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THE LINEAR CANONICAL HANKEL WAVELET TRANSFORM ON GELFAND–SHILOV SPACES

UJJAWALA SINGH AND MANOJ KUMAR MISHRA*

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Abstract. In this article, we discussed some fruitful estimates for linear canonical Hankel transform on some Gelfand-Shilov spaces of type W. Also boundedness result of wavelet transform involving the linear canonical Hankel transform on certain W-type spaces.

1. Introduction

The Gelfand and Shilov [9], gives an introduction about generalized functions space of W-type and discusses various applications to analysis, PDE, stochastic processes, and representation theory. Chung [4] provide symmetric descriptions of the Gelfand-Shilov spaces of types S and W with regard to the Fourier transformation. These findings provide a clear explanation of why these spaces are invariant to Fourier transformations. The Gelfand and Shilov [9], Friedman [8] and Gurevich [3] investigated the W-type spaces. They examined the behaviour of Fourier transform in W-type spaces. Pathak and Upadhyay [14] discussed the spaces generalizing the spaces of type W in L^p norm. Pathak and Pandey [13] examined certain Gelfand-Shilov spaces of type W using the continuous wavelet transform. They properly constructed spaces of type W defined on $\mathbb{R}\times\mathbb{R}_+$, $\mathbb{C}\times\mathbb{R}_+$ and $\mathbb{C}\times\mathbb{C}$, the continuity and boundedness results for continuous wavelet transform was obtained. Pilipovic et al. [15] studied the local and global properties of wavelet transforms on Gelfand-Shilov type spaces. Upadhyaya et al. [18] and Prasad and Mahato [16], discussed the characterization of W-type spaces by using wavelet transform associated with the fractional Fourier transform. For the more details of W-type spaces Cordero et al. [6] investigated localization operators in the context of ultra-distributions.

The main objective of this paper is to investigate the nature of linear canonical Hankel wavelet transform on Gelfand-Shilov type spaces of W-type. This work is motivated by the work of Mahato and Singh [11], Pathak [16] and Van [19]. In their work they presented the results for characterizing the inverse of the fractional Hankel transform on some Gelfand-Shilov spaces of type W. Furthermore they constructed some

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* Corresponding author.

spaces of type W, on which they studied the nature of wavelet transform associated with fractional Hankel transform.

As per [2, 7], the continuous wavelet transform (CWT) $W_\psi(b,a)$ is a function of two parameters and, therefore, contains a high amount of extra (redundant) information when analyzing a function is defined as:

$$\left(W_\psi \phi\right)(b,a) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \phi(t)\,\bar{\psi}\left(\frac{t-b}{a}\right) dt \qquad (1)$$
$$= \int_{\mathbb{R}} \phi\, \overline{\psi_{b,a}(t)} dt$$

where $\phi, \psi_{b,a}(t) \in L^2(\mathbb{R})$.

The space $L^p_{\mu,\nu,\alpha}$, $1 \leqslant p \leqslant \infty$ as the space of all those real valued measurable function ϕ on I such that

$$\|\phi\|_{L^p_{\mu,\nu,\alpha}} = \left| \int_0^\infty |\phi(x)|^p x^{\nu\mu-\alpha+2\nu+1} dx \right|^{\frac{1}{p}} < \infty.$$

The concept of linear canonical transformation (LCT) defined with four parameters a,b,c,d was developed by the two projects, Collins [5] on the field of paraxial optics and on the other hand, Moshinsky and Quesne [12] in the field of nuclear physics in mid seventies. Wolf [20] presented the canonical Hankel transformation of function f for n-dimension and $\nu \geqslant 1-n$. Bultheel et al. [1] introduced $\mathscr{K}(y,x)$ to the kernel of fractional Hankel transform by replacing $a = d = \cos\theta$ and $b = -c = \sin\theta$ as:

$$\left[\mathscr{H}^\theta f\right](\xi) = \frac{e^{i(1+\nu)\left(\frac{\pi}{2}-\theta\right)}}{\sin\theta} \int_0^\infty f(x) e^{-i\frac{\xi^2+x^2}{2}\cot\theta} J_\nu\left(\frac{x\xi}{\sin\theta}\right) x dx.$$

Utilizing the hypothesis of Bultheel [1], Prasad and Kumar [17], characterized linear canonical Hankel transformation of the integrable function f over positive real line. Like theory of LCT this transform can be states as depending on three more real parameters ν, α, β with uni-modular matrix A of order 2×2 along with condition $\nu\mu + 2\nu - \alpha \geqslant 1$ as

$$\left(\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f\right)(y) = \int_0^\infty K^A(y,x) f(x) dx, \qquad (2)$$

where, the kernel of the transformations are given as:

$$K^A(y,x) = \nu\beta \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} x^{-1-2\alpha+2\nu} e^{\frac{i\beta}{2b}\left(ax^{2\nu}+dy^{2\nu}\right)} (xy)^\alpha J_\mu\left(\frac{\beta}{b}(xy)^\nu\right), \quad b \neq 0. \qquad (3)$$

The inversion formula of (1.1) is given by:

$$f(x) = \left(\mathscr{H}^{A^{-1}}_{\mu,\nu,\alpha,\beta}\left(\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f\right)(y)\right)(x) = \int_0^\infty K^{A^{-1}}(x,y)\left(\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f\right)(y) dy,$$

where A^{-1} denotes inverse of the matrix A.

As per [10], the linear canonical Hankel wavelet $\psi_{m,n,A}$ of any function $\psi \in L^2_{\mu,\nu,\alpha}(I)$ by using the LCH-translation and dilation D_m defined as

$$\begin{aligned}\psi_{m,n,A} &= D_m\left(\tau_n^A\psi\right)(t) = D_m\psi^A(n,t)\\ &= m^{-2\nu+2\alpha}e^{\frac{\iota\beta}{2b}a\left(\frac{1}{m^{2\nu}}-1\right)t^{2\nu}}e^{\frac{\iota\beta}{2b}a\left(\frac{1}{m^{2\nu}}+1\right)n^{2\nu}}\psi^A\left(\frac{n}{m},\frac{t}{m}\right),\\ &\text{for}\quad m \geqslant 0,\ \ n > 0.\end{aligned} \tag{4}$$

LEMMA 1. *Let ψ be any arbitrary function belong to $L^2_{\mu,\nu,\alpha}$. Then the linear canonical Hankel transform of $\psi_{m,n,A}$ is given by*

$$\begin{aligned}\left(\mathscr{H}^A_{\mu,\nu,\alpha,\nu}\psi_{n,m,A}\right)(\omega) &= e^{-\frac{\iota\beta}{2b}\left[\left(m^{2\nu}-1\right)d\omega^{2\nu}-an^{2\nu}\right]}(m\omega)^{-\nu\mu-\alpha}(\omega n)^{\alpha}J_\mu\left(\frac{\beta}{b}(\omega n)^{\nu}\right)\\ &\quad\times\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\nu\mu+\alpha}\psi(z)e^{-\frac{\iota\beta}{2b}az^{2\nu}})(m\omega)}.\end{aligned}$$

Now by using Parseval's relation and Lemma 1, the above defined continuous wavelet transform $\left(W^A_\psi f\right)(n,m)$ becomes

$$\begin{aligned}\left(W^A_\psi f\right)(n,m) &= \frac{b}{\nu\beta}e^{-\iota\frac{\pi}{2}(1+\mu)}\int_0^\infty K^{A^{-1}}(n,\omega)(m\omega)^{-\nu\mu-\alpha}e^{\frac{\iota\beta}{2b}d(m\omega)^{2\nu}}\\ &\quad\times\left(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f\right)(\omega)\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\omega)}d\omega.\end{aligned} \tag{5}$$

2. The spaces $W_{M,\sigma}, W^{\Omega,\eta}$ and $W^{\Omega,\eta}_{M,\sigma}$

In this section, we discuss the definition and characterizations of W-type Gelfand-Shilov spaces that will be employed in our study of the linear canonical wavelet transform. For defining the spaces $W_{M,\sigma}$, $W^{\Omega,\eta}$ and $W^{\Omega,\eta}_{M,\sigma}$ we need two functions $m(x)$, $(0 \leqslant x < \infty)$ and $\omega(y)$, $(0 \leqslant y < \infty)$, on I be continuous increasing function such that $m(0) = 0 = \omega(0)$ and $m(x) \to \infty$ as $x \to \infty$ and $\omega(y) \to \infty$ as $y \to \infty$, the function $M(\zeta)$ and $\Omega(\eta)$ for each $\zeta, \eta \geqslant 0$ are defined as [9],

$$M(\zeta) = \int_0^\zeta m(x)dx, \tag{6}$$

and

$$\Omega(\eta) = \int_0^\eta \omega(y)dy. \tag{7}$$

The function $M(\zeta)$ and $\Omega(\eta)$ are continuous and increasing, and satisfy with the value $M(0) = 0$, $M(\zeta) \to \infty$ for $\zeta \to \infty$ and $\Omega(0) = 0$, $\Omega(\eta) \to \infty$ for $\eta \to \infty$ by using these value we can developed the condition for convex inequality which are the following

$$M(\zeta_1+\zeta_2) \geqslant M(\zeta_1)+M(\zeta_2), \quad \Omega(\eta_1+\eta_2) \geqslant \Omega(\eta_1)+\Omega(\eta_2). \tag{8}$$

If the function $m(x)$ and $\omega(y)$ are mutually inverse, that is, $m(\omega(x)) = x$ and $\omega(m(y)) = y$. Consequently, the functions M and Ω described above are referred to as dual in the Young sense. The Young inequalities in this instance is given by

$$\zeta\eta \leqslant M(\zeta) + \Omega(\eta), \quad \text{for each } \zeta \geqslant 0, \ \eta \geqslant 0. \tag{9}$$

Now, as per [11, 16, 19] we define the W-type spaces as:

DEFINITION 1. Let $q,k \in \mathbb{N}_0$. A smooth function $\phi(x)$ belongs to $W_{M,\sigma,A}$ $(\sigma > 0)$ if for every $\delta > 0$ there exist $C_{q,\delta} > 0$ depending on $\phi(x)$ such that

$$|x^k (x^{1-2\nu} D_x)^q (e^{\pm \frac{\iota\beta}{2b} a x^{2\nu}} x^{-\nu\mu-\alpha} \phi(x))| \leqslant C_{q,\delta} \exp[-M(\sigma - \delta)x].$$

DEFINITION 2. The spaces $W^{\Omega,\eta,A}$, $(\eta > 0)$ contains all smooth function $\psi(z)$, $(z = x + \iota y \in \mathbb{C})$ that for any $\rho > 0$ satisfy the following inequality

$$|z^k e^{\pm \frac{\iota\beta}{2b} a z^{2\nu}} \psi(z)| \leqslant C_{k,\rho} \exp[\Omega(\eta + \rho)y], \ k = 0, 1, 2, \ldots$$

where $C_{k,\rho} > 0$ depends on $\psi(z)$.

DEFINITION 3. Let $M(x)$ be dual to $\Omega(y)$ in the Young sense. We define the space $W^{\Omega,\eta}_{M,\sigma,A}$, (σ, η) as the collection of all entire analytic functions $\phi(z)$, $(z = x + \iota y \in \mathbb{C})$ that for any $\rho, \delta > 0$ satisfy the inequality

$$|z^k e^{\pm \frac{\iota\beta}{2b} a z^{2\nu}} \phi(z)| \leqslant C_{\delta,\rho} \exp[-M(\sigma - \delta)x + \Omega(\eta + \rho)y], \ k = 0, 1, 2, \ldots,$$

where $C_{\delta,\rho}$ is a positive constant depends on $\phi(z)$.

The following recurrence relation [17] we will use in further investigations:

$$\left(x^{1-2\nu} D_x\right)^m \left[x^{-\nu\mu} J_\mu\left(\beta x^\nu\right)\right] = (-\nu\beta)^m x^{-\nu(\mu+m)} J_{\mu+m}(\beta x)^\nu. \tag{10}$$

3. Linear canonical Hankel transform on W type spaces

In this section, we have studied about the nature of linear canonical Hankel transform on $W_{M,\sigma,A}$, $W^{\Omega,\eta,A}$, $W^{\Omega,\eta}_{M,\sigma,A}$ type spaces and will be employed in our study of wavelet transform.

THEOREM 2. *Let $M(x)$ be the function which is dual to the function $\Omega(y)$ in the Young sense. Then the linear canonical Hankel transform $\mathscr{H}^A_{\mu,\nu,\alpha,\beta}$ is defined as above is continuous linear mapping from $W^{\Omega,\eta,A}$ into $W_{M,\frac{1}{\eta},A}$.*

Proof. Let $q,k \in \mathbb{N}$, $z = x + \iota y$ and A is the uni-modular matrix defined as earlier and $\phi \in W^{\Omega,\eta,A}$. Then from Definition 2

$$|z^k e^{\pm \frac{\iota\beta}{2b} a z^{2\nu}} \phi(z)| \leqslant C_{k,\rho} \exp[\Omega(\eta + \rho)y], \ k = 0, 1, 2, \ldots$$

Now using, definition of LCHT

$$
\begin{aligned}
&\left|\omega^k(\omega^{1-2\nu}D_\omega)^q(e^{-\frac{\iota\beta}{2b}a\omega^{2\nu}}\omega^{-\nu\mu-\alpha}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(\omega))\right|\\
&=\left|\omega^k(\omega^{1-2\nu}D_\omega)^q\left\{e^{-\frac{\iota\beta}{2b}a\omega^{2\nu}}\omega^{-\nu\mu-\alpha}\frac{\nu\beta}{b}e^{-\iota\frac{\pi}{2}(1+\mu)}\int_0^\infty z^{-1-2\alpha+2\nu}\right.\right.\\
&\quad\left.\left.\times e^{\frac{\iota\beta}{2b}(a\omega^{2\nu}+dz^{2\nu})}(z\omega)^\alpha J_\mu\left(\frac{\beta}{b}(z\omega)^\nu\right)dx\right\}\right|\\
&=\left|\omega^k(\omega^{1-2\nu}D_\omega)^q\omega^{-\nu\mu}\frac{\nu\beta}{b}e^{-\frac{\pi}{2}(1+\mu)}\int_0^\infty z^{-1-\alpha+2\nu}e^{\frac{\iota\beta}{2b}dz^{2\nu}}J_\mu\left(\frac{\beta}{b}(z\omega)^\nu\right)dx\right|\\
&=\left|\frac{\nu\beta}{b}\right|\left|\omega^k(\omega^{1-2\nu}D_\omega)^q\omega^{-\nu\mu}\int_0^\infty z^{-1-\alpha-2\nu}e^{\frac{\iota\beta}{2b}dz^{2\nu}}J_\mu\left(\frac{\beta}{b}(z\omega)^\nu\right)dx\right|.
\end{aligned}
$$

Using recurrence relation (10) in the above equation

$$
\begin{aligned}
&=\left|\frac{\nu\beta}{b}\right|\left|\omega^k\int_0^\infty\left(-\frac{\nu\beta}{b}z^\nu\right)^q(z\omega)^{-\nu(\mu+q)}J_{\mu+q}\left(\frac{\beta}{b}(z\omega)^\nu\right)z^{-1-\alpha+2\nu+\nu\mu+\nu q}\right.\\
&\quad\left.\times e^{\frac{\iota\beta}{2b}dz^{2\nu}}dx\right|\\
&=\left|\frac{\nu\beta}{b}\right|^{1+q}\left|\omega^k\int_0^\infty(z\omega)^{-\nu(\mu+q)}J_{\mu+q}\left(\frac{\beta}{b}(z\omega)^\nu\right)z^{-1-\alpha+2\nu+\nu\mu+2\nu q}e^{\frac{\iota\beta}{2b}dz^{2\nu}}dx\right|\\
&\leqslant\left|\frac{\nu\beta}{b}\right|^{1+q+k}\left|\int_0^\infty(z\omega)^{-\nu(\mu+q)+k}J_{\mu+q}\left(\frac{\beta}{b}(z\omega)^\nu\right)z^{-1-\alpha+2\nu+\nu\mu+2\nu q-k}e^{\frac{\iota\beta}{2b}dz^{2\nu}}dx\right|.
\end{aligned}
$$

Since $\mu\nu+2\nu-\alpha\geqslant 1$, where $\alpha,\nu\in\mathbb{R}$ and $\left|(\omega z)^{-\nu(\mu+q)+k}J_\mu\left(\frac{\beta}{b}(\omega z)^\nu\right)\right|$ is bounded on $0\leqslant|\omega z|<\infty$ by $B^A_{\mu,\nu,\alpha,\beta}\exp(-Im(\omega z))$ (say).

In viewing Definition 2 and using the inequality $|z|^l\leqslant\frac{(|z|^{l+2}+|z|^l)}{1+x^2}$ the above expression becomes

$$
\begin{aligned}
&\left|\omega^k(\omega^{1-2\nu}D_\omega)^q e^{-\frac{\iota\beta}{2b}a\omega^{2\nu}}\omega^{-\nu\mu-\alpha}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(\omega)\right|\\
&\leqslant D\int_0^\infty B^A_{\mu,\nu,\alpha,\beta}(C_{-1-\alpha+2\nu+\nu\mu+2\nu q-k,\rho}+C_{-1-\alpha+2\nu+\nu\mu+2\nu q-k+2,\rho})\\
&\quad\times\exp(-Im(\omega z))\exp(\Omega(\eta+\rho)(y))\frac{dx}{1+x^2}\\
&\leqslant DB^A_{\mu,\nu,\alpha,\beta}(C_{-1-\alpha+2\nu+\nu\mu+2\nu q-k,\rho}+C_{-1-\alpha+2\nu+\nu\mu+2\nu q-k+2,\rho})\\
&\quad\times\exp(-\omega y+\Omega(\eta+\rho)(y))\int_0^\infty\frac{dx}{1+x^2}.
\end{aligned}
$$

Now, consider the Young inequality for ωy and replace ω, y by $\frac{\omega}{(\eta+\rho)}$ and $(\eta+\rho)y$, respectively

$$\begin{aligned} \omega y &= M\left(\frac{\omega}{\eta+\rho}\right) + \Omega((\eta+\rho)y) \\ \exp(-\omega y + \Omega(\eta+\rho)(y)) &= \exp[-|\omega||y| + \Omega(\eta+\rho)(y))] \\ &= \exp\left[-M\left(\frac{\omega}{(\eta+\rho)}\right)\right]. \end{aligned}$$

Assume $\frac{1}{\eta+\rho} = \frac{1}{\eta} - \delta$, where δ is arbitrary small number. Then the above inequality becomes

$$\begin{aligned} &\left|\omega^k(\omega^{1-2\nu}D_\omega)^q(e^{-\frac{\iota\beta}{2b}ax^{2\nu}}\omega^{-\nu\mu-\alpha}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(\omega))\right| \\ &\leqslant C'_{-1-\alpha+2\nu+\nu\mu+2\nu q-k,\rho}\exp\left[-M\left(\frac{1}{\eta}-\delta\right)\omega\right]. \end{aligned}$$

This completes the proof. □

THEOREM 3. *Let $M(x)$ and $\Omega(y)$ be same as in the above theorem, then the linear canonical Hankel transform $\mathscr{H}^A_{\mu,\nu,\alpha,\beta}$ is continuous linear mapping $W_{M,\sigma,A}$ into $W^{\Omega,1/\sigma,A}$.*

Proof. Let $\phi \in W_{M,\sigma,A}$, then the definition 1 gives

$$\left|\omega^k(\omega^{1-2\nu}D_\omega)^q e^{-\frac{\iota\beta}{2b}a\omega^{2\nu}}\omega^{-\nu\mu-\alpha}\phi(\omega)\right| \leqslant C_{q,\delta}[-M(\sigma-\delta)\omega], \quad k,q = 0,1,2,3,\ldots$$

Now, we see that

$$\begin{aligned} &|z^{-\nu\mu-\alpha}e^{-\frac{\iota\beta}{2b}az^{2\nu}}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(z)| \\ &= \left|z^{-\nu\mu-\alpha}e^{-(\frac{\iota\beta}{2b})az^{2\nu}}\frac{\nu\beta}{b}e^{-\iota\frac{\pi}{2}(1+\mu)}\int_0^\infty e^{\frac{\iota\beta}{2b}(az^{2\nu}+d\omega^{2\nu})}(z\omega)^\alpha J_\mu\left(\frac{\beta}{b}(z\omega)^\nu\right)\right. \\ &\quad \left.\times\omega^{-1-2\alpha+2\nu}\phi(\omega)d\omega\right| \\ &\leqslant \left|\frac{\nu\beta}{b}\right|\left|z^{-\nu\mu}\int_0^\infty e^{\frac{\iota\beta}{2b}d\omega^{2\nu}}J_\mu\left(\frac{\beta}{b}(z\omega)^{2\nu}\right)\omega^{-1-\alpha+2\nu}\phi(\omega)d\omega\right| \\ &\leqslant \left|\frac{\nu\beta}{b}\right|\left|\int_0^\infty\left\{(z\omega)^{-\nu\mu}J_\mu\left(\frac{\beta}{b}(z\omega)^\nu\right)\right\}\omega^{-1-\alpha+2\nu+\nu\mu}e^{\frac{\iota\beta}{2b}d\omega^{2\nu}}\phi(\omega)d\omega\right| \end{aligned}$$

$$\leqslant \left|\frac{\nu\beta}{b}\right| |z^{-k}| \left| \int_0^\infty \left\{ (z\omega)^{-\nu\mu+k} J_\mu\left(\frac{\beta}{b}(z\omega)^\nu\right) \right\} \omega^{-1+2\nu-k} \right.$$
$$\left. \times \left(\omega^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b} d\omega^{2\nu}} \phi(\omega) \right) d\omega \right|$$
$$\leqslant \left|\frac{\nu\beta}{b}\right| |z^{-k}| \left| \int_0^\infty \omega^k (\omega^{1-2\nu} D_\omega)^{-k} (z\omega)^{-\nu\mu+k} J_\mu\left(\frac{\beta}{b}(z\omega)^\nu\right) \right.$$
$$\left. \times \omega^{-1+2\nu} \left\{ (\omega^{1-2\nu} D_\omega)^k \omega^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b} d\omega^{2\nu}} \phi(\omega) \right\} d\omega \right|.$$

Using recurrence relation equation (10)

$$\leqslant \left|\frac{\nu\beta}{b}\right| |z|^{-k} \left| \int_0^\infty \left(-\frac{\nu\beta}{b} \omega^{-\nu} \right)^{-k} (z\omega)^{-\nu(\mu-k+k)} J_{\mu+k}\left(\frac{\beta}{b}(z\omega)^\nu\right) \omega^{-1+2\nu} \right.$$
$$\left. \times \left\{ (\omega^{1-2\nu} D_\omega)^k \omega^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b} d\omega^{2\nu}} \phi(\omega) \right\} d\omega \right|$$
$$\leqslant \left|\frac{\nu\beta}{b}\right|^{1+k} \int_0^\infty \left| (z\omega)^{-\nu\mu} J_{\mu+k}\left(\frac{\beta}{b}(z\omega)^\nu\right) \right|$$
$$\times \left| \omega^{-1+2\nu+k} \left\{ (\omega^{1-2\nu} D_\omega)^k \omega^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b} d\omega^{2\nu}} \phi(\omega) \right\} d\omega \right|.$$

Since $\nu\mu + 2\nu - \alpha \geqslant 1$, where $\mu, \alpha \in \mathbb{R}$, $\left| (z\omega)^{-\nu\mu} J_{\mu+k}\left(\frac{\beta}{b}(z\omega)^\nu\right) \right|$ is bounded on $0 \leqslant |(z\omega)| < \infty$ by $C^A_{\mu,\nu,\alpha,\beta} \exp(-Im(z))$ (say). Then the above expression estimate as

$$\left| z^{-\nu\mu-\alpha+k} e^{-\frac{\iota\beta}{2b} az^{2\nu}} (\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(z) \right|$$
$$\leqslant D \int_0^\infty C_{k,\delta} \exp[-M(\sigma-\delta)\omega] C^A_{\mu,\nu,\alpha,\beta} \exp(-\omega y) \omega^{1+2\nu} d\omega$$
$$\leqslant D C^A_{\mu,\nu,\alpha,\beta} C_{k,\delta} \int_0^\infty \exp[-M(\sigma-\delta)\omega] \exp(\omega y) \omega^{1+2\nu} d\omega$$
$$\leqslant D C^A_{\mu,\nu,\alpha,\beta} C_{k,\delta} \int_0^\infty \exp[-M(\sigma-\delta)\omega] \exp(\omega y) \omega^{1+2\nu} d\omega$$
$$\leqslant D C^A_{\mu,\nu,\alpha,\beta} C_{k,\delta} \int_0^\infty \exp[\omega y - M(\sigma-2\delta)\omega] \exp[\delta\omega] \omega^{1+2\nu} d\omega.$$

We can set a real positive number δ, such that $\frac{1}{(\sigma-2\delta)} = \frac{1}{\sigma} + \rho$, where ρ is arbitrarily small together with δ. Finally we have

$$\left| z^{-\nu\mu-\alpha+k} e^{-\frac{\iota\beta}{2b} az^{2\nu}} (\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(z) \right| \leqslant D_{k,\sigma} \exp\left[\Omega\left(\frac{1}{\sigma} + \rho \right) y \right],$$

where $D_{k,\sigma} = DC^{A'}_{\mu,\nu,\alpha,\beta} C_{k,\delta} \int_0^\infty \exp[\delta\omega]\omega^{1+2\nu} d\omega$. $\square$

THEOREM 4. *Let $M(x)$ and $M_1(x)$ are dual to $\Omega_1(y)$ and $\Omega(y)$, respectively, in the Young sense. Then the linear canonical Hankel transform $\mathscr{H}^A_{\mu,\nu,\alpha,\beta}$ is a continuous linear mapping from $W^{\Omega,\eta}_{M,\sigma,A}$ into $W^{\Omega_1,1/\sigma}_{M_1,1/\eta,A}$.*

Proof. Assume that $z = u + \imath v$, $\omega = x + \imath y$ and $\phi \in W^{\Omega,\eta}_{M,\sigma,A}$. Then we obtain

$$\begin{aligned}
|(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(z)| &= \left| \frac{\nu\beta}{b} e^{-\imath\frac{\pi}{2}(1+\mu)} \int_0^\infty e^{\frac{\imath\beta}{2b}(a\omega^{2\nu}+dz^{2\nu})} (\omega z)^\alpha J_\mu\left(\frac{\beta}{b}(\omega z)^\alpha\right) \right. \\
&\quad \left. \times \omega^{-1-2\alpha+2\nu}\phi(\omega)dx \right| \\
&\leqslant \left|\frac{\nu\beta}{b}\right| \int_0^\infty |e^{\frac{\imath\beta}{2b}(a\omega^{2\nu}+dz^{2\nu})}(\omega z)^\alpha J_\mu\left(\frac{\beta}{b}(\omega z)^\alpha\right)\omega^{-1-2\alpha+2\nu}\phi(\omega)dx| \\
&\leqslant \left|\frac{\nu\beta}{b}\right| \int_0^\infty |(\omega z)^{-\nu\mu} J_\mu\left(\frac{\beta}{b}(\omega z)^\alpha\right)||e^{\frac{\imath\beta}{2b}dz^{2\nu}} z^{\nu\mu+\alpha}| \\
&\quad \times |\omega^{-1-\alpha+2\nu+\nu\mu} e^{\frac{\imath\beta}{2b}d\omega^{2\nu}} \phi(\omega)|dx.
\end{aligned}$$

Since $\nu\mu + 2\alpha - \alpha \geqslant 1$, $\mu, \alpha \in \mathbb{R}$ and $\left|(\omega z)^{\nu\mu} J_\mu\left(\frac{\beta}{b}(\omega z)^\alpha\right)\right|$ is bounded on $0 \leqslant |(\omega z)| < \infty$ by $C^A_{\mu,\nu,\alpha,\beta}\exp(-Im(\omega z))$ (say).

$$\begin{aligned}
|(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(z)| &\leqslant \left|\frac{\nu\beta}{b}\right| \int_0^\infty C^A_{\mu,\nu,\alpha,\beta}\exp(-xv-uy)|e^{\frac{\imath\beta}{2b}dz^{2\nu}} z^{\nu\mu+\alpha}| \\
&\quad \times |\omega^{-\nu\mu-\alpha} e^{\frac{\imath\beta}{2b}a\omega^{2\nu}}\phi(\omega)||\omega^{-1+2\nu+2\nu\mu}|dx \\
&\leqslant \left|\frac{\nu\beta}{b}\right| C^A_{\mu,\nu,\alpha,\beta} \int_0^\infty \exp(-xv-uy)|e^{\frac{\imath\beta}{2b}az^{2\nu}} z^{\nu\mu+\alpha}| \\
&\quad \times C_{\delta,\rho}\exp[-M(\sigma-\delta)x + \Omega(\eta+\rho)y]|\omega^{-1+2\nu+2\nu\mu}|dx \\
&\leqslant D'|e^{\frac{\imath\beta}{2b}az^{2\nu}} z^{\nu\mu+\alpha}| \int_0^\infty \exp(-xv-uy) \\
&\quad \times \exp[-M(\sigma-\delta)x + \Omega(\eta+\rho)y]|\omega^{-1+2\nu+\nu\mu}|dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|e^{-\frac{\imath\beta}{2b}az^{2\nu}} z^{-\nu\mu-\alpha}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\phi)(z)| \\
&\leqslant D' \int_0^\infty \exp[xv - M(\sigma-\delta)x]\exp[-uy+\Omega(\eta+\rho)y]|\omega^{-1+2\nu+\nu\mu}|dx
\end{aligned}$$

$$= D' \int_0^\infty \exp\left[\Omega_1\left(\frac{y}{\sigma - 2\delta}\right)\right] \exp\left[-M_1 \frac{u}{\eta + \rho}\right] |\exp[-M(\delta x)] \omega^{-1+2\nu+\nu\mu}| dx$$
$$\leqslant C_{\delta',\rho'} \exp\left[-M_1\left(\frac{1}{\eta} - \delta'\right) u + \Omega_1\left(\frac{1}{\sigma} + \rho'\right) v\right],$$

where $C_{\delta',\rho'} = D' \int_0^\infty |\exp[-M(\delta x)] \omega^{-1+2\nu+\nu\mu}| dx$. □

4. Wavelat transform on W-type spaces

In this section, we have studied about the continuity and boundedness properties of LCH wavelet transform on suitably constructed Gelfand-Shilov space of type W. In order to continue our study about LCH wavelet transform on the above mentioned space, we shall need to introduce the following function spaces.

DEFINITION 4. The space $\tilde{W}_{M,\sigma,A}$, $\sigma > 0$ is defined to be the collection of all complex valued infinitely differentiable functions $\phi(n,m) \in C^\infty(\mathbb{C} \times \mathbb{R}^+)$, for $\delta > 0$, and ν as earlier satisfy the following inequality,

$$\left|\left(n^{1-2\nu} \frac{\partial}{\partial n}\right)^k \left(\frac{\partial}{\partial m}\right)^l \left\{n^{-\nu\mu-\alpha} e^{-\frac{\iota\beta}{2b} a n^{2\nu}} \phi(n,m)\right\}\right|$$
$$\leqslant C_{k,l,\delta} \exp\left[-M\left\{\left(\frac{n}{1+m}\right)(\sigma - \delta)\right\}\right], \text{ where } \; k,l = 1,2,3\ldots$$

and $C_{k,l,\delta}$ are positive constant depends on the function ϕ.

DEFINITION 5. The spaces $\tilde{W}^{\Omega,\sigma,m\sigma,A}$, $\sigma > 0$ and ν as earlier contains of the function $\phi(s,m) \in C^\infty(\mathbb{C} \times \mathbb{R}^+)$ entirely analytic with respect to $s = b + \iota\lambda$ which for any $\rho, \rho' > 0$ satisfy inequality

$$\left|\frac{1}{(1+|m|^{-t})} \left(m^{1-2\nu} \frac{\partial}{\partial m}\right)^t \phi(s,m)\right|$$
$$\leqslant C_{t,\rho} \exp\Big[\Omega(\sigma + \rho)\lambda + \Omega(\sigma\Omega + \rho')\lambda\Big], \text{ with } \; t = 0,1,2\ldots$$

where all positive constant $C_{t,\rho}$ depend on ϕ.

THEOREM 5. *Let $\Omega(y)$ is dual to $M(x)$ in the Young sense. Suppose that*

$$\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\left((.)^{-\nu\mu-\alpha} e^{-\frac{\iota\beta}{2b} d(.)^{2\nu}} \psi(.)\right)(m\omega) \in W_{M,\sigma,A} \quad \textit{and} \quad \mathscr{H}^A_{\mu,\nu,\alpha,\beta}(f) \in W_{M,\sigma,A},$$

then the linear canonical Hankel wavelet transform is a continuous linear mapping from $W_{M,\sigma,A}$ into $\tilde{W}^{\Omega,1/\sigma,1/m\sigma,A}$.

Proof. Since $\mathscr{H}^A_{\mu,\nu,\alpha,\beta}\left((.)^{-\nu\mu-\alpha}e^{-\frac{\iota\beta}{2b}d(.)^{2\nu}}\psi(.)\right)(n\omega)\in W_{M,\sigma,A}$, $\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(f)\in W_{M,\sigma,A}$, therefore LCHT can extended to the complex value of $s=n+\iota\lambda$ according to the definition 5, thus we obtain

$$\begin{aligned}
&\left|\left(m^{1-2\nu}\frac{\partial}{\partial m}\right)^t (W^A_\psi f)(s,m))\right|\\
&=\left|\frac{b}{\nu\beta}e^{-\frac{\iota\pi}{2}(1+\mu)}\left(m^{1-2\nu}\frac{\partial}{\partial m}\right)^t\int_0^\infty K^{A^{-1}}(\omega,s)(m\omega)^{-\nu\mu-\alpha}e^{\frac{\iota\beta}{2b}d(m\omega)^{2\nu}}\right.\\
&\quad\left.\times\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(f)\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\omega)d\omega\right|\\
&=\left|\frac{b}{\nu\beta}e^{-\frac{\iota\pi}{2}(1+\mu)}\int_0^\infty\left(m^{1-2\nu}\frac{\partial}{\partial m}\right)^t\frac{\nu\beta}{b}e^{\iota\frac{\pi}{2}(1+\mu)}e^{-\frac{\iota\beta}{2b}(a\omega^{2\nu}+ds^{2\nu})}(\omega s)^\alpha\right.\\
&\quad\times J_\mu\left(\frac{\beta}{b}(\omega s)^\nu\right)\omega^{-1-2\alpha+2\nu}(m\omega)^{-\nu\mu-\alpha}e^{\frac{\iota\beta}{2b}d(m\omega)^{2\nu}}\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(f)\\
&\quad\left.\times\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\omega)}d\omega\right|\\
&=\left|\int_0^\infty\left[e^{-\frac{\iota\beta}{2b}a\omega^{2\nu}}\omega^{-1-\alpha+2\nu}\mathscr{H}^A_{\mu,\nu,\alpha\beta}(f)e^{-\frac{\iota\beta}{2b}as^{2\nu}}J_\mu\left(\frac{\beta}{b}(s\omega)^\nu\right)\right]\left(m^{1-2\nu}\frac{\partial}{\partial m}\right)^t\right.\\
&\quad\left.\times\left\{\overline{e^{\frac{\iota\beta}{2b}d(m\omega)^{2\nu}}\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\omega)}\,\overline{(m\omega)^{-\nu\mu-\alpha}}\right\}d\omega\, s^\alpha\right|\\
&=\left|\int_0^\infty\left[e^{-\frac{\iota\beta}{2b}a\omega^{2\nu}}\omega^{-1-\alpha+2\nu+\mu}\mathscr{H}^A_{\mu,\nu,\alpha\beta}(f)e^{-\frac{\iota\beta}{2b}as^{2\nu}}(\omega s)^{-\mu}J_\mu\left(\frac{\beta}{b}(s\omega)^\nu\right)\right]\left(m^{1-2\nu}\frac{\partial}{\partial m}\right)^t\right.\\
&\quad\left.\times\overline{\left\{(m\omega)^{-\nu\mu-\alpha}e^{\frac{\iota\beta}{2b}d(m\omega)^{2\nu}}\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\omega)\right\}}d\omega s^{\alpha+\mu}\right|.
\end{aligned}$$

Since $\left|(\omega s)^{-\mu}J_\mu\left(\frac{\beta}{b}(s\omega)^\nu\right)\right|$ is bounded by $0\leqslant|(\omega s)|<\infty$ by $D^A_{\mu,\nu,\alpha,\beta}\exp(-Im(s\omega))$ (say), the above inequality

$$\begin{aligned}
&\left|\left(m^{1-2\nu}\frac{\partial}{\partial m}\right)^t (W^A_\psi f)(s,m))\right|\\
&\leqslant\left|\int_0^\infty|e^{-\frac{\iota\beta}{2b}a\omega^{2\nu}}\omega^{-1-\alpha+2\nu+\mu+t}\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f(\omega)|D^A_{\mu,\nu,\alpha,\beta}\exp(-\lambda\omega)\right.\\
&\quad\times\left|(m\omega)^t\left((m\omega)^{-t}\frac{\partial}{\partial(m\omega)}\right)^t\overline{\left\{(m\omega)^{-\nu\mu-\alpha}e^{\frac{\iota\beta}{2b}d(m\omega)^{2\nu}}\right.}\right.\\
&\quad\left.\left.\overline{\times\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\omega)\right\}}\right|\left|s^{\alpha+\mu}e^{-\frac{\iota\beta}{2b}as^{2\nu}}\right||m|^{-t}d\omega\\
&\leqslant\int_0^\infty|e^{-\frac{\iota\beta}{2b}a\omega^{2\nu}}\omega^{-\nu\mu-\alpha}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta})f(\omega)||\omega^{-1-2\nu+\mu+t+\nu\mu}||m|^{-t}\\
&\quad\times\left\{D^A_{\mu,\nu,\alpha,\beta}\exp(-\lambda\omega)\right\}\left|\left((m\omega)^{-t}\frac{\partial}{\partial(m\omega)}\right)^t\overline{\left\{(m\omega)^{-\nu\mu-\alpha}e^{\frac{\iota\beta}{2b}d(m\omega)^{2\nu}}\right|}\right.\\
&\quad\overline{\times\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\omega)}\Bigg\}(m\omega)^t\left|s^{\alpha+\mu}e^{-\frac{\iota\beta}{2b}as^{2\nu}}\right|d\omega.
\end{aligned}$$

Now using the Definition 1, we got

$$\begin{aligned}
&\left|\left(m^{1-2\nu}\frac{\partial}{\partial m}\right)^{t}(W_{\psi}^{A}f)(s,m)\right| \\
&\leqslant D_{\mu,\nu,\alpha,\beta}^{A}(1+|m|^{-t})\left|s^{\alpha+\mu}e^{-\frac{\iota\beta}{2b}as^{2\nu}}\right|\int_{0}^{\infty}\exp(\lambda\omega)\,C_{\delta,\alpha}\exp[-M(\sigma-\delta)\omega] \\
&\quad\times C_{\delta,\alpha'}[-M(\sigma-\delta')(m\omega)]\,\omega^{-1-2\nu+\mu+t+\nu\mu}\,d\omega \\
&\leqslant D'^{A}_{\mu,\nu,\alpha,\beta}(1+|m|^{-t})\left|s^{\sigma+\mu}e^{-\frac{\iota\beta}{2b}as^{2\nu}}\right|\int_{0}^{\infty}\exp\Big[2\lambda\omega-M(\sigma-\delta)\omega \\
&\quad-M(\sigma-\delta')(m\omega)\Big]\,\omega^{-1-2\nu+\mu+t+\nu\mu}\,d\omega.
\end{aligned}$$

Applying the Young's inequality properties, the above expression can be written as:

$$\begin{aligned}
-M[(\sigma-\delta)\omega]+|\lambda\omega| &\leqslant -M[\delta\omega]+\Omega\Big[\frac{\lambda}{\sigma-2\delta}\Big] \\
-M[(\sigma-\delta')m\omega]+|\lambda\omega| &\leqslant -M[\delta' m\omega]+\Omega\Big[\frac{\lambda}{m(\sigma-2\delta')}\Big].
\end{aligned}$$

Therefore, we obtain the above expression

$$\begin{aligned}
&\leqslant D'^{A}_{\mu,\nu,\alpha,\beta}(1+|m|^{-t})\left|s^{\alpha+\mu}e^{-\frac{\iota\beta}{2b}as^{2\nu}}\right|\exp\Big[\Omega\Big(\frac{\lambda}{\sigma-2\delta}\Big)+\Omega\Big(\frac{\lambda}{m}\frac{1}{\sigma-\delta'}\Big)\Big] \\
&\quad\times\int_{0}^{\infty}\omega^{-1-2\nu+\mu+t+\nu\mu}\exp[-M(\delta\omega)]\,d\omega.
\end{aligned}$$

Since $\int_0^\infty \omega^{-1-2\nu+\mu+t+\nu\mu}\exp[-M(\delta\omega)]\,d\omega<\infty$ and we can choose real number ρ, ρ' such that

$$\frac{1}{m\sigma-\delta'}=\frac{1}{m\sigma}+\rho' \quad\text{and}\quad \frac{1}{\sigma-2\delta}=\frac{1}{\sigma}+\rho.$$

We thus obtain the above expression bounded by

$$\begin{aligned}
&\left|\frac{1}{(1+|m|^{-t})}\left(m^{1-2\nu}\frac{\partial}{\partial m}\right)^{t}\Big\{s^{\alpha+\mu}e^{\frac{\iota\beta}{2b}as^{2\nu}}\Big\}(W_{\psi}^{A}f)(s,m)\right| \\
&\leqslant C_{\alpha,\rho,\rho'}\exp\Big[\Omega\Big(\frac{1}{\sigma}+\rho\Big)\lambda+\Omega\Big(\frac{1}{m\sigma}+\rho'\Big)\lambda\Big],
\end{aligned}$$

where $C_{\alpha,\rho,\rho'}=D'\int_0^\infty \omega^{-1-2\nu+\mu+t+\nu\mu}\exp[-M(\delta\omega)]\,d\omega$. □

THEOREM 6. *Let $\Omega(y)$ is dual to $M(x)$ in the Young sense, and suppose $\mathscr{H}^{A}_{\mu,\nu,\alpha,\beta}$ $\in W^{\Omega,\eta,A}$ and $\mathscr{H}^{A}_{\mu,\nu,\alpha,\beta}\left((.)^{-\nu\mu-\alpha}e^{-\frac{\iota\beta}{2b}d(.)^{2\nu}}\psi(.)\right)(m\omega)\in W^{\Omega,\eta,A}$.*

Then the linear canonical wavelet transform $(W_{\psi}^{A}f)(n,m)$ is a continuous linear mapping from $W^{M,\eta,A}$ into $\tilde{W}^{\Omega,1/\eta,A}$.

Proof. Since $\phi, \psi \in W^{\Omega,\eta,A}$, following the technique of Gelfand and Shilov [9], the expression for the linear canonical wavelet transform defined by (5) can be written as $(\gamma = \eta + \iota\omega)$

$$
\begin{aligned}
&(W_\psi^A f)(n,m)\\
&= \frac{b}{\nu\beta} e^{-\iota\frac{\pi}{2}(1+\mu)} \int_0^\infty K^{A^{-1}}((\eta+\iota\omega),n)((\eta+\iota\omega)m)^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b}d(m(\eta+\iota\omega))^{2\nu}}\\
&\quad \times (\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f)(\eta+\iota\omega)\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu} e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m(\eta+\iota\omega)}d\eta\\
&= \frac{b}{\nu\beta} e^{-\iota\frac{\pi}{2}(1+\mu)} \int_0^\infty K^{A^{-1}}(\gamma,n)(\gamma m)^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}\\
&\quad \times (\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f)(\gamma)\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu} e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))}(m\gamma)d\eta.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\left|\left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k\left(\frac{\partial}{\partial m}\right)^l n^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b}an^{2\nu}}(W_\psi^A f)(n,m)\right|\\
&= \left|\frac{b}{\nu\beta}\left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k\left(\frac{\partial}{\partial m}\right)^l \int_0^\infty n^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b}an^{2\nu}} K^{A^{-1}}(\gamma,n)(\gamma m)^{-\nu\mu-\alpha}\right.\\
&\quad \left.\times e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f)(\gamma)\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu} e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))}(m\gamma)d\eta\right|\\
&= \left|\int_0^\infty \left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k\left(\frac{\partial}{\partial m}\right)^l \left[n^{-\nu\mu-\alpha} e^{-\frac{\iota\beta}{2b}an^{2\nu}}\gamma^{-1-2\alpha+2\nu} e^{\frac{\iota\beta}{2b}(an^{2\nu}+d\gamma^{2\nu})}\right.\right.\\
&\quad \times J_\mu\left(\frac{\beta}{b}(\gamma n)^\nu\right)(m\gamma)^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu} e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))}(m\gamma)\\
&\quad \left.\left.\times (\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f)(\gamma)d\eta\right]\right|\\
&= \int_0^\infty \left|\left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k\left[n^{-\nu\mu} J_\mu\left(\frac{\beta}{b}(\gamma n)^\nu\right)\right] e^{\frac{\iota\beta}{2b}m\gamma^{2\nu}}\gamma^{-1-\alpha+2\nu}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f)(\gamma)\right.\\
&\quad \left.\times\left(\frac{\partial}{\partial m}\right)^l\left[(m\gamma)^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu} e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))}(m\gamma)d\eta\right]\right|\\
&\leqslant \int_0^\infty \left|\left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k\left[(\gamma n)^{-\nu\mu} J_\mu\left(\frac{\beta}{b}(\gamma n)^\nu\right)\right] e^{\frac{\iota\beta}{2b}a\gamma^{2\nu}}\gamma^{-1-\alpha+2\nu+\nu\mu}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta} f)(\gamma)\right.\\
&\quad \left.\times\left(\frac{\partial}{\partial m}\right)^l\left[e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}(m\gamma)^{-\nu\mu-\alpha}\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu} e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))}(m\gamma)\right]d\eta\right|.
\end{aligned}
$$

Therefore the above expression becomes,

$$\begin{aligned}
&\left|\left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k\left(\frac{\partial}{\partial m}\right)^l n^{-\nu\mu-\alpha}e^{\frac{\iota\beta}{2b}an^{2\nu}}(W_\psi^A f)(n,m)\right| \\
&\leqslant D\int_0^\infty\left|\left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k\left[n^{-\nu\mu}J_\mu\left(\frac{\beta}{b}(\gamma n)^\nu\right)\right]\right|\left|e^{\frac{\iota\beta}{2b}a\gamma^{2\nu}}\gamma^{-1-\alpha+2\nu+\nu\mu}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f)(\gamma)\right| \\
&\quad\times\left|\left(\frac{\partial}{\partial m}\right)^l\left[e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}(m\gamma)^{-\nu\mu-\alpha}\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\gamma)}\right]d\eta\right|.
\end{aligned}$$

Now, using recurrence relation equation 1, we obtain

$$\begin{aligned}
&=D\int_0^\infty\left|\left(-\nu\gamma^\nu\right)^k\left[(\gamma n)^{-\nu(\mu+k)}J_{\mu+k}\left(\frac{\beta}{b}(\gamma n)^\nu\right)\right]\right| \\
&\quad\times\left|e^{\frac{\iota\beta}{2b}a\gamma^{2\nu}}\gamma^{-1-\alpha+2\nu+\nu\mu+\nu k}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f)(\gamma)\right| \\
&\quad\times\left|\left(\frac{\partial}{\partial m}\right)^l\left[e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}(m\gamma)^{-\nu\mu-\alpha}\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\gamma)}\right]d\eta\right| \\
&=D\int_0^\infty\left|\left(-\nu\right)^k\left[(\gamma n)^{-\nu(\mu+k)}J_{\mu+k}\left(\frac{\beta}{b}(\gamma n)^\nu\right)\right]\right| \\
&\quad\times\left|e^{\frac{\iota\beta}{2b}a\gamma^{2\nu}}\gamma^{-1-\alpha+2\nu+\nu\mu+2\nu k}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f)(\gamma)\right| \\
&\quad\times\left|\left(\frac{\partial}{\partial m}\right)^l\left[e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}(m\gamma)^{-\nu\mu-\alpha}\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\gamma)}\right]d\eta\right|.
\end{aligned}$$

Since $\left|(\gamma n)^{-\nu(\mu+k)}J_{\mu+k}\left(\frac{\beta}{b}(\gamma n)^\nu\right)\right|$ is bounded on $0<|(\gamma n)|<\infty$ by $E^A_{\mu,\nu,\alpha,\beta}$ $\exp(-Im(\gamma n))$, using Definition 4 and the inequality $|z|^l\leqslant\frac{(|z|^{l+2}+|z|^l)}{1+x^2}$ the above expression becomes

$$\begin{aligned}
&\left|\left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k\left(\frac{\partial}{\partial m}\right)^l n^{-\nu\mu-\alpha}e^{\frac{\iota\beta}{2b}an^{2\nu}}(W_\psi^A f)(n,m)\right| \\
&\leqslant D'\int_0^\infty E^A_{\mu,\nu,\alpha,\beta}\exp(-Im(\omega n))\left|e^{\frac{\iota\beta}{2b}a\gamma^{2\nu}}\gamma^{-1-\alpha+2\nu+\nu\mu+2\nu k}(\mathscr{H}^A_{\mu,\nu,\alpha,\beta}f)(\gamma)\right| \\
&\quad\times\left|\left(\frac{\partial}{\partial m}\right)^l\left[e^{\frac{\iota\beta}{2b}d(m\gamma)^{2\nu}}(m\gamma)^{-\nu\mu-\alpha}\overline{\mathscr{H}^A_{\mu,\nu,\alpha,\beta}(z^{\alpha+\nu\mu}e^{-\frac{\iota\beta}{2b}az^{2\nu}}\psi(z))(m\gamma)}\right]d\eta\right| \\
&\leqslant D''\int_0^\infty\exp(-Im(\omega n))\left\{C_{k,-1-\alpha+2\nu+\nu\mu+2\nu k}+C_{k,-1-\alpha+2\nu+\nu\mu+2\nu k+2}\right\} \\
&\quad\times\exp[\Omega(\zeta+\rho)\omega]C_{l,\nu,\mu,\alpha}\exp[\Omega(\zeta+\rho')(m\omega)]\frac{d\eta}{1+|\eta|^2}
\end{aligned}$$

$$\leqslant D'' \exp[-\omega n + \Omega((\zeta+\rho)(1+m)\omega)] \int_0^\infty \frac{d\eta}{1+|\eta|^2}, \text{ if } \rho = \rho'$$
$$\leqslant D''' \exp\left[-M\left(\frac{n}{1+m}\frac{1}{\zeta+\rho}\right)\right] \int_0^\infty \frac{d\eta}{1+|\eta|^2}.$$

We can set a real number $\delta > 0$ such that $\frac{1}{\zeta+\rho} = \frac{1}{\zeta} - \delta$, we get,

$$\left| \left(n^{1-2\nu}\frac{\partial}{\partial n}\right)^k \left(\frac{\partial}{\partial m}\right)^l n^{-\nu\mu-\alpha} e^{\frac{\iota\beta}{2b}an^{2\nu}} (W_\psi^A f)(n,m) \right|$$
$$\leqslant C_{k,l,\zeta,\delta} \exp\left[-M\left(\frac{n}{1+m}\frac{1}{\zeta+\rho}\right)\right],$$

where $C_{k,l,\zeta,\delta} = D''' \int_0^\infty \frac{d\eta}{1+|\eta|^2}$. $\square$

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Ujjawala Singh
Department of Mathematics
VIT-AP University
Beside AP Secretariat, Amaravati 522237, Andhra Pradesh, India
e-mail: ujjwalasingh52011@gmail.com

Manoj Kumar Mishra
Department of Mathematics
VIT-AP University
Beside AP Secretariat, Amaravati 522237, Andhra Pradesh, India
e-mail: manojmishra.iitg@gmail.com