumqi 830017
e-mail: zhoujiang@xju.edu.cn