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ON NEW LYAPUNOV–TYPE INEQUALITY FOR THE DIRICHLET PROBLEM OF THE FRACTIONAL BAGLEY–TORVIK DIFFERENTIAL EQUATION

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Abstract. This paper discusses a class of fractional Bagley-Torvik differential equations under Dirichlet boundary conditions and establishes a new Lyapunov-type inequality. Firstly, by proving an auxiliary lemma, the discussed boundary value problem is effectively transformed into an integral equation involving the Green's function. Secondly, an upper bound estimate for the Green's function is provided. Finally, using a priori estimation methods, the corresponding Lyapunov-type inequality is derived.

1. Introduction

Fractional calculus is a branch of mathematics that studies integrals and derivatives of arbitrary order, extending traditional integer-order calculus. Fractional differential equations (FDEs) are equations that include fractional differential operators. Over the past few decades, FDEs have been recognized as particularly effective in describing real-world phenomena characterized by memory and hereditary effects, thereby serving as a fundamental tool in the mathematical modeling of complex mechanical and physical processes. At present, FDEs are extensively employed in diverse scientific domains, including dispersion processes in fractal and porous media, capacitor theory, electrochemical systems, semiconductor physics, turbulence modeling, condensed matter theory, viscoelasticity, biomathematics, and statistical mechanics [7, 9, 20]. For example, Bagley and Torvik [22] explored the application of FDEs in modeling the behavior of viscoelastic materials and proposed the following model:

$$mx''(t)+2A\sqrt{\mu\rho}D_{0+}^{3/2}x(t)+Kx(t)=0, \tag{1.1}$$

where $D_{0+}^{3/2}$ is the Riemann-Liouville fractional derivative of order $3/2$. Additionally, ρ denotes the fluid density, μ is the viscosity, m and A are the mass and area of the thin rigid plate immersed in the viscous fluid, respectively, and K is the stiffness of the string. The variable x represents the motion of the plate. The authors

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demonstrate that the constitutive relations established through fractional derivatives effectively describe the frequency-dependent behavior of viscoelastic polymers and exhibit excellent applicability in finite element analysis. Equation (1.1) is the well-known Bagley-Torvik equation. Given its extensive practical applications, recent years have seen widespread academic interest in the study of the Bagley-Torvik equation and its generalized forms [2, 24].

On the other hand, the study of the Lyapunov inequality can be traced back to 1892, as Lyapunov proved the following result:

THEOREM 1.1. *Let* $q(t) \in C([a,b],\mathbb{R})$. *If the Hill equation*

$$x''(t) + q(t)x(t) = 0, \quad t \in (a,b),$$

admits a non-trivial continuous solution under the Dirichlet boundary conditions

$$x(a) = x(b) = 0,$$

then $q(t)$ *satisfies the following inequality:*

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1.2}$$

This result is optimal, as the constant 4 cannot be replaced by a larger number [11]. Inequality (1.2) is referred to as the classical Lyapunov inequality. Lyapunov inequality and its generalizations have been demonstrated to be valuable tools in the study of eigenvalue problems, disconjugacy, oscillation theory, and various other applications within the theories of ordinary differential equations, partial differential equations, difference equations, impulsive differential equations, and dynamic equations on time scales [1, 8, 14, 25].

In recent years, the advancement of fractional calculus theory has ignited significant interest among scholars in the study of fractional Lyapunov-type inequalities [4, 5, 15, 16, 21, 23, 27]. Ferreira [5] was a pioneer in deriving a Lyapunov-type inequality for fractional boundary value problem (BVP) involving the Caputo fractional derivative. The main result was stated as follows:

THEOREM 1.2. *Let* $q(t) \in C([a,b],\mathbb{R})$. *If the fractional BVP:*

$$\begin{cases} ({}^C D^{\alpha}_{a+}x)(t) + q(t)x(t) = 0, \quad t \in (a,b), \\ x(a) = x(b) = 0, \end{cases}$$

has a nontrivial continuous solution, where $1 < \alpha \leqslant 2$, *and* ${}^C D^{\alpha}_{a+}$ *is the Caputo fractional derivative of order* α, *then* $q(t)$ *satisfies the following inequality:*

$$\int_a^b |q(s)|ds > \frac{\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.3}$$

More recently, due to the diversity of definitions in fractional calculus, some scholars have derived a series of fractional Lyapunov-type inequalities based on different fractional calculus. Toprakseven [21] discussed Lyapunov-type inequalities for a class of fractional differential equations with integral boundary conditions involving the Caputo-Fabrizio fractional calculus. Srivastava [19] investigated Lyapunov-type inequalities for a class of fractional differential equations with Riemann-Stieltjes integral boundary conditions including the Caputo fractional calculus. Dien and Nieto [3] explored Lyapunov-type inequalities for a class of sequential fractional differential equations with mixed boundary conditions using the ψ-Hilfer fractional calculus. Łupińska [13] examined Lyapunov-type inequalities for a class of fractional differential equations with mixed boundary conditions based on the Katugampola fractional calculus. Hamiaz [6] studied Lyapunov-type inequalities for a class of fractional differential equations with anti-periodic boundary conditions within the Atangana-Baleanu-Caputo fractional calculus. Liu and Wang [12] discussed Lyapunov-type inequalities for a class of fractional p-Laplacian differential equations with Dirichlet boundary conditions concerning the local fractional calculus. For more recent work, please refer to the latest review article on fractional Lyapunov-type inequalities by Bashir et al. [17].

Note that $(D^{\alpha}_{a+}u)(t)+q(t)u(t)=0$ is referred to as a single-term fractional differential equation. In some cases, differential equations contain multiple derivatives of the function. Such differential equations are called multi-term differential equations. For example, the previously mentioned Bagley-Torvik equation and the Langevin equation are both multi-term differential equations. In recent years, numerous scholars have investigated Lyapunov-type inequalities for fractional BVPs. However, up to now, only a limited amount of literature has addressed Lyapunov-type inequalities for BVPs of multi-term fractional differential equations [10, 18, 26]. Among them, the authors of [26] considered Lyapunov-type inequalities for fractional Langevin-type equations involving the Caputo-Hadamard fractional derivative subject to mixed boundary conditions. In [10], the authors studied Lyapunov-type inequalities for a class of Langevin-type equations involving the Caputo fractional derivative under Dirichlet boundary conditions and mixed boundary conditions, respectively. In [18], Pourhadi and Mursaleen discussed Lyapunov-type inequalities for a class of multi-term fractional differential equations with mixed boundary conditions. The main result was presented as follows:

THEOREM 1.3. *Let $p(t)\in C^1([a,b])$ and $q(t)\in C([a,b])$. If the fractional BVP:*

$$\begin{cases} {}^{C}D^{\alpha}_{a+}y(t)+p(t)y'(t)+q(t)y(t)=0, \quad a<t<b, \\ y(a)=y'(a)=y(b)=0, \end{cases} \tag{1.4}$$

has a nontrivial continuous solution, where $2<\alpha\leqslant 3$, and ${}^{C}D^{\alpha}_{a+}$ is the Caputo fractional derivative of order α, then the following inequality holds:

(i) If $\alpha\leqslant b-a+1$, then

$$\int_a^b (|p(s)|+|q(s)|+|p'(s)|)ds \geqslant \frac{\Gamma(\alpha)(b-a)^{1-\alpha}}{\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}.$$

(ii) If $\alpha\geqslant b-a+1$, then

$$\int_a^b (|p(s)|+|q(s)|+|p'(s)|)ds \geqslant \frac{\Gamma(\alpha)(b-a)^{2-\alpha}}{(\alpha-1)\max\{g(\alpha),h(\alpha),A(\alpha+1)\}},$$

where

$$g(\alpha)=\frac{1}{4}(4-\alpha)^2,\ A(\alpha)=4\alpha^{-\alpha}(\alpha-2)^{\alpha-2},$$
$$h(\alpha)=\left(\frac{\alpha-2}{2}\right)^{\frac{(\alpha-2)(3-\alpha)}{4-\alpha}}-\left(\frac{\alpha-2}{2}\right)^{\frac{2-(\alpha-2)^2}{4-\alpha}}.$$

It is worth noting that there is currently no literature mentioning the study of Lyapunov-type inequalities for BVPs of fractional Bagley-Torvik differential equations. Therefore, inspired by existing literature, in this paper, we focus on the Lyapunov-type inequalities for the following fractional Bagley-Torvik differential equation:

$$^{C}D^{\alpha}_{a+}y(t)+\mu\,^{C}D^{3/2}_{a+}y(t)+q(t)y(t)=0,\quad a<t<b, \tag{1.5}$$

subject to Dirichlet boundary conditions:

$$y(a)=y(b)=0, \tag{1.6}$$

where $3/2<\alpha\leqslant 2$, $^{C}D^{\kappa}_{a+}$ is the Caputo fractional derivative of order $\kappa=\alpha$ or $3/2$, $q(t)$ is a real-valued continuous function, and $\mu\geqslant 0$ is a constant. The key highlights of the paper can be summarized as follows:

- The Bagley-Torvik equation we study has practical significance, and discussing the Lyapunov-type inequality for the Bagley-Torvik equation is of great importance for its qualitative analysis.
- The Bagley-Torvik equation is a type of multi-term differential equation, and discussing its Lyapunov-type inequality is more complex compared to single-term equations. For instance, the Bagley-Torvik equation (1.5) includes a dissipative term $^{C}D^{3/2}_{a+}$, which directly increases the difficulty of transforming problem (1.5)–(1.6) into an integral equation with Green's functions (for this, we prove auxiliary Lemma 3.1 in this paper), and also complicates the discussion of the properties of the Green's function (see Remark 3.1).
- The equations discussed in this paper involve two fractional derivatives, which are more general compared to the multi-term equations discussed in [18]. Additionally, as $\mu\to 0$, the results obtained in this paper can degenerate to those in [5]. Therefore, the results of this paper extend and enrich the existing literature, offering broader applicability.

The remainder of the paper is organized as follows: In Section 2, we review the definitions and basic properties of Caputo fractional calculus. In Section 3, we use the conclusions from Section 2 to transform the boundary value problem (1.5)–(1.6) into an equivalent integral equation with a Green's function, and provide relevant estimates for the Green's function using a combination of numerical and graphical methods. Additionally, we establish a Lyapunov-type inequality for the problem (1.5)–(1.6) using a priori estimation methods. In Section 4, we provide an example to verify the validity of the obtained results. Finally, in Section 5, we provide a brief conclusion and an outlook on future work.

2. Preliminaries

We begin this section by recalling the definitions and associated properties of fractional calculus.

DEFINITION 2.1. ([9]) Let $[a,b]$ $(-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. Then, the α-th $(\alpha > 0)$ order Riemann-Liouville fractional integral $I^{\alpha}_{a+}x(t)$ of an integrable real-valued function x defined on $[a,b]$ is given by

$$I^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds, \quad t > a,$$

provided the right-hand sides are pointwise defined on $[a,b]$.

DEFINITION 2.2. ([9]) Let $[a,b]$ $(-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The α $(\alpha > 0)$ order Caputo fractional derivative ${}^{C}D^{\alpha}_{a+}x(t)$ of a function $x \in AC^n([a,b],\mathbb{R})$ is defined as

$${}^{C}D^{\alpha}_{a+}x(t) = (I^{n-\alpha}_{a+}D^n x)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds,$$

where $n = [\alpha]+1$, $D = d/dt$, $AC^n[a,b] = \{x : [a,b] \to \mathbb{R} | D^{n-1}x(t) \in AC[a,b]\}$, $AC[a,b]$ denotes the set of all absolutely continuous functions on $[a,b]$.

LEMMA 2.1. ([9]) *Let $\alpha > 0$. Suppose $x \in AC^n[a,b]$, then*

$$I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+}x(t) = x(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1},$$

where $n = [\alpha]+1$, $c_i = -\frac{x^{(i)}(a)}{i!}$ $(i = 0,1,2,\cdots,n-1)$.

LEMMA 2.2. ([9]) *Let $\alpha,\beta > 0$. Suppose $x \in L^{\infty}(a,b)$, then*

$$I^{\alpha}_{a+}I^{\beta}_{a+}x(t) = I^{\alpha+\beta}_{a+}x(t), \quad {}^{C}D^{\alpha}_{a+}I^{\alpha}_{a+}x(t) = x(t).$$

LEMMA 2.3. ([9]) *Let $\alpha > 0$, $\lambda > -1$, $t > a$, then*

$$I^{\alpha}_{a+}(t-a)^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)}(t-a)^{\alpha+\lambda}, \quad {}^{C}D^{\alpha}_{a+}(t-a)^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}(t-a)^{\lambda-\alpha}.$$

In particular, ${}^{C}D^{\alpha}_{a+}(t-a)^k = 0$, for $k = 0,1,2,\cdots,n-1$, where $n = [\alpha]+1$.

3. Main results

3.1. Green's functions and their properties for BVP (1.5)–(1.6)

In this subsection, we first prove an auxiliary lemma. Then, using the preliminary knowledge from Section 2, we transform the boundary value problem (1.5)–(1.6) into an integral equation involving the Green's function. Additionally, we present the relevant properties of the Green's function.

LEMMA 3.1. *Let* $\frac{3}{2} < \alpha \leqslant 2$ *and* $y(t) \in AC^2[a,b]$, *then*

$$I_{a+}^{\alpha}\, {}^{C}D_{a+}^{3/2} y(t) = I_{a+}^{\alpha-(3/2)} y(t) - \frac{y(a)(t-a)^{\alpha-(3/2)}}{\Gamma\left[\alpha-(1/2)\right]} - \frac{y'(a)}{\Gamma\left[\alpha+(1/2)\right]}(t-a)^{\alpha-(1/2)}.$$

Proof. In fact, it follows from the Definition 2.1 and Lemmas 2.1–2.3 that

$$\begin{aligned} I_{a+}^{\alpha}\, {}^{C}D_{a+}^{3/2} y(t) &= I_{a+}^{\alpha-(3/2)} I_{a+}^{3/2}\, {}^{C}D_{a+}^{3/2} y(t) \\ &= I_{a+}^{\alpha-(3/2)} \left[y(t) - y(a) - y'(a)(t-a)\right] \\ &= I_{a+}^{\alpha-(3/2)} y(t) - \frac{y(a)}{\Gamma\left[\alpha-(3/2)\right]} \int_a^t (t-s)^{\alpha-(5/2)} ds \\ &\quad - \frac{y'(a)\Gamma(2)}{\Gamma\left[\alpha+(1/2)\right]}(t-a)^{\alpha-(1/2)} \\ &= I_{a+}^{\alpha-(3/2)} y(t) - \frac{y(a)(t-a)^{\alpha-(3/2)}}{\Gamma\left[\alpha-(1/2)\right]} - \frac{y'(a)}{\Gamma\left[\alpha+(1/2)\right]}(t-a)^{\alpha-(1/2)}. \end{aligned}$$

The proof is complete. □

LEMMA 3.2. *A function* $y(t) \in C[a,b]$ *is a solution to the boundary value problem* (1.5)–(1.6) *if and only if* $y(t)$ *satisfies the following integral equation*

$$y(t) = \frac{\mu}{\Gamma\left[\alpha-(3/2)\right]} \int_a^b G_1(t,s) y(s) ds + \frac{1}{\Gamma(\alpha)} \int_a^b G_2(t,s) q(s) y(s) ds,$$

where

$$G_1(t,s) = \begin{cases} \dfrac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}(b-s)^{\alpha-\frac{5}{2}} - (t-s)^{\alpha-\frac{5}{2}}, & a \leqslant s \leqslant t \leqslant b, \\ \dfrac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}(b-s)^{\alpha-\frac{5}{2}}, & a \leqslant t \leqslant s \leqslant b, \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leqslant s \leqslant t \leqslant b, \\ \dfrac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}(b-s)^{\alpha-1}, & a \leqslant t \leqslant s \leqslant b. \end{cases}$$

Proof. Applying the operator I_{a+}^{α} to both sides of equation (1.5), we have

$$I_{a+}^{\alpha}\, {}^{C}D_{a+}^{\alpha} y(t) + \mu I_{a+}^{\alpha}\, {}^{C}D_{a+}^{3/2} y(t) + I_{a+}^{\alpha} q(t) y(t) = 0.$$

By using Lemma 2.1 and Lemma 3.1, and incorporating the boundary condition $y(a) = 0$, we can derive

$$y(t) = y'(a)(t-a) + \frac{\mu y'(a)}{\Gamma[\alpha+(1/2)]}(t-a)^{\alpha-(1/2)} - \mu I_{a+}^{\alpha-(3/2)} y(t) - I_{a+}^{\alpha} q(t) y(t). \quad (3.1)$$

Taking into account the boundary condition $y(b)=0$, we obtain

$$y'(a)=\frac{\mu I_{a+}^{\alpha-(3/2)}y(t)\,|_{t=b}+I_{a+}^{\alpha}q(t)y(t)|_{t=b}}{(b-a)+\frac{\mu}{\Gamma[\alpha+(1/2)]}(b-a)^{\alpha-(1/2)}}. \tag{3.2}$$

Substituting (3.2) into (3.1), we get

$$\begin{aligned} y(t)=&\frac{\mu I_{a+}^{\alpha-(3/2)}y(t)|_{t=b}+I_{a+}^{\alpha}q(t)y(t)|_{t=b}}{(b-a)+\frac{\mu}{\Gamma[\alpha+(1/2)]}(b-a)^{\alpha-(1/2)}}\left[(t-a)+\mu I_{a+}^{\alpha-(3/2)}(t-a)\right]\\ &-\mu I_{a+}^{\alpha-(3/2)}y(t)-I_{a+}^{\alpha}q(t)y(t). \end{aligned}$$

It follows that

$$\begin{aligned} y(t)=&\frac{\mu}{\Gamma\left[\alpha-(3/2)\right]}\left\{\int_a^b\frac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}(b-s)^{\alpha-\frac{5}{2}}y(s)ds\right.\\ &\left.-\int_a^t(t-s)^{\alpha-\frac{5}{2}}y(s)ds\right\}+\frac{1}{\Gamma(\alpha)}\left\{-\int_a^t(t-s)^{\alpha-1}y(s)q(s)ds\right.\\ &\left.+\int_a^b\frac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-\frac{1}{2}}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-\frac{1}{2}}}(b-s)^{\alpha-1}y(s)q(s)ds\right\}, \end{aligned}$$

that is,

$$y(t)=\frac{\mu}{\Gamma\left[\alpha-(3/2)\right]}\int_a^b G_1(t,s)y(s)ds+\frac{1}{\Gamma(\alpha)}\int_a^b G_2(t,s)q(s)y(s)ds. \tag{3.3}$$

Conversely, by utilizing Lemma 2.2 and Lemma 2.3, it is easy to verify that (3.3) satisfy the equation (1.5) and the boundary conditions (1.6). Hence, Lemma 3.2 is proved. □

LEMMA 3.3. ([5]) *Let* $1<\delta<2$, *then*

$$(2-\delta)(\delta-1)^{\frac{(\delta-1)}{(2-\delta)}}\leqslant\frac{(\delta-1)^{\delta-1}}{\delta^{\delta}}.$$

The following we present the properties of the integral kernel function as stated in Lemma 3.2.

LEMMA 3.4. *The integral kernel functions* $G_1(t,s)$ *and* $G_2(t,s)$ *in Lemma* 3.2 *satisfy the following properties:*

(i) $|G_2(t,s)|\leqslant N_\alpha$, $(t,s)\in[a,b]\times[a,b]$, *where*

$$N_\alpha=\frac{\Gamma\left[\alpha+(1/2)\right]\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}(b-a)^{\alpha}+\mu\frac{(\alpha-1)^{\alpha-1}[\alpha-(1/2)]^{\alpha-(1/2)}}{[2\alpha-(3/2)]^{2\alpha-(3/2)}}(b-a)^{2\alpha-(3/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}.$$

(ii) $\int_a^b |G_1(t,s)|ds = k(t)$, *where*

$$k(t) = \frac{1}{\alpha-(3/2)}\left\{\frac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}\right.$$
$$\left.\times\left[2(b-t)^{\alpha-(3/2)}-(b-a)^{\alpha-(3/2)}\right]+(t-a)^{\alpha-(3/2)}\right\},\quad t\in[a,b].$$

Proof. First, we prove property (i). To begin with, we define the following functions:

$$\begin{aligned}
&f_1(t,s) = (t-a)^{\alpha-(1/2)}(b-s)^{\alpha-1}-(b-a)^{\alpha-(1/2)}(t-s)^{\alpha-1},\quad a\leqslant s\leqslant t\leqslant b,\\
&f_2(t,s) = (t-a)(b-s)^{\alpha-1}-(b-a)(t-s)^{\alpha-1},\quad a\leqslant s\leqslant t\leqslant b,\\
&g_1(t,s) = \mu f_1(t,s)+\Gamma\left[\alpha+(1/2)\right]f_2(t,s),\quad a\leqslant s\leqslant t\leqslant b,\\
&L_1(t,s) = (t-a)(b-s)^{\alpha-1},\quad a\leqslant t\leqslant s\leqslant b,\\
&L_2(t,s) = (t-a)^{\alpha-(1/2)}(b-s)^{\alpha-1},\quad a\leqslant t\leqslant s\leqslant b,\\
&g_2(t,s) = \Gamma\left[\alpha+(1/2)\right]L_1(t,s)+\mu L_2(t,s),\quad a\leqslant t\leqslant s\leqslant b.
\end{aligned}$$

Then $G_2(t,s)$ can be rewritten as:

$$G_2(t,s)=\begin{cases}\left\{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}\right\}^{-1}g_1(t,s), & a\leqslant s\leqslant t\leqslant b,\\ \left\{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}\right\}^{-1}g_2(t,s), & a\leqslant t\leqslant s\leqslant b.\end{cases}$$

We now spread our proof in two steps.

Step 1. we estimate the upper bound of the function $g_1(t,s)$. To this end, we only need to estimate the upper bounds of the functions $f_1(t,s)$ and $f_2(t,s)$ separately. In fact, from the definition of the function $f_1(t,s)$:

$$f_1(t,s) = (t-a)^{\alpha-(1/2)}(b-s)^{\alpha-1}-(b-a)^{\alpha-(1/2)}(t-s)^{\alpha-1},\quad a\leqslant s\leqslant t\leqslant b.$$

Fixing the variable $t\in[a,b]$, we differentiate $f_1(t,s)$ with respect to the variable s, it follows,

$$\begin{aligned}\frac{\partial f_1(t,s)}{\partial s} =& -(\alpha-1)(t-a)^{\alpha-(1/2)}(b-s)^{\alpha-2}+(\alpha-1)(b-a)^{\alpha-(1/2)}(t-s)^{\alpha-2}\\
=&(\alpha-1)(b-a)^{\alpha-(1/2)}(t-s)^{\alpha-2}\left[1-\left(\frac{t-a}{b-a}\right)^{\alpha-(1/2)}\left(\frac{t-s}{b-s}\right)^{2-\alpha}\right]\geqslant 0,\end{aligned}$$

that is, $f_1(t,s)$ is monotonically increasing with respect to s on $[a,b]$, which implies

$$f_1(t,a)\leqslant f_1(t,s)\leqslant f_1(t,t).$$

Note that

$$\begin{aligned} f_1(t,a) &= (t-a)^{\alpha-(1/2)}(b-a)^{\alpha-1} - (b-a)^{\alpha-(1/2)}(t-a)^{\alpha-1} \\ &= (b-a)^{\alpha-(1/2)}(t-a)^{\alpha-1}\left(\sqrt{\frac{t-a}{b-a}}-1\right) \leqslant 0, \quad t\in[a,b], \end{aligned}$$

and

$$f_1(t,t) = (t-a)^{\alpha-(1/2)}(b-t)^{\alpha-1} \geqslant 0, \quad t\in[a,b].$$

Hence,

$$|f_1(t,s)| \leqslant \max\left\{\max_{t\in[a,b]} f_1(t,t),\ \max_{t\in[a,b]} -f_1(t,a)\right\}.$$

Let

$$K_1(t) = f_1(t,t), \quad t\in[a,b].$$

It is easy to see that $K_1(t)$ is continuous on $[a,b]$ and differentiable within (a,b), we obtain

$$\begin{aligned} K_1'(t) &= [\alpha-(1/2)](t-a)^{\alpha-(3/2)}(b-t)^{\alpha-1} - (\alpha-1)(t-a)^{\alpha-(1/2)}(b-t)^{\alpha-2} \\ &= (t-a)^{\alpha-(3/2)}(b-t)^{\alpha-2}\left\{[\alpha-(1/2)](b-t)-(\alpha-1)(t-a)\right\}. \end{aligned}$$

Let $K_1'(t)=0$, then the function $K_1'(t)$ has a unique zero t_1^* in the interval (a,b),

$$\begin{aligned} t_1^* = \frac{[\alpha-(1/2)]b+(\alpha-1)a}{2\alpha-(3/2)} &= a+\frac{[\alpha-(1/2)](b-a)}{2\alpha-(3/2)} \\ &= b-\frac{(\alpha-1)(b-a)}{2\alpha-(3/2)} \in (a,b). \end{aligned}$$

Note that $K_1(a)=K_1(b)=0$ and $K_1(t)\geqslant 0$ for $t\in[a,b]$, therefore,

$$\max_{t\in[a,b]} K_1(t) = K_1(t_1^*) = \frac{(\alpha-1)^{\alpha-1}\left[\alpha-(1/2)\right]^{\alpha-(1/2)}}{[2\alpha-(3/2)]^{2\alpha-(3/2)}}(b-a)^{2\alpha-(3/2)}. \tag{3.4}$$

Let

$$K_2(t) = -f_1(t,a) = (b-a)^{\alpha-(1/2)}(t-a)^{\alpha-1} - (t-a)^{\alpha-(1/2)}(b-a)^{\alpha-1}, \quad t\in[a,b].$$

It is not difficult to see that $K_2(t)$ is non-negative and continuous on $[a,b]$, and differentiable within (a,b), we obtain

$$\begin{aligned} K_2'(t) &= (\alpha-1)(b-a)^{\alpha-(1/2)}(t-a)^{\alpha-2} - [\alpha-(1/2)](t-a)^{\alpha-(3/2)}(b-a)^{\alpha-1} \\ &= (b-a)^{\alpha-1}(t-a)^{\alpha-2}\left\{(\alpha-1)(b-a)^{1/2} - [\alpha-(1/2)](t-a)^{1/2}\right\}. \end{aligned}$$

Let $K_2'(t)=0$, then $K_2'(t)$ has a unique zero t_2^* in the interval (a,b),

$$t_2^* = a+\left[\frac{\alpha-1}{\alpha-(1/2)}\right]^2(b-a)\in(a,b).$$

Since $K_2(a) = K_2(b) = 0$ and $K_2(t) \geqslant 0$ for $t \in [a,b]$, it follows that

$$\max_{t\in[a,b]} K_2(t) = K_2(t_2^*) = \frac{1}{2\alpha-1}\left[\frac{\alpha-1}{\alpha-(1/2)}\right]^{2(\alpha-1)}(b-a)^{2\alpha-(3/2)}. \tag{3.5}$$

Define

$$H(\alpha) = \frac{\max\limits_{t\in[a,b]} K_1(t)}{\max\limits_{t\in[a,b]} K_2(t)} = \frac{2[\alpha-(1/2)]^{3\alpha-(3/2)}}{(\alpha-1)^{\alpha-1}[2\alpha-(3/2)]^{2\alpha-(3/2)}}, \quad \frac{3}{2} < \alpha \leqslant 2.$$

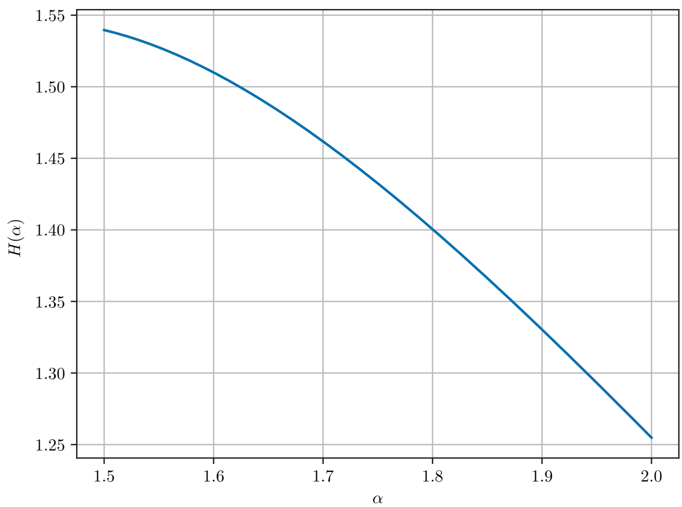


Figure 1: *The figure shows the graph of the function* $H(\alpha)$ *over the interval* $[3/2,2]$.

Then, it is evident from Figure 1 that

$$\max_{t\in[a,b]} K_1(t) \geqslant \max_{t\in[a,b]} K_2(t). \tag{3.6}$$

Combining equations (3.4)–(3.6), we obtain

$$\begin{aligned} |f_1(t,s)| &\leqslant \max\left\{\max_{t\in[a,b]} f_1(t,t), -\max_{t\in[a,b]} f_1(t,a)\right\} \\ &= \max_{t\in[a,b]} f_1(t,t) \\ &= \frac{\left[\alpha-(1/2)\right]^{\alpha-(1/2)}(\alpha-1)^{\alpha-1}}{\left[2\alpha-(3/2)\right]^{2\alpha-(3/2)}}(b-a)^{2\alpha-(3/2)}. \end{aligned}$$

If $a \leqslant s \leqslant t \leqslant b$, it can be proven that

$$|f_2(t,\mathrm{s})| \leqslant \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b-a)^\alpha.$$

The detailed proof can be found in the literature [5, Lemma 2], and will not be elaborated here. Therefore, we can conclude that

$$\begin{aligned} |g_1(t,s)| &= \mu |f_1(t,s)| + \Gamma\left[\alpha + (1/2)\right] |f_2(t,s)| \\ &\leqslant \mu \frac{\left[\alpha - (1/2)\right]^{\alpha-(1/2)} (\alpha-1)^{\alpha-1}}{\left[2\alpha - (3/2)\right]^{2\alpha-(3/2)}} (b-a)^{2\alpha-(3/2)} \\ &\quad + \Gamma\left[\alpha + (1/2)\right] \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} (b-a)^\alpha. \end{aligned} \tag{3.7}$$

Step 2. We estimate the upper bound of $g_2(t,s)$. In fact, from the definition of $L_1(t,s)$, it is easy to obtain that $L_1(t,s)$ satisfies the following inequality

$$0 \leqslant L_1(t,s) \leqslant L_1(s,s) = f_2(s,s) \leqslant \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b-a)^\alpha. \tag{3.8}$$

Based on the definition of $L_2(t,s)$, it is easy to prove that $L_2(t,s)$ satisfies the following inequality:

$$0 \leqslant L_2(t,s) \leqslant L_2(s,s) = K_1(s) \leqslant \frac{(\alpha-1)^{\alpha-1}\left[\alpha-(1/2)\right]^{\alpha-(1/2)}}{\left[2\alpha-(3/2)\right]^{2\alpha-(3/2)}}(b-a)^{2\alpha-(3/2)}. \tag{3.9}$$

Combining equations (3.8) and (3.9), we obtain

$$\begin{aligned} 0 \leqslant g_2(t,s) =& \Gamma\left[\alpha + (1/2)\right] L_1(t,s) + \mu L_2(t,s) \\ \leqslant& \Gamma\left[\alpha + (1/2)\right] \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b-a)^\alpha \\ &+ \mu \frac{(\alpha-1)^{\alpha-1}\left[\alpha-(1/2)\right]^{\alpha-(1/2)}}{\left[2\alpha-(3/2)\right]^{2\alpha-(3/2)}}(b-a)^{2\alpha-(3/2)}. \end{aligned} \tag{3.10}$$

In view of equations (3.7) and (3.10), we can derive that

$$\begin{aligned} |G_2(t,s)| &\leqslant \left[\Gamma\left[\alpha + (1/2)\right](b-a) + \mu(b-a)^{\alpha-(1/2)}\right]^{-1} \\ &\quad \times \max\left\{\max_{t,s\in[a,b]} |g_1(t,s)|, \max_{t,s\in[a,b]} g_2(t,s)\right\} \\ &= \frac{\Gamma\left[\alpha+(1/2)\right]\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b-a)^\alpha + \mu\frac{(\alpha-1)^{\alpha-1}[\alpha-(1/2)]^{\alpha-(1/2)}}{[2\alpha-(3/2)]^{2\alpha-(3/2)}}(b-a)^{2\alpha-(3/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a) + \mu(b-a)^{\alpha-(1/2)}} \\ &= N_\alpha. \end{aligned}$$

Finally, we also need to prove that Property (ii) holds. In fact, since $\frac{3}{2} < \alpha \leqslant 2$, if

$a \leqslant s \leqslant t \leqslant b$, then we can derive

$$\begin{aligned}
&\frac{\Gamma[\alpha+(1/2)](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma[\alpha+(1/2)](b-a)+\mu(b-a)^{\alpha-(1/2)}}(b-s)^{\alpha-(5/2)}-(t-s)^{\alpha-(5/2)}\\
&=(t-s)^{\alpha-(5/2)}\left\{\frac{\Gamma[\alpha+(1/2)](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma[\alpha+(1/2)](b-a)+\mu(b-a)^{\alpha-(1/2)}}\left(\frac{t-s}{b-s}\right)^{(5/2)-\alpha}-1\right\}\leqslant 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_a^b |G_1(t,s)|\,ds =&-\int_a^t \frac{\Gamma[\alpha+(1/2)](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma[\alpha+(1/2)](b-a)+\mu(b-a)^{\alpha-(1/2)}}(b-s)^{\alpha-(5/2)}ds\\
&+\int_t^b \frac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}(b-s)^{\alpha-(5/2)}ds\\
&+\int_a^t (t-s)^{\alpha-(5/2)}ds\\
=&\frac{1}{\alpha-(3/2)}\frac{\Gamma\left[\alpha+(1/2)\right](t-a)+\mu(t-a)^{\alpha-(1/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}\\
&\times\left[2(b-t)^{\alpha-(3/2)}-(b-a)^{\alpha-(3/2)}\right]\\
&+\frac{1}{\alpha-(3/2)}(t-a)^{\alpha-(3/2)}=k(t),
\end{aligned}$$

which completes the proof of Lemma 3.4. □

REMARK 3.1. In existing literature, the relationship between $\max_{t\in[a,b]} f_1(t,t)$ and $\max_{t\in[a,b]} -f_1(t,a)$ is typically established using the conclusion of Lemma 3.3, as seen in [5, 23, 27]. However, the conclusion of Lemma 3.3 is no longer applicable when addressing the relationship between $\max_{t\in[a,b]} f_1(t,t)$ and $\max_{t\in[a,b]} -f_1(t,a)$ in this paper.

In fact, let $\delta=\dfrac{2\alpha-(3/2)}{\alpha-(1/2)}$, then $1<\delta<2$. By using Lemma 3.3, we find that

$$\begin{aligned}
\max_{t\in[a,b]} K_2(t) &= \frac{1}{2\alpha-1}\left[\frac{\alpha-1}{\alpha-(1/2)}\right]^{2\alpha-2}(b-a)^{2\alpha-(3/2)}\\
&=(2-\delta)(\delta-1)^{\frac{\delta-1}{2-\delta}}(b-a)^{2\alpha-(3/2)}\\
&\leqslant \frac{(\delta-1)^{\delta-1}}{\delta^{\delta}}(b-a)^{2\alpha-(3/2)}\\
&=\left\{\frac{\left[\alpha-(1/2)\right]^{\alpha-(1/2)}(\alpha-1)^{\alpha-1}}{\left[2\alpha-(3/2)\right]^{2\alpha-(3/2)}}\right\}^{\frac{1}{\alpha-(1/2)}}(b-a)^{2\alpha-(3/2)}
\end{aligned}$$

$$\geqslant \frac{\left[\alpha-(1/2)\right]^{\alpha-(1/2)}(\alpha-1)^{\alpha-1}}{\left[2\alpha-(3/2)\right]^{2\alpha-(3/2)}}(b-a)^{2\alpha-(3/2)}$$
$$= \max_{t\in[a,b]} K_1(t).$$

This further highlights the novelty of this paper, rather than simply repeating the work of existing literature.

3.2. The Lyapunov-type inequality for the BVP (1.5)–(1.6)

In this subsection we present a Lyapunov-type inequality for the boundary value problem (1.5)–(1.6). To this end, we define the Banach space $C[a,b]$ endowed with the norm $||x||_\infty = \max_{t\in[a,b]} |x(t)|$, $x(t) \in C[a,b]$.

THEOREM 3.1. *Let $q(t)\in C([a,b],\mathbb{R})$. If the boundary value problem* (1.5)–(1.6) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|ds \geqslant \frac{\Gamma(\alpha)}{N_\alpha}\left\{1-\frac{\mu}{\Gamma\left[\alpha-(3/2)\right]}||k||_\infty\right\}. \tag{3.11}$$

Proof. In fact, by Lemma 3.2, the solution $y(t)$ of the boundary value problem (1.5)–(1.6) satisfies the following integral equation

$$y(t) = \frac{\mu}{\Gamma\left[\alpha-(3/2)\right]}\int_a^b G_1(t,s)y(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^b G_2(t,s)q(s)y(s)ds, \quad t\in[a,b].$$

By applying Lemma 3.4, it follows that

$$\begin{aligned}|y(t)| &\leqslant \frac{\mu}{\Gamma[\alpha-(3/2)]}\int_a^b |G_1(t,s)|\,|y(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_a^b |G_2(t,s)|\,|q(s)|\,|y(s)|\,ds \\ &\leqslant \left\{\frac{\mu}{\Gamma[\alpha-(3/2)]}\int_a^b |G_1(t,s)|ds + \frac{N_\alpha}{\Gamma(\alpha)}\int_a^b |q(s)|ds\right\}||y||_\infty \\ &\leqslant \left\{\frac{\mu}{\Gamma[\alpha-(3/2)]}||k||_\infty + \frac{N_\alpha}{\Gamma(\alpha)}\int_a^b |q(s)|ds\right\}||y||_\infty.\end{aligned}$$

Hence,

$$||y||_\infty \leqslant \left\{\frac{\mu}{\Gamma[\alpha-(3/2)]}||k||_\infty + \frac{N_\alpha}{\Gamma(\alpha)}\int_a^b |q(s)|ds\right\}||y||_\infty. \tag{3.12}$$

By solving inequality (3.12), we can obtain (3.11). Therefore, Theorem 3.1 is proved. □

COROLLARY 3.1. *Let $q(t) \in C([a,b],\mathbb{R})$. If the fractional boundary value problem*

$$\begin{cases} {}^{C}D_{a+}^{\alpha}y(t)+q(t)y(t)=0, \quad a<t<b, \\ u(a)=u(b)=0, \end{cases}$$

admits a nontrivial continuous solution, then

$$\int_a^b |q(s)|ds \geqslant \frac{\Gamma(\alpha)}{N_\alpha}. \tag{3.13}$$

Proof. In fact,

$$\lim_{\mu\to 0}\frac{\Gamma(\alpha)}{N_\alpha}\left\{1-\frac{\mu}{\Gamma[\alpha-(3/2)]}||k||_\infty\right\}=\frac{\Gamma(\alpha)}{N_\alpha}.$$

Utilizing Theorem 3.1, it is known that equation (3.13) holds. Clearly, this coincides with the results obtained in [5]. □

REMARK 3.2. In this paper, the condition $\alpha > \frac{3}{2}$ is imposed to ensure that ${}^{C}D_{a+}^{\alpha}y(t)$ in equation (1.5) represents the highest-order derivative term. As $\alpha \to 2$, equation (1.5) degenerates into the model proposed in [22].

REMARK 3.3. If $\alpha < \frac{3}{2}$, the model (1.5) cannot be referred to as the Torvik-Bagley equation. However, we can still consider its Lyapunov-type inequality. By simply interchanging the terms ${}^{C}D_{a+}^{3/2}y(t)$ and ${}^{C}D_{a+}^{\alpha}y(t)$ in model (1.5) and applying the same analytical methods used in this paper, a new Lyapunov-type inequality can be obtained. Interested readers may further discuss this topic, but it will not be elaborated upon in this paper.

4. Example

EXAMPLE 4.1. Consider the following fractional boundary value problem

$$\begin{cases} {}^{C}D_{a+}^{7/4}y(t)+\frac{1}{2}{}^{C}D_{a+}^{3/2}y(t)+t^{1/2}y(t)=0, \quad 1<t<2, \\ y(1)=y(2)=0. \end{cases} \tag{4.1}$$

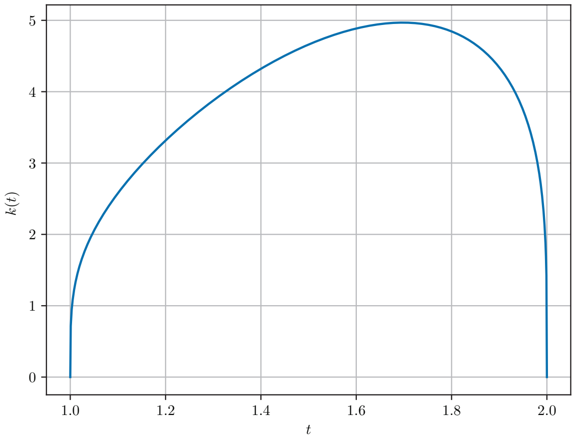
Corresponding to problem (1.5)–(1.6), here

$$\alpha=\frac{7}{4}, \quad \mu=\frac{1}{2}, \quad q(t)=t^{1/2}, \quad a=1, \quad b=2.$$

Through direct calculation, we can obtain

$$k(t)=\frac{8(t-1)\Gamma(9/4)+4(t-1)^{5/4}}{2\Gamma(9/4)+1}[2(2-t)^{1/4}-1]+4(t-1)^{1/4}, \quad 1\leqslant t\leqslant 2.$$

$$N_\alpha = \frac{\Gamma\left[\alpha+(1/2)\right]\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b-a)^\alpha + \mu\frac{(\alpha-1)^{\alpha-1}[\alpha-(1/2)]^{\alpha-(1/2)}}{[2\alpha-(3/2)]^{2\alpha-(3/2)}}(b-a)^{2\alpha-(3/2)}}{\Gamma\left[\alpha+(1/2)\right](b-a)+\mu(b-a)^{\alpha-(1/2)}}$$
$$\approx 0.2915.$$



t	$k(t)$
1.6954	4.96814
1.6964	4.96817
1.6974	4.96818
1.6984	4.96817
1.6994	4.96814

Figtab 1: *The figures respectively show the graph of the function* $k(t)$ *on the interval* $[1,2]$ *and the maximum value of* $k(t)$ *on the interval* $[1,2]$ *as* $||k||_\infty \approx 4.96818$.

According to Figtab 1, $k(t)$ attains its maximum value of $\|k\|_\infty \approx 4.96818$ at $t = 1.6974$. Then we have

$$1.2190 \approx \int_1^2 |t^{1/2}|dt \geqslant \frac{\Gamma(\alpha)}{N_\alpha}\left\{1 - \frac{\mu||k||_\infty}{\Gamma[\alpha-(3/2)]}\right\}$$
$$\approx \frac{0.9191}{0.2915}\left(1-\frac{2.4841}{3.6256}\right) \approx 0.9927.$$

This indicates that the condition in Theorem 3.1 is satisfied.

5. Conclusion

This paper delves into the Lyapunov-type inequalities for fractional Bagley-Torvik differential equations under Dirichlet boundary conditions. By transforming the BVP into an integral equation with a Green's function and employing a priori estimation methods, we successfully established the corresponding Lyapunov-type inequality. The results of this paper not only provide a new theoretical perspective for studying Lyapunov-type inequalities for fractional BVPs but also enrich the existing literature. This has significant theoretical implications for the qualitative analysis of BVPs for fractional Bagley-Torvik differential equations. We expect that future research will further explore this foundation, particularly in studying Lyapunov-type inequalities for fractional

Bagley-Torvik differential equations under nonlocal boundary conditions and their applications involving generalized fractional calculus.

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