Journal of
Mathematical
Inequalities
Volume 19, Number 4 (2025), 1093–1107 doi:10.7153/jmi-2025-19-70

COMPLETE PICTURE ON THE MONOTONICITY CHARACTER OF A CLASS OF SEQUENCES WITH A PARAMETER

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(*Communicated by L. Mihoković*)

Abstract. We describe the monotonicity character of the class of sequences $z_n^{(\alpha)} := (1-\alpha)x_n + \alpha y_n$, $n \in \mathbb{N}$, where the parameter α belongs to the interval $[0,1]$, $x_n = \sum_{j=1}^n \frac{1}{\sqrt{j}} - 2\sqrt{n}$, and $y_n = \sum_{j=1}^n \frac{1}{\sqrt{j}} - 2\sqrt{n+1}$, on the whole domain, that is, on the set $\mathbb{N}$. If for some value of the parameter α the sequence is not strictly decreasing or strictly increasing on the whole domain, we determine the exact value of the index n where the monotonicity is changed, as well as the types of the monotonicity before and after the value of the index. A comparison of the problem of describing the monotonicity character of the sequence on the whole domain and the problem of describing its eventual monotonicity, as well as some methods for dealing with the problems, is also given.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{Z}$ the set of whole numbers, $\mathbb{N}_k$ be the set $\{j \in \mathbb{Z} : j \geqslant k\}$ where $k \in \mathbb{Z}$ is fixed, and $\mathbb{R}$ be the set of real numbers.

A sequence $(x_n)_{n\in\mathbb{N}_l} \subset \mathbb{R}$ is monotone if it is nondecreasing, or strictly increasing, or nonincreasing, or strictly decreasing on the whole domain (i.e., on the set $\mathbb{N}_l$), that is, if the inequality $x_{n+1} \geqslant x_n$, or $x_{n+1} > x_n$, or $x_{n+1} \leqslant x_n$, or $x_{n+1} < x_n$, hold for every $n \geqslant l$, respectively. A sequence $(x_n)_{n\in\mathbb{N}_l} \subset \mathbb{R}$ is called eventually monotone if there is index $n_0 \in \mathbb{N}_{l+1}$ (i.e., $n_0 \geqslant l+1$) such that the subsequence $(x_n)_{n\in\mathbb{N}_{n_0}}$ is monotone, but the subsequence $(x_n)_{n\in\mathbb{N}_{n_0-1}}$ is not.

The problem of describing the monotonicity character of real sequences is one of the basic problems in the research area, which is frequently connected to the convergence. For some problems and results in this direction see, for instance, [1, 2, 3, 4, 5, 6, 8, 9, 11, 15, 17, 18, 19, 20, 21, 22, 23, 29, 30, 31, 32, 34, 36, 37, 41, 42, 43] and the related references therein, where can be found many methods and tricks for dealing with the problem of monotonicity of various sequences of real numbers, which can be given by concrete formulas or by some recursive relations/difference equations.

Since researches are usually interested in the long term behavior of sequences of real numbers, the problem of determining their monotonicity character essentially

Mathematics subject classification (2020): Primary 40A05; Secondary 52A41.
Keywords and phrases: Real sequence, monotone sequence, eventual monotonicity, convex combinations.

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reduces to determining their eventual monotonicity. Because of this fact, the problem of describing the monotonicity character of a real sequence on its whole domain of definition is neglected. However, it turned out that there are some classes of sequences of real numbers of some interest, for which it is possible to describe the monotonicity character on the whole domain, as it was the case in [39].

In many cases it is suitable to study some pairs of real sequences which are in a sense related to each other. For instance, in [2, 5, 7, 11, 14, 15, 17, 18, 23, 26, 27, 28, 29, 33, 34, 35, 36, 38, 39, 40, 41, 42] can be found such pairs of sequences, which in many cases are monotone and converge to the same limit from which some other facts or characteristics can be obtained.

Let us mention now some results, problems and studies, which motivated us for the investigation in this paper.

Every mathematician certainly has seen the following pair of sequences

$$a_n = \left(1+\frac{1}{n}\right)^n \quad \text{and} \quad b_n = \left(1+\frac{1}{n}\right)^{n+1}, \quad n \in \mathbb{N}, \tag{1}$$

and knows that a_n is strictly increasing, whereas b_n is strictly decreasing, which can be proved by several methods. Since $0 < b_n - a_n = \frac{a_n}{n} < \frac{b_1}{n}$, $n \in \mathbb{N}$, they converge to the same limit, the base of the natural logarithm e, so we also have

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}, \tag{2}$$

for every $n \in \mathbb{N}$ ([16, 18, 23, 29, 43]).

From (2), it follows that

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}, \quad n \in \mathbb{N}. \tag{3}$$

The estimates for the number e given in (2) can be improved. Namely, it can be proved that

$$\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{4n}\right) < e < \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right), \tag{4}$$

for every $n \in \mathbb{N}$. The estimates can be found, for example, in [29] as Problem 171. They have a geometric interpretation connected to the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. Namely, if the interval $[a_n, b_n]$ is divided into four equal subintervals, then the number e occurs in the second one from the left for each $n \in \mathbb{N}$. Some better estimates than the ones in (2) and (4) can be found in [11] and [19]. However, one of the sequences therein estimating the number e need not be monotone on the whole domain, that is, on the set $\mathbb{N}$. This means that it is only an eventually monotone sequence.

The following pair of sequences

$$\widehat{a}_n = \sum_{j=n+1}^{2n} \frac{1}{j} \quad \text{and} \quad \widehat{b}_n = \sum_{j=n}^{2n} \frac{1}{j}, \quad n \in \mathbb{N}, \tag{5}$$

can be also found frequently in the literature and problem books. It is easy to show that the sequence $\widehat{a}_n$ is strictly increasing, whereas the sequence $\widehat{b}_n$ is strictly decreasing. Employing the inequalities in (3) it is shown that they both converge to $\ln 2$, so we also have

$$\sum_{j=n+1}^{2n} \frac{1}{j} < \ln 2 < \sum_{j=n}^{2n} \frac{1}{j}, \tag{6}$$

for every $n \in \mathbb{N}$ (see, e.g., [15, 23, 29]; for some related sequences see [10, 24]).

The estimates for the number $\ln 2$ given in (6) can be improved. Namely, it can be proved that

$$\frac{1}{8n} + \widehat{a}_n < \ln 2 < \frac{1}{4n} + \widehat{b}_n, \tag{7}$$

for every $n \in \mathbb{N}$. The estimates in (7) can be found in [41]. They have a geometric interpretation connected to the sequences $(\widehat{a}_n)_{n\in\mathbb{N}}$ and $(\widehat{b}_n)_{n\in\mathbb{N}}$. Namely, if the interval $[\widehat{a}_n, \widehat{b}_n]$ is divided into eight equal subintervals, then the number $\ln 2$ occurs in the second one from the left for each $n \in \mathbb{N}$. An elementary proof based on investigating the monotonicity of these two sequences, can be found in [39] and [42]. A more complex proof based on some geometric considerations and the Hermite-Hadamard inequalities ([12, 13, 21]), can be found in [39].

In [39] we presented several results and gave many interesting comments and remarks. Among other ones, we proved therein a bit surprising result. Namely, for the sequence $c_n^{(\alpha)} = (1-\alpha)\widehat{a}_n + \alpha\widehat{b}_n$, $n \in \mathbb{N}$, where $\alpha \in [0,1]$, and the sequences $(\widehat{a}_n)_{n\in\mathbb{N}}$ and $(\widehat{b}_n)_{n\in\mathbb{N}}$ are defined in (5), we managed to describe the monotonicity character of the sequence $(c_n^{(\alpha)})_{n\in\mathbb{N}}$ for each value of the parameter α on the whole domain of definition (i.e., on the set $\mathbb{N}$). This means that if for some value of the parameter α the sequence is not strictly decreasing or strictly increasing on the whole domain, we determined the exact value of the index n where the monotonicity is changed, as well as the types of the monotonicity before and after the value of the index. The cases of strict monotonicity also occur. Namely, it was shown that if $\alpha \in [1/4, 1]$, the sequence $(c_n^{(\alpha)})_{n\in\mathbb{N}}$ is strictly decreasing, whereas if $\alpha \in [0, 1/6)$ that the sequence is strictly increasing.

In [39] were also studied the sequences

$$\widetilde{a}_n = \sum_{j=n+1}^{3n} \frac{1}{j} \quad \text{and} \quad \widetilde{b}_n = \sum_{j=n}^{3n} \frac{1}{j}, \quad n \in \mathbb{N}, \tag{8}$$

and was proved a similar theorem which describes the monotonicity character of the sequence $\widetilde{c}_n^{(\alpha)} = (1-\alpha)\widetilde{a}_n + \alpha\widetilde{b}_n$, $n \in \mathbb{N}$, for each value of the parameter α on the whole domain of definition. The proof of the theorem is similar to the one dealing with the sequence $(c_n^{(\alpha)})_{n\in\mathbb{N}}$, but is much more technical. The theorem reveals an interesting thing related to the dyadic divisions of the intervals mentioned above. Namely, it shows that the dyadic divisions of the intervals $[\widetilde{a}_n, \widetilde{b}_n]$ $n \in \mathbb{N}$, are not naturally connected to the location of their joint limit, that is, the number $\ln 3$. In fact, more suitable divisions are the triadic ones.

These two theorems presented in [39] also show that, in fact, it is more natural to study monotonicity of the convex combinations of the sequences, that is, of the sequences $(c_n^{(\alpha)})_{n\in\mathbb{N}}$ and $(\widetilde{c}_n^{(\alpha)})_{n\in\mathbb{N}}$, than the monotonicity of the sequences which are obtained by the dyadic divisions of the intervals $[\widehat{a}_n,\widehat{b}_n]$ and $[\widetilde{a}_n,\widetilde{b}_n]$. This observation suggests investigation of the convex combinations of other known pairs of sequences of real numbers.

Here we continue this line of investigations by studying the monotonicity character of the convex combinations of the following pair of sequences

$$x_n = 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2\sqrt{n}, \tag{9}$$

$$y_n = 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2\sqrt{n+1}, \tag{10}$$

for $n\in\mathbb{N}$. These sequences are related to the two pairs of sequences given in (5) and (8), and can be also found in the literature and many problem books (see, for instance, [15, 23]).

Let us mention some known facts on the sequences (9) and (10). Note that

$$x_n - y_n = 2(\sqrt{n+1}-\sqrt{n}) > 0,$$

for every $n\in\mathbb{N}$,

$$x_{n+1}-x_n = \frac{1}{\sqrt{n+1}}-2(\sqrt{n+1}-\sqrt{n}) = \frac{1}{\sqrt{n+1}}-\frac{2}{\sqrt{n+1}+\sqrt{n}} < 0,$$

for every $n\in\mathbb{N}$, and

$$y_{n+1}-y_n = \frac{1}{\sqrt{n+1}}-2(\sqrt{n+2}-\sqrt{n+1}) = \frac{1}{\sqrt{n+1}}-\frac{2}{\sqrt{n+2}+\sqrt{n+1}} > 0,$$

for every $n\in\mathbb{N}$, that is, the sequence $(x_n)_{n\in\mathbb{N}}$ is strictly decreasing, the sequence $(y_n)_{n\in\mathbb{N}}$ is strictly increasing and we also have that

$$y_n < x_n$$

for every $n\in\mathbb{N}$.

From this and since

$$x_n - y_n = \frac{2}{\sqrt{n+1}+\sqrt{n}},$$

for every $n\in\mathbb{N}$, the sequences x_n and y_n converge to the same limit, that is,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c_{1/2}, \tag{11}$$

for some $c_{1/2}>0$. So, we have

$$y_1 < \cdots < y_n < c_{1/2} < x_n < \cdots < x_1,$$

for every $n \in \mathbb{N}$ (see, e.g., [15, 21, 23]).

REMARK 1. Note that the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, are obtained from the sequences

$$a_n(f) := \sum_{j=1}^{n} f(j) - \int_0^n f(t)dt,$$
$$b_n(f) := \sum_{j=1}^{n} f(j) - \int_0^{n+1} f(t)dt,$$

for $n \in \mathbb{N}$, respectively, where f is a real decreasing function on the interval $(0,+\infty)$. Indeed, it should be taken the function

$$f(t) = \frac{1}{\sqrt{t}}. \tag{12}$$

This is not a unique case. For example, the sequences converging to the Euler gamma constant [10, 15, 25] are obtained in a similar way, by taking the function

$$f(t) = \frac{1}{t}.$$

But, since unlike the function (12), it is not integrable on the interval $(0,\delta]$, $\delta > 0$, it is considered on the interval $[1,+\infty)$, and the integrals $\int_0^n f(t)dt$ and $\int_0^{n+1} f(t)dt$ are replaced by $\int_1^n f(t)dt$ and $\int_1^{n+1} f(t)dt$, respectively.

Motivated by above-mentioned investigations here we consider the following sequences

$$z_n^{(\alpha)} := \alpha x_n + (1-\alpha)y_n, \quad n \in \mathbb{N}, \tag{13}$$

where $\alpha \in [0,1]$.

Note also that combining the relations in (11) and the definition of the sequence (13), it follows that

$$\lim_{n\to\infty} z_n^{(\alpha)} = c_{1/2},$$

for each $\alpha \in [0,1]$, so the problem of convergence for the class of sequences is not of a special interest.

The purpose of this investigation is to present a complete picture on the monotonicity character of the sequences defined in (13), for each value of the parameter α in the interval $[0,1]$. If for some value of the parameter α the sequence is not strictly decreasing or strictly increasing on the whole domain, we determine the exact value of the index n where the monotonicity is changed, as well as the types of the monotonicity before and after the value of the index. A comparison of the problem of describing the monotonicity of the sequence on the whole domain and the problem of describing its eventual monotonicity is also given, where it is explained why the first problem is more difficult than the second one.

2. Two examples of the sequences in (13)

Beside above-mentioned studies and problems, two special cases of the sequence (13) also served as a natural motivation for the investigation in this paper, namely, the cases $\alpha = 1/2$ and $\alpha = 1/4$. We present them here as a good motivation and for the benefit of the reader. The reason why these special cases were taken for a preliminary investigation of the problem of describing the monotonicity character of the class of the sequences in (13) was based on our expectation that in these two cases it is possible to solve the problem by using some relatively simple algebraic calculations and use of some simple algebraic formulas, especially, the following one

$$\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}}, \tag{14}$$

when $\max\{x,y\} > 0$ and $\min\{x,y\} \geqslant 0$.

These two cases are incorporated in two examples. The first example deals with the case $\alpha = 1/2$, whereas the second example deals with the case $\alpha = 1/4$.

EXAMPLE 1. In this example we consider the case $\alpha = 1/2$. By using some standard calculations and (14), we have

$$\begin{aligned} z_{n+1}^{(1/2)} - z_n^{(1/2)} =& \frac{1}{\sqrt{n+1}} - 2\Big(\frac{\sqrt{n+1}+\sqrt{n+2}}{2} - \frac{\sqrt{n}+\sqrt{n+1}}{2}\Big) \\ =& \frac{1}{\sqrt{n+1}} - (\sqrt{n+2}-\sqrt{n}) = \frac{1}{\sqrt{n+1}} - \frac{2}{\sqrt{n+2}+\sqrt{n}} \\ =& \frac{\sqrt{n+2}-\sqrt{n+1}-(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}(\sqrt{n+2}+\sqrt{n})} \\ =& \Big(\frac{1}{\sqrt{n+2}+\sqrt{n+1}} - \frac{1}{\sqrt{n+1}+\sqrt{n}}\Big)\frac{1}{\sqrt{n+1}(\sqrt{n+2}+\sqrt{n})} \\ =& \frac{\sqrt{n}-\sqrt{n+2}}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})\sqrt{n+1}(\sqrt{n+2}+\sqrt{n})} < 0, \end{aligned} \tag{15}$$

for every $n \in \mathbb{N}$. From (15) we immediately obtain that the sequence $(z_n^{(1/2)})_{n\in\mathbb{N}}$ is strictly decreasing on the whole domain.

EXAMPLE 2. In this example we consider the case $\alpha = 1/4$. By using some standard calculations and (14), we have

$$\begin{aligned} z_{n+1}^{(1/4)} - z_n^{(1/4)} =& \frac{1}{\sqrt{n+1}} - 2\Big(\frac{\sqrt{n+1}+3\sqrt{n+2}}{4} - \frac{\sqrt{n}+3\sqrt{n+1}}{4}\Big) \\ =& \frac{1}{\sqrt{n+1}} - \frac{1}{2(\sqrt{n+1}+\sqrt{n})} - \frac{3}{2(\sqrt{n+2}+\sqrt{n+1})} \\ =& \frac{1}{4\sqrt{n+1}} - \frac{1}{2(\sqrt{n+1}+\sqrt{n})} + \frac{3}{4\sqrt{n+1}} - \frac{3}{2(\sqrt{n+2}+\sqrt{n+1})} \end{aligned}$$

$$
\begin{aligned}
&=\frac{\sqrt{n}-\sqrt{n+1}}{4\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})}+\frac{3(\sqrt{n+2}-\sqrt{n+1})}{4\sqrt{n+1}(\sqrt{n+2}+\sqrt{n+1})}\\
&=\frac{1}{4\sqrt{n+1}}\Big(\frac{3}{(\sqrt{n+2}+\sqrt{n+1})^2}-\frac{1}{(\sqrt{n+1}+\sqrt{n})^2}\Big)\\
&=\frac{4n+2\sqrt{n+1}(3\sqrt{n}-\sqrt{n+2})}{4\sqrt{n+1}(\sqrt{n+2}+\sqrt{n+1})^2(\sqrt{n+1}+\sqrt{n})^2}\\
&=\frac{4n+2\sqrt{n+1}\frac{8n-2}{3\sqrt{n}+\sqrt{n+2}}}{4\sqrt{n+1}(\sqrt{n+2}+\sqrt{n+1})^2(\sqrt{n+1}+\sqrt{n})^2}>0,
\end{aligned}
\tag{16}
$$

for every $n \in \mathbb{N}$. From (16) we immediately obtain that the sequence $(z_n^{(1/4)})_{n\in\mathbb{N}}$ is strictly increasing on the whole domain.

These two examples together with the above-mentioned description of the monotonicity of the sequences x_n and y_n give some hints about the monotonicity character of the other sequences belonging to this class. It is clear that the method dealing with the cases $\alpha = 1/4$ and $\alpha = 1/2$ cannot be applied in many cases, say, in the case $\alpha = \frac{e}{\pi}$. So, a more complex method can be developed to include all the cases when $\alpha \in [0,1]$. In the sections that follow we state and prove our main results which completely solve the problem of describing the monotonicity character of the sequences in (13).

3. An auxiliary result

In this section we present an auxiliary result, which is incorporated in the following lemma. The lemma will be used in the proof of the main result, and is one of the important parts of the proof.

LEMMA 1. *Let*

$$
g(t) := \frac{(\sqrt{t+1}+\sqrt{t})(\sqrt{t+2}+\sqrt{t})}{4\sqrt{t+1}(\sqrt{t+2}+\sqrt{t+1})}. \tag{17}
$$

Then,

$$
\lim_{t\to+\infty} g(t) = \frac{1}{2}, \tag{18}
$$

and the function g is strictly increasing on the interval $[1,+\infty)$.

Proof. We have

$$
\begin{aligned}
\lim_{t\to+\infty} g(t) &= \lim_{t\to+\infty}\frac{(\sqrt{t+1}+\sqrt{t})(\sqrt{t+2}+\sqrt{t})}{4\sqrt{t+1}(\sqrt{t+2}+\sqrt{t+1})}\\
&=\lim_{t\to+\infty}\frac{\Big(\sqrt{1+\frac{1}{t}}+1\Big)\Big(\sqrt{1+\frac{2}{t}}+1\Big)}{4\sqrt{1+\frac{1}{t}}\Big(\sqrt{1+\frac{2}{t}}+\sqrt{1+\frac{1}{t}}\Big)}=\frac{1}{2},
\end{aligned}
\tag{19}
$$

proving the relation in (18).

To prove the second claim in the lemma, we use a useful representation of the function g. Note that the function g can be written in the form

$$\begin{aligned} g(t) &= \frac{1}{4}\left(1+\sqrt{\frac{t}{t+1}}\right)\left(1+\frac{\sqrt{t}-\sqrt{t+1}}{\sqrt{t+2}+\sqrt{t+1}}\right) \\ &= \frac{1}{4}\left(1+\sqrt{1-\frac{1}{t+1}}\right)\left(1-\frac{1}{(\sqrt{t+2}+\sqrt{t+1})(\sqrt{t+1}+\sqrt{t})}\right), \end{aligned} \tag{20}$$

from which it follows that the function is strictly increasing for $t \geqslant 1$ (moreover, for $t \geqslant 0$), due to the fact that the functions

$$1+\sqrt{1-\frac{1}{t+1}} \quad \text{and} \quad 1-\frac{1}{(\sqrt{t+2}+\sqrt{t+1})(\sqrt{t+1}+\sqrt{t})}$$

are obviously strictly increasing and positive on the set. □

REMARK 2. Note that we have managed to prove the monotonicity of the function (17) in an elegant and elementary way. It is expected that this can be also done by inspecting the derivatives of the function, but due to the complexity of the function this is a much more complicated way to prove the monotonicity of the function.

4. Main result

In this section we formulate and prove our main result.

THEOREM 1. *Let* $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ *be the sequence defined in* (13)*, where* $\alpha \in [0,1]$*, and*

$$\alpha_k = \frac{(\sqrt{k+1}+\sqrt{k})(\sqrt{k+2}+\sqrt{k})}{4\sqrt{k+1}(\sqrt{k+2}+\sqrt{k+1})}, \quad k \in \mathbb{N}_0. \tag{21}$$

Then,

(a) if $\alpha \in [1/2,1]$*, the sequence* $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ *is strictly decreasing;*

(b) if $\alpha \in [0,\alpha_1)$*, the sequence* $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ *is strictly increasing;*

(c) if $\alpha = \alpha_1$*, the sequence* $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ *is nondecreasing;*

(d) if $\alpha \in (\alpha_k,\alpha_{k+1}]$ *for a fixed* $k \in \mathbb{N}$*, the sequence is nondecreasing for* $n \geqslant k+1$ *and strictly decreasing for* $1 \leqslant n \leqslant k+1$*.*

Proof. (a) By using some standard calculations and employing formula (14) several times, we have

$$
\begin{aligned}
z_{n+1}^{(\alpha)} - z_n^{(\alpha)} =&(1-\alpha)(y_{n+1}-y_n)+\alpha(x_{n+1}-x_n)\\
=&\frac{1}{\sqrt{n+1}}-2\big((2\alpha-1)\sqrt{n+1}+(1-\alpha)\sqrt{n+2}-\alpha\sqrt{n}\big)\\
=&\frac{1}{\sqrt{n+1}}-2(\sqrt{n+2}-\sqrt{n+1})\\
&+2\alpha\big(\sqrt{n+2}-\sqrt{n+1}-(\sqrt{n+1}-\sqrt{n})\big)\\
=&\frac{1}{\sqrt{n+1}}-\frac{2}{\sqrt{n+2}+\sqrt{n+1}}\\
&+2\alpha\Big(\frac{1}{\sqrt{n+2}+\sqrt{n+1}}-\frac{1}{\sqrt{n+1}+\sqrt{n}}\Big)\\
=&\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}(\sqrt{n+2}+\sqrt{n+1})}\\
&+2\alpha\Big(\frac{\sqrt{n}-\sqrt{n+2}}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})}\Big)\\
=&\frac{1}{\sqrt{n+1}(\sqrt{n+2}+\sqrt{n+1})^2}\\
&-\frac{4\alpha}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})},
\end{aligned}
\tag{22}
$$

for every $n\in\mathbb{N}$.

Since due to Example 1 the sequence $(z_n^{(1/2)})_{n\in\mathbb{N}}$ is strictly decreasing, and the function

$$
\begin{aligned}
f_\alpha(t) =&\frac{1}{\sqrt{t+1}(\sqrt{t+2}+\sqrt{t+1})^2}\\
&-\frac{4\alpha}{(\sqrt{t+2}+\sqrt{t+1})(\sqrt{t+1}+\sqrt{t})(\sqrt{t+2}+\sqrt{t})},
\end{aligned}
\tag{23}
$$

is strictly decreasing in α for each fixed $t\geqslant 0$ (if we regard α as a variable), we have that the sequence $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ is also strictly decreasing for each $\alpha\in[1/2,1]$.

From (23) it follows that a zero t_α of the function $f_\alpha(t)$ satisfies the relation

$$
\alpha=\frac{(\sqrt{t_\alpha+1}+\sqrt{t_\alpha})(\sqrt{t_\alpha+2}+\sqrt{t_\alpha})}{4\sqrt{t_\alpha+1}(\sqrt{t_\alpha+2}+\sqrt{t_\alpha+1})},
\tag{24}
$$

for a given $\alpha\in[0,1]$.

Now we show that for each $\alpha\in[\alpha_0,1/2)$ there is a unique zero t_α of the function $f_\alpha(t)$.

Let g be the function defined in (17). The function is obviously continuous on the interval $[0,+\infty)$. By Lemma 1 the function is strictly increasing and tends to $1/2$ as

$t \to +\infty$. By using these facts it follows that for each $\alpha \in [\alpha_0, 1/2)$, there is a unique $t_\alpha \geqslant 0$ such that

$$g(t_\alpha) = \alpha. \tag{25}$$

From this and since

$$f_\alpha(t) = \frac{4}{(\sqrt{t+2}+\sqrt{t+1})(\sqrt{t+1}+\sqrt{t})(\sqrt{t+2}+\sqrt{t})}(g(t)-\alpha), \tag{26}$$

it follows that for each $\alpha \in [\alpha_0, 1/2)$ a unique zero of the function $g(t)-\alpha$ is also a unique zero of the function $f_\alpha(t)$, which means, that t_α is the zero of the function $f_\alpha(t)$.

Above-mentioned facts also imply that there is a continuous inverse of the function g on the interval $[\alpha_0, 1/2)$, from which together with (25) it follows that

$$t_\alpha = g^{-1}(\alpha).$$

Since the function g satisfies the condition (18), we also have that $t_\alpha \to +\infty$ as $\alpha \to 1/2-0$.

Let the sequence $(\alpha_k)_{k\in\mathbb{N}_0}$, be defined in (21). Then from the definition of the sequence we have that $\alpha_k = g(k)$ for every $k \in \mathbb{N}_0$. This means that the number k is a unique zero of the function $g(t)-\alpha_k$ for each $k \in \mathbb{N}_0$, that is, $k = t_{\alpha_k}$ for $k \in \mathbb{N}_0$.

(b) Let $\alpha \in [0, \alpha_1)$. Then by using (26), it follows that

$$f_\alpha(n) = \frac{4}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})}(g(n)-\alpha), \tag{27}$$

for each $n \in \mathbb{N}$.

Since the function g is strictly increasing on the interval $[1, +\infty)$, we have that

$$g(n) > g(1), \tag{28}$$

for each $n \in \mathbb{N}$.

Combining the relations in (27) and (28) it follows that

$$f_\alpha(n) \geqslant \frac{4}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})}(g(1)-\alpha), \tag{29}$$

for each $n \in \mathbb{N}$.

From (29) and since

$$\alpha_1 = g(1), \tag{30}$$

we have

$$f_\alpha(n) \geqslant \frac{4}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})}(\alpha_1-\alpha), \tag{31}$$

for each $n \in \mathbb{N}$.

Since $\alpha \in [0, \alpha_1)$, then from (31) we have

$$f_\alpha(n) \geqslant \frac{4}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})}(\alpha_1 - \alpha) > 0$$

for each $n \in \mathbb{N}$, from which together with (22) it follows that the sequence $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ is strictly increasing.

(c) We have $\alpha = \alpha_1$. From (27), (28) and (30), we have

$$f_{\alpha_1}(n) = \frac{4}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})}(g(n) - \alpha_1) > 0$$

for each $n \in \mathbb{N}_2$, from which together with (22) it follows that the subsequence $(z_n^{(\alpha_1)})_{n\in\mathbb{N}_2}$ is strictly increasing.

Beside this, from (27) and (30), we have

$$f_{\alpha_1}(1) = \frac{4}{(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)(\sqrt{3}+1)}(g(1) - \alpha_1) = 0,$$

from which together with (22) it follows that $z_1^{(\alpha_1)} = z_2^{(\alpha_1)}$.

All these facts show that the sequence $(z_n^{(\alpha_1)})_{n\in\mathbb{N}_2}$ is really nondecreasing, as claimed.

(d) By Lemma 1 we have that the sequence $(\alpha_k)_{k\in\mathbb{N}_0}$ is strictly increasing and that $\lim_{k\to+\infty} \alpha_k = 1/2$. Hence, we have

$$\bigcup_{k=1}^{\infty} (\alpha_k, \alpha_{k+1}] = \left(\alpha_1, \frac{1}{2}\right).$$

This means that the union of all the intervals $(\alpha_k, \alpha_{k+1}]$ when $k \in \mathbb{N}$, is the set of all the values of parameter α which has not been considered yet. Hence, when we finish dealing with this case, all the possible cases are taken into the consideration.

Assume that $\alpha \in (\alpha_k, \alpha_{k+1}]$ for a fixed $k \in \mathbb{N}$. Since

$$g(k) = \alpha_k \quad \text{and} \quad g(k+1) = \alpha_{k+1},$$

and due to the monotonicity of the function g, we have that in this case a unique zero t_α of the function $g(t) - \alpha$ belongs to the interval $(k, k+1]$. Hence, as a consequence we have that

$$g(n) \geqslant \alpha \quad \text{for} \quad n \geqslant k+1, \tag{32}$$

and

$$g(n) < \alpha \quad \text{for} \quad n \leqslant k. \tag{33}$$

From (27) and (32) we have

$$f_\alpha(n)=\frac{4}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})}(g(n)-\alpha)\geqslant 0,$$

for $n\geqslant k+1$, from which together with (22) it follows that the subsequence $(z_n^{(\alpha)})_{n\in\mathbb{N}_{k+1}}$ is nondecreasing.

From (27) and (33) we have

$$f_\alpha(n)=\frac{4}{(\sqrt{n+2}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n})(\sqrt{n+2}+\sqrt{n})}(g(n)-\alpha)<0,$$

for $n\leqslant k$, from which together with (22) it follows that the subsequence $(z_n^{(\alpha)})_{n=\overline{1,k+1}}$ is decreasing, completing the proof of the theorem. □

REMARK 3. From the proof of Theorem 1 we see that, in fact, more is proved. Namely, an inspection of the inequalities in the proof of the theorem shows that the following theorem was obtained.

THEOREM 2. *Let $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ be the sequence defined in (13), where $\alpha\in[0,1]$, and $(\alpha_k)_{k\in\mathbb{N}_0}$ be the sequence defined in (21). Then,*

(a) if $\alpha\in[1/2,1]$, the sequence $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ is strictly decreasing;

(b) if $\alpha\in[0,\alpha_1)$, the sequence $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ is strictly increasing;

(c) if $\alpha=\alpha_1$, the sequence $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ is nondecreasing;

(d) if $\alpha\in(\alpha_k,\alpha_{k+1})$ for a fixed $k\in\mathbb{N}$, the sequence is strictly increasing for $n\geqslant k+1$ and strictly decreasing for $1\leqslant n\leqslant k+1$;

(e) if $\alpha=\alpha_{k+1}$ for a fixed $k\in\mathbb{N}$, then the sequence is strictly increasing for $n\geqslant k+2$, $z_{k+1}^{(\alpha)}=z_{k+2}^{(\alpha)}$, and it is strictly decreasing for $1\leqslant n\leqslant k+1$.

REMARK 4. Note that since by the result in Example 2 the sequence $(z_n^{(1/4)})_{n\in\mathbb{N}}$ is strictly increasing, and the function in (23) is decreasing in α for each $t\geqslant 0$, we have that the sequences $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ are also strictly increasing for each $\alpha\in[0,1/4]$. However, since

$$\alpha_1=\frac{(\sqrt{2}+1)(\sqrt{3}+1)}{4\sqrt{2}(\sqrt{3}+\sqrt{2})}=\frac{\sqrt{6}+\sqrt{3}+\sqrt{2}+1}{4(\sqrt{6}+\sqrt{2})}>\frac{1}{4},$$

the result in Example 2 is not enough for describing all the sequences in (13) which are strictly increasing on the whole domain.

Nevertheless, the sequence $(z_n^{(1/4)})_{n\in\mathbb{N}}$ really served as a good motivation for the investigation. Note also that $f_{1/4}(t)=0$ implies $t_{1/4}\in(0,1)$.

On the other hand, it happened that the result in Example 1 is enough for describing all the sequences in (13) which are strictly decreasing on the whole domain, but, of course, it is only a sufficient case to be proved, and only Theorem 1 shows that it is also the necessary one.

5. Eventual monotonicity of the sequence (13)

The problem of determining eventual monotonicity of a sequence of real numbers is much simpler, since there are some standard methods for dealing with the problem. Namely, in order to determine the type of eventual monotonicity of a sequence it should be found $n_0 \in \mathbb{N}$ such that the sequence is monotone on the set $\mathbb{N}_{n_0}$. In many cases this can be done by some asymptotic formulas.

The method is quite old. For example, several problems in [29] are solved by using some asymptotic formulas (see, also [18, 23]). However, the formulas are usually not suitable for solving the problem of determining monotonicity of a sequence on the whole domain of the definition. But, the formulas are frequently useful for establishing the type of eventual monotonicity of a sequence, which can suggests the type of the monotonicity of the sequence on the whole domain (in the case if a sequence is really monotone on the domain).

Let us see what we can get by using the asymptotic relation

$$(1+x)^\beta = 1+\beta x+\frac{\beta(\beta-1)}{2}x^2+\frac{\beta(\beta-1)(\beta-2)}{6}x^3+O(x^4)$$

as $x\to 0$, with $\beta\notin\mathbb{Z}$ (see, e.g., [18, 23, 43]), in the case of the sequences given in (13).

We have

$$\begin{aligned}
&z_{n+1}^{(\alpha)}-z_n^{(\alpha)} = \alpha(x_{n+1}-x_n)+(1-\alpha)(y_{n+1}-y_n)\\
=&\frac{1}{\sqrt{n+1}}-2\big(\alpha\sqrt{n+1}+(1-\alpha)\sqrt{n+2}-\alpha\sqrt{n}-(1-\alpha)\sqrt{n+1}\big)\\
=&\frac{1}{\sqrt{n}}\Big(1+\frac{1}{n}\Big)^{-1/2}\\
&-2\sqrt{n}\Big(\alpha\Big(1+\frac{1}{n}\Big)^{1/2}+(1-\alpha)\Big(1+\frac{2}{n}\Big)^{1/2}-\alpha-(1-\alpha)\Big(1+\frac{1}{n}\Big)^{1/2}\Big)\\
=&\frac{1}{\sqrt{n}}\Big(1-\frac{1}{2n}+\frac{3}{8n^2}-\frac{5}{16n^3}+O\Big(\frac{1}{n^4}\Big)\Big)\\
&-2\sqrt{n}\Big(\alpha\Big(1+\frac{1}{2n}-\frac{1}{8n^2}+\frac{1}{16n^3}+O\Big(\frac{1}{n^4}\Big)\Big)\\
&+(1-\alpha)\Big(1+\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{2n^3}+O\Big(\frac{1}{n^4}\Big)\Big)-\alpha\\
&-(1-\alpha)\Big(1+\frac{1}{2n}-\frac{1}{8n^2}+\frac{1}{16n^3}+O\Big(\frac{1}{n^4}\Big)\Big)\Big)\\
=&\frac{(\frac{1}{2}-\alpha)}{2n^{3/2}}+\frac{3\alpha-2}{4n^{5/2}}+O\Big(\frac{1}{n^{7/2}}\Big),
\end{aligned}$$

from which it immediately follows that the sequence $z_n^{(\alpha)}$ is eventually increasing for each $\alpha\in[0,\frac{1}{2})$, whereas it is eventually decreasing for each $\alpha\in(\frac{1}{2},1]$.

If $\alpha=\frac{1}{2}$, then we have

$$z_{n+1}^{(1/2)}-z_n^{(1/2)}=-\frac{1}{8n^{5/2}}+O\Big(\frac{1}{n^{7/2}}\Big)$$

from which it follows that the sequence $z_n^{(1/2)}$ is eventually decreasing.

So, we have that the sequence $z_n^{(\alpha)}$ is eventually decreasing when $\alpha \in [1/2, 1]$, whereas it is eventually increasing when $\alpha \in [0, 1/2)$.

These results match with the ones obtained in Theorem 1 in the sense of eventual monotonicity. So, they can serve as some suggestions for the type of the monotonicity of the sequences defined in (13). However, as they hold for sufficiently large values of index n, they are not suitable for the determining the monotonicity character of the sequences on the whole domain.

Authors contributions. This work is based on some ideas by the author of this paper, who has done all the research and writing by himself.

Use of AI tools declaration. The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest. The author declares no conflicts of interest.

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(Received February 21, 2025)

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