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NEW WEIGHT CHARACTERIZATIONS FOR THE DISCRETE HARDY INEQUALITY WITH KERNEL

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Abstract. In this work, we present new pairs of equivalent conditions for the validity of discrete weighted Hardy inequality with kernel satisfying Oinarov's conditions for the parameters $1 < p \leqslant q < \infty$. In addition, we give lower and upper estimates for the best constant of the inequality.

1. Introduction

Let $1 \leqslant p, q < \infty$ and $u = \{u_n\}_{n=1}^{\infty}$, $v = \{v_n\}_{n=1}^{\infty}$ be positive sequences of real numbers, which we in the sequel call weighted sequences. $l_{p,v}$ denote the space of all sequence $f = \{f_n\}_{n=1}^{\infty}$ of real numbers whose norm $||f||_{l_{p,v}} = (\sum_{n=1}^{\infty} |v_n f_n|^p)^{1/p}$ is finite. Let us consider the following discrete weighted Hardy inequality for nonnegative sequence $f = \{f_n\}_{n=1}^{\infty}$ in the form

$$\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} a_{n,k} f_k\right)^q\right)^{\frac{1}{q}} \leqslant C \left(\sum_{n=1}^{\infty} v_n^p f_n^p\right)^{\frac{1}{p}}, \tag{1}$$

where $\{a_{n,k}\}$ is a nonnegative triangular matrix (i.e., $a_{n,k} \geqslant 0$, $n \geqslant k \geqslant 1$ and $a_{n,k} = 0$, $k > n \geqslant 1$) is called kernel of the inequality, and C is a constant independent of f.

Various expressions related to the kernel and the weight sequences occur as a result of the estimation process for the best constant C of inequality (1). When we obtain the desired forms from these expressions, there are certain losses. The estimation of the best constant is adversely affected by this. At the expense of minor losses, it is required to analyze the expressions themselves and derive estimations from them. Consequently, new equivalent conditions are created. The Hardy inequalities' theory has already identified instances of this type.

If $a_{n,k} \equiv 1$, then (1) takes the form

$$\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} f_k\right)^q\right)^{\frac{1}{q}} \leqslant C \left(\sum_{n=1}^{\infty} v_n^p f_n^p\right)^{\frac{1}{p}}. \tag{2}$$

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G. Bennet [1, 2] proved that, if $1 < p \leqslant q < \infty$ then inequality (2) holds for all nonnegative sequence $f = \{f_n\}_{n=1}^{\infty}$ if and only if either

$$A_1 := \sup_{n\in\mathbb{N}} \left(\sum_{k=n}^{\infty} u_k^q\right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} v_k^{-p'}\right)^{\frac{1}{p'}} < \infty \tag{3}$$

or

$$A_2 := \sup_{n\in\mathbb{N}} \left(\sum_{k=1}^{n} v_k^{-p'}\right)^{-\frac{1}{p}} \left(\sum_{k=1}^{n} u_k^q \left(\sum_{m=1}^{k} v_m^{p'}\right)^{q}\right)^{\frac{1}{q}} < \infty$$

or

$$A_3 := \sup_{n\in\mathbb{N}} \left(\sum_{k=n}^{\infty} u_k^q\right)^{-\frac{1}{q'}} \left(\sum_{k=n}^{\infty} v_k^{-p'} \left(\sum_{m=k}^{\infty} u_m^q\right)^{p'}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, the best constant is estimated as

$$\max\left\{q^{-1/q}A_2, (p')^{-1/p'}A_3\right\} \leqslant A_1 \leqslant C \leqslant \min\left\{p'A_2, qA_3\right\} \leqslant \min\left\{p'q^{1/q}, q(p')^{1/p'}\right\}A_1.$$

This finding implies that any new equivalent condition can be useful in estimating the best constant. L.-E. Persson, A. Wedestig and Ch. A. Okpoti [15] provided another equivalent conditions depending on a parameter, where were shown that estimates for the best constant could be significantly improved due to a convenient choice of the parameter. In general, the equivalent conditions ensuring the validity of inequality (2) and the various estimates for the best constant are sufficiently well investigated for the values of the parameters $0 < p, q < \infty$. For more information, see [5]. Let us present the following estimates

$$A_1 \leqslant C \leqslant \tilde{C} A_1, \tag{4}$$

which will be used in the proofs of the main results of this work, where $\tilde{C} = \left(1 + \frac{q}{p'}\right)^{\frac{1}{q}} \left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}}$, see [2].

If the kernel $\{a_{n,k}\}$ of inequality (1) is different from constant sequence, then the process becomes a little more complicated. In this case, inequality (1) with kernel satisfying certain conditions has also been investigated. One such kernels is the kernel $a_{n,k}$, which is increasing in n and decreasing in k, and satisfies the condition that there exists a constant $h \geqslant 1$ such that

$$a_{n,k} \leqslant h\left(a_{n,s} + a_{s,k}\right) \quad \text{for all} \quad n \geqslant s \geqslant k \geqslant 1, \tag{5}$$

which is referred to as *Oinarov's kernel*. In the early 21st century, R. Oinarov and S. Kh. Shalginbayeva were among the first to establish necessary and sufficient conditions for the validity of inequality (1) for a class of kernels broader than the Oinarov kernel; see [10, 11, 12] for details. However, estimates for the sharp (i.e., best possible) constant in inequality (1) for such kernels had remained scarce until 2024, when A. Kalybay and S. Kh. Shalginbayeva [4] addressed this issue and established the following theorem:

THEOREM A. *Let* $1 < p \leqslant q < \infty$ *and a matrix* $\{a_{n,k}\}$ *be Oinarov's kernel. Then for any nonnegative* $f \in l_{p,v}$ *the inequality* (1) *holds if and only if* $\bar{A} = \max\{\bar{A}_1, \bar{A}_2\} < \infty$, *where*

$$\bar{A}_1 = \sup_{s\in\mathbb{N}} \left(\sum_{n=s}^{\infty} a_{n,s}^q u_n^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{s} v_n^{-p'}\right)^{\frac{1}{p'}};$$

$$\bar{A}_2 = \sup_{s\in\mathbb{N}} \left(\sum_{n=s}^{\infty} u_n^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{s} a_{s,n}^{p'} v_n^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover,

$$\bar{A} \leqslant C \leqslant \left(2(h+1)^q + (h+1)^{2q}(1+h\tilde{C}^q)\right)^{\frac{1}{q}} \bar{A}$$

and C *is the best constant in* (1).

L.-E. Persson, A. Wedestig and Ch. A. Okpoti have also made contributions to establishing parameter dependent conditions that ensure the validity of inequality (1) for separable kernels, as well as parameter-dependent sufficient conditions for its validity in the case of general kernels; see [13, 14, 15] for details. New equivalent conditions for the integral form of (1) and estimates for its best constant were obtained; for further information, see [6, 7, 8, 9]. In [3] were also given parameter dependent sufficient condition for satisfying of integral inequality with rather general kernel and upper estimate for its best constant.

In this paper, we present new equivalent conditions for the validity of inequality (1) with Oinarov's kernel and, as a key advantage of these conditions, we derive convenient estimates for the sharp constant. The paper is organized as follows: the first section is Introduction. In the second section we give our main results, proof of which will be given in the next section.

2. Main part

Let us denote:

$$\tilde{A}_1(s) = \left(\sum_{n=s}^{\infty} a_{n,s}^q u_n^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{s-1} v_n^{-p'}\right)^{\frac{1}{p'}};$$

$$\tilde{A}_2(s) = \left(\sum_{n=s}^{\infty} u_n^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{s-1} a_{s,n}^{p'} v_n^{-p'}\right)^{\frac{1}{p'}};$$

$$B_1(s) = \left(\sum_{n=s}^{\infty} v_n^{-p'} \left(\sum_{k=n}^{\infty} a_{k,n}^q u_k^q\right)^{p'}\right)^{\frac{1}{p'q}} \left(\sum_{n=1}^{s} v_n^{-p'}\right)^{\frac{1}{p'q'}};$$

$$B_2(s) = \left(\sum_{n=s}^{\infty} v_n^{-p'} \left(\sum_{k=n}^{\infty} a_{k,n} u_k^q\right)^{p'}\right)^{\frac{1}{p'}} \left(\sum_{n=s}^{\infty} u_n^q\right)^{-\frac{1}{q'}};$$

$$B_3(s) = \left(\sum_{n=1}^{s} u_n^q \left(\sum_{k=1}^{n} a_{n,k} v_k^{-p'}\right)^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{s} v_n^{-p'}\right)^{-\frac{1}{p}};$$

$$B_4(s) = \left(\sum_{n=1}^{s} u_n^q \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{-p'}\right)^q\right)^{\frac{1}{qp'}} \left(\sum_{n=s}^{\infty} u_n^q\right)^{\frac{1}{qp}}.$$

THEOREM 1. *Let $1 < p \leqslant q < \infty$ and a matrix $\{a_{n,k}\}$ be Oinarov's kernel. Then for any nonnegative $f \in l_{p,v}$ inequality (1) holds if and only if*

$$B_1 = \sup_{s\in\mathbb{N}} B_1(s) < \infty, \qquad B_2 = \sup_{s\in\mathbb{N}} B_2(s) < \infty. \tag{6}$$

Moreover, the best constant of the inequality (1) satisfies

$$\max\left\{\sup_{s\in\mathbb{N}}\left[\tilde{A}_1^{p'q}(s) + B_1^{p'q}(s)\right]^{\frac{1}{p'q}}, \sup_{s\in\mathbb{N}}\left[\tilde{A}_2^{p'}(s) + B_2^{p'}(s)\right]^{\frac{1}{p'}}\right\} \leqslant C \leqslant X, \tag{7}$$

where X is a positive solution of the corresponding nonlinear equation

$$X^q - h^{q-1} q B_2 X^{q-1} = h^{q-1} q^{\frac{p'+1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q \qquad \text{if} \quad 1 < q \leqslant 2,$$

$$X^{q'} - h(qB_2)^{q'-1} X = hq^{\frac{p'+1}{p'(q-1)}} (q')^{\frac{1}{p'}} B_1^{q'} \qquad \text{if} \qquad q \geqslant 2. \tag{8}$$

THEOREM 2. *Let $1 < p \leqslant q < \infty$ and a matrix $\{a_{n,k}\}$ be Oinarov's kernel. Then for any nonnegative $f \in l_{p,v}$ inequality (1) holds if and only if*

$$B_3 = \sup_{s\in\mathbb{N}} B_3(s) < \infty, \quad B_4 = \sup_{s\in\mathbb{N}} B_4(s) < \infty. \tag{9}$$

Moreover, the best constant of the inequality (1) satisfies

$$\max\left\{\sup_{s\in\mathbb{N}}\left[\tilde{A}_1^q(s+1) + B_3^q(s)\right]^{\frac{1}{q}}, \sup_{s\in\mathbb{N}}\left[\tilde{A}_2^{p'q}(s+1) + B_4^{p'q}(s)\right]^{\frac{1}{p'q}}\right\} \leqslant C \leqslant X, \tag{10}$$

where X is a positive solution of the corresponding nonlinear equation

$$X^{p'} - h^{p'-1} p' B_3 X^{p'-1} = h^{p'-1} (p')^{\frac{q+1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} \qquad \text{if} \quad 1 \leqslant p' \leqslant 2,$$

$$X^p - h(p'B_3)^{p-1} X = hp^{\frac{1}{q}} (p')^{\frac{q+1}{q(p'-1)}} B_4^p \qquad \text{if} \quad p' \geqslant 2. \tag{11}$$

EXAMPLE 1. (i) If $1 < p \leqslant q = 2$ then equation (8) and its positive solution take the forms

$$X^2 - 2hB_2X = 2^{(p'+2)/p'} hB_1^2 \quad \text{and} \quad X = \left(hB_2 + \sqrt{h^2B_2^2 + 2^{(p'+2)/p'} hB_1^2}\right),$$

respectively. Therefore, the following upper estimate holds

$$C \leqslant \left(hB_2 + \sqrt{h^2B_2^2 + 2^{(p'+2)/p'}hB_1^2}\right).$$

(ii) If $2 = p \leqslant q < \infty$ then equation (11) and its positive solution take the forms

$$X^2 - 2hB_3X = 2^{(q+2)/q}hB_4^2 \quad \text{and} \quad X = \left(hB_3 + \sqrt{h^2B_3^2 + 2^{(q+2)/q}hB_4^2}\right),$$

respectively. Therefore, the following upper estimate holds

$$C \leqslant \left(hB_3 + \sqrt{h^2B_3^2 + 2^{(q+2)/q}hB_4^2}\right).$$

EXAMPLE 2. Let $p = q = 2$, $\{a_{n,k}\} = \{n-k\}$ (i.e., $h = 1$) and the weight sequences $u_n = v_n = 2^{-\frac{n}{2}}$ then we have

(i) according to A. Kalybay and S. Shalginbayeva's estimates: $3.46 \leqslant C \leqslant 32.49$, $d = 29.026$;

(ii) according to Theorem 2.1: $4 \leqslant C \leqslant 8.76$, $d = 4.756$;

(iii) according to Theorem 2.2: $5.18 \leqslant C \leqslant 12$, $d = 6.82$;

(iv) according to Theorems 2.1 and 2.2: $5.18 \leqslant C \leqslant 8.76$, $d = 3.58$,

where d is the difference between the upper and the lower estimates.

3. Proofs

First we deal with the duality principle, which will be used in the proofs of the main results.

Duality principle. When it comes to creating new conditions and working with conjugate inequality, the concept of duality is critical.

LEMMA 1. *Let $1 < p,q < \infty$ and $0 < C < \infty$. Then the inequality (1) holds if and only if "dual" inequality*

$$\left(\sum_{n=1}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n}g_k\right)^{p'}\right)^{\frac{1}{p'}} \leqslant C\left(\sum_{n=1}^{\infty} u_n^{-q'}g_n^{q'}\right)^{\frac{1}{q'}} \tag{12}$$

holds for all nonnegative $g = \{g_n\} \in l_{q',u^{-1}}$.

Proof. Suppose that the inequality (1) holds. Then using succevely the definition of the norm, Fubini's theorem and Hölder's inequality we have

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n}g_k\right)^{p'}\right)^{\frac{1}{p'}} &= \sup_{||f||_{l_{p,v}}=1}\left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} a_{k,n}g_k\right) f_n\right) \\
&= \sup_{||f||_{l_{p,v}}=1}\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k} a_{k,n}f_n\right) g_k\right) \\
&\leqslant \sup_{||f||_{l_{p,v}}=1}\left(\sum_{k=1}^{\infty} u_k^q\left|\sum_{n=1}^{k} a_{k,n}f_k\right|^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{\infty} u_k^{-q'}g_k^{q'}\right)^{\frac{1}{q'}} \\
&\leqslant \sup_{||f||_{l_{p,v}}=1}\left(\sum_{k=1}^{\infty} u_k^q\left(\sum_{n=1}^{k} a_{k,n}|f_k|\right)^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{\infty} u_k^{-q'}g_k^{q'}\right)^{\frac{1}{q'}} \\
&\leqslant C\sup_{||f||_{l_{p,v}}=1}\left(\sum_{k=1}^{\infty} v_k^p|f_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{\infty} u_k^{-q'}g_k^{q'}\right)^{\frac{1}{q'}} \\
&= C\left(\sum_{k=1}^{\infty} u_k^{-q'}g_k^{q'}\right)^{\frac{1}{q'}},
\end{aligned}$$

which implies (12). The converse inequality is shown analogously. Lemma is proved. □

3.1. Proof of theorem 1

Proof. Necessity and lower estimate. Assume that the inequality (1) holds, where C is the best constant. Then by lemma 1 we have that the inequality (12) is also hold with the same constant C. Let $k_0 \in \mathbb{N}$ be a fixed. Then choosing test sequence

$$g_{n,k_0} = a_{n,k_0}^{q-1}u_n^q, \qquad n = 1,2,\ldots$$

we get for the right hand side of (12)

$$\left(\sum_{n=1}^{\infty} g_{n,k_0}^{q'}u_n^{-q'}\right)^{\frac{1}{q'}} = \left(\sum_{n=k_0}^{\infty} a_{n,k_0}^{q}u_n^{q}\right)^{\frac{1}{q'}} \tag{13}$$

and for the left hand side of (12)

$$\left(\sum_{n=1}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n}g_{k,k_0}\right)^{p'}\right)^{\frac{1}{p'}} = \left(\sum_{n=1}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n}a_{k,k_0}^{q-1}u_k^q\right)^{p'}\right)^{\frac{1}{p'}}$$

$$= \left(\sum_{n=1}^{k_0-1} v_n^{-p'}\left(\sum_{k=k_0}^{\infty} a_{k,n}a_{k,k_0}^{q-1}u_k^q\right)^{p'} + \sum_{n=k_0}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n}a_{k,k_0}^{q-1}u_k^q\right)^{p'}\right)^{\frac{1}{p'}}$$

[using the monotonicity of $a_{n,k}$ in the first sum as $a_{k,n} \geqslant a_{k,k_0}$ for $1 \leqslant n \leqslant k_0$ and in the second sum as $a_{k,k_0} \geqslant a_{k,n}$ for $k_0 \leqslant n$]

$$\geqslant \left(\left(\sum_{n=1}^{k_0-1} v_n^{-p'} \right) \left(\sum_{k=k_0}^{\infty} a_{k,k_0}^q u_k^q \right)^{p'} + \sum_{n=k_0}^{\infty} v_n^{-p'} \left(\sum_{k=n}^{\infty} a_{k,n}^q u_k^q \right)^{p'} \right)^{\frac{1}{p'}}. \tag{14}$$

Taking into account (13) and (14) in (12) we obtain

$$\left(\sum_{n=1}^{k_0-1} v_n^{-p'} \right) \left(\sum_{k=k_0}^{\infty} a_{k,k_0}^q u_k^q \right)^{p'} + \sum_{n=k_0}^{\infty} v_n^{-p'} \left(\sum_{k=n}^{\infty} a_{k,n}^q u_k^q \right)^{p'} \leqslant C^{p'} \left(\sum_{n=k_0}^{\infty} a_{n,k_0}^q u_n^q \right)^{\frac{p'}{q'}}.$$

Multiplying both sides of the estimate by $\left[\sum_{n=1}^{k_0} v_n^{-p'} \right]^{q-1}$ we have

$$\left(\sum_{n=1}^{k_0-1} v_n^{-p'} \right) \left(\sum_{n=1}^{k_0} v_n^{-p'} \right)^{q-1} \left(\sum_{k=k_0}^{\infty} a_{k,k_0}^q u_k^q \right)^{p'}$$
$$+ \left(\sum_{n=k_0}^{\infty} v_n^{-p'} \left(\sum_{k=n}^{\infty} a_{k,n}^q u_k^q \right)^{p'} \right) \left(\sum_{n=1}^{k_0} v_n^{-p'} \right)^{q-1}$$
$$\leqslant C^{p'} \left(\sum_{n=k_0}^{\infty} a_{n,k_0}^q u_n^q \right)^{\frac{p'}{q'}} \left(\sum_{n=1}^{k_0} v_n^{-p'} \right)^{q-1}$$

and then

$$\left(\sum_{n=1}^{k_0-1} v_n^{-p'} \right)^q \left(\sum_{k=k_0}^{\infty} a_{k,k_0}^q u_k^q \right)^{p'} + \left(\sum_{n=k_0}^{\infty} v_n^{-p'} \left(\sum_{k=n}^{\infty} a_{k,n}^q u_k^q \right)^{p'} \right) \left(\sum_{n=1}^{k_0} v_n^{-p'} \right)^{q-1}$$
$$\leqslant C^{p'} \left(\sum_{n=k_0}^{\infty} a_{n,k_0}^q u_n^q \right)^{\frac{p'}{q'}} \left(\sum_{n=1}^{k_0} v_n^{-p'} \right)^{q-1},$$

i.e.,

$$\tilde{A}_1^{p'q}(k_0) + B_1^{p'q}(k_0) \leqslant C^{p'} \overline{A}_1^{p'(q-1)}(k_0).$$

Using the lower estimate in theorem A, i.e., $\overline{A}_1 \leqslant C$, we get

$$\tilde{A}_1^{p'q}(k_0) + B_1^{p'q}(k_0) \leqslant C^{p'q}.$$

This proves the necessity of the condition $B_1 < \infty$ in (6) and the first part of the lower estimate in (7).

To prove the rest part of the necessity we choose the test sequence in (12) as

$$g_{n,k_0} = u_n^q \chi_{[k_0,\infty)}(n), \quad n = 1, 2, \ldots,$$

where $k_0 \in \mathbb{N}$ is a fixed and $\chi_{[k_0,\infty)}(n)$ is characteristic sequence.

Then we get

$$\left(\sum_{n=1}^{\infty} u_n^{-q'} g_{n,k_0}^{q'}\right)^{\frac{1}{q'}} = \left(\sum_{n=k_0}^{\infty} u_n^q\right)^{\frac{1}{q'}}$$

and

$$\left(\sum_{n=1}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n} g_{k,k_0}\right)^{p'}\right)^{\frac{1}{p'}}$$
$$= \left(\sum_{n=1}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n} u_k^q \chi_{[k_0,\infty)}(k)\right)^{p'}\right)^{\frac{1}{p'}}$$
$$= \left(\sum_{n=1}^{k_0-1} v_n^{-p'}\left(\sum_{k=k_0}^{\infty} a_{k,n} u_k^q\right)^{p'} + \sum_{n=k_0}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n} u_k^q\right)^{p'}\right)^{\frac{1}{p'}}.$$

From these and (12) we obtain

$$\sum_{n=1}^{k_0-1} v_n^{-p'}\left(\sum_{k=k_0}^{\infty} a_{k,n} u_k^q\right)^{p'} + \sum_{n=k_0}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n} u_k^q\right)^{p'} \leqslant C^{p'}\left(\sum_{n=k_0}^{\infty} u_n^q\right)^{\frac{p'}{q'}},$$

i.e.,

$$\left(\sum_{n=1}^{k_0-1} v_n^{-p'}\left(\sum_{k=k_0}^{\infty} a_{k,n} u_k^q\right)^{p'}\right)\left(\sum_{n=k_0}^{\infty} u_n^q\right)^{-\frac{p'}{q'}}$$
$$+\sum_{n=k_0}^{\infty} v_n^{-p'}\left(\sum_{k=n}^{\infty} a_{k,n} u_k^q\right)^{p'}\left(\sum_{n=k_0}^{\infty} u_n^q\right)^{-\frac{p'}{q'}} \leqslant C^{p'}.$$

Using monotonicity of the kernel as $a_{k,n} \geqslant a_{k_0,n}$ for $k \geqslant k_0$ in the first sum we obtain

$$\left\{\tilde{A}_2^{p'}(k_0) + B_2^{p'}(k_0)\right\}^{\frac{1}{p'}} \leqslant C.$$

This proves the necessity of the condition $B_2 < \infty$ and the second part of the lower estimate in (7).

Sufficiency and upper estimate. Let us denote

$$S = \sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} a_{n,k} f_k\right)^q,$$

then consequently using Lagrange's mean value theorem, Fubini's theorem and Hölder's inequality we get (here we suppose that $\sum_{k=1}^{0} a_{n,k}f_k = 0$)

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} u_n^q \left(\sum_{m=1}^{n} \left[\left(\sum_{k=1}^{m} a_{n,k}f_k \right)^q - \left(\sum_{k=1}^{m-1} a_{n,k}f_k \right)^q \right] \right) \\
&= q \sum_{n=1}^{\infty} u_n^q \left(\sum_{m=1}^{n} a_{n,m}f_m \left(\sum_{k=1}^{m-1} a_{n,k}f_k + \xi_m a_{n,m}f_m \right)^{q-1} \right) \\
&\leqslant q \sum_{n=1}^{\infty} u_n^q \left(\sum_{m=1}^{n} a_{n,m}f_m \left(\sum_{k=1}^{m} a_{n,k}f_k \right)^{q-1} \right) \\
&= q \sum_{m=1}^{\infty} f_m \left[\sum_{n=m}^{\infty} a_{n,m}u_n^q \left(\sum_{k=1}^{m} a_{n,k}f_k \right)^{q-1} \right] \\
&= q \sum_{m=1}^{\infty} [v_m f_m] \left[v_m^{-1} \sum_{n=m}^{\infty} a_{n,m}u_n^q \left(\sum_{k=1}^{m} a_{n,k}f_k \right)^{q-1} \right] \\
&\leqslant q \left(\sum_{m=1}^{\infty} v_m^p f_m^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} v_m^{-p'} \left(\sum_{n=m}^{\infty} a_{n,m}u_n^q \left(\sum_{k=1}^{m} a_{n,k}f_k \right)^{q-1} \right)^{p'} \right)^{\frac{1}{p'}} \\
&= q\|f\|_{l_{p,v}} \overline{S}^{\,\frac{1}{p'}},
\end{aligned}
$$

where $\xi_m \in (0,1)$, $m = 1,\ldots,n$ and

$$
\overline{S} = \sum_{m=1}^{\infty} v_m^{-p'} \left(\sum_{n=m}^{\infty} a_{n,m}u_n^q \left(\sum_{k=1}^{m} a_{n,k}f_k \right)^{q-1} \right)^{p'}. \tag{15}
$$

So, we get the following estimate

$$
S \leqslant q\|f\|_{l_{p,v}} \overline{S}^{\,\frac{1}{p'}}. \tag{16}
$$

Further, we estimate $\overline{S}$. We divide the proof into two cases.

Case 1. Let $q \geqslant 2$. Using (5) and Minkowski's inequality we estimate for the inner sum of $\overline{S}$ as

$$
\begin{aligned}
&\left(\sum_{n=m}^{\infty} a_{n,m}u_n^q \left(\sum_{k=1}^{m} a_{n,k}f_k \right)^{q-1} \right)^{\frac{1}{q-1}} \\
&\leqslant h \left(\sum_{n=m}^{\infty} a_{n,m}u_n^q \left(a_{n,m} \sum_{k=1}^{m} f_m + \sum_{k=1}^{m} a_{m,k}f_k \right)^{q-1} \right)^{\frac{1}{q-1}}
\end{aligned}
$$

$$\leqslant h\left[\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n^q \left(\sum_{k=1}^{m} f_k\right)^{q-1}\right)^{\frac{1}{q-1}} + \left(\sum_{n=m}^{\infty} a_{n,m} u_n^q \left(\sum_{k=1}^{m} a_{m,k} f_k\right)^{q-1}\right)^{\frac{1}{q-1}}\right]$$

$$= h\left[\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n^q\right)^{\frac{1}{q-1}} \left(\sum_{k=1}^{m} f_k\right) + \left(\sum_{n=m}^{\infty} a_{n,m} u_n^q\right)^{\frac{1}{q-1}} \left(\sum_{k=1}^{m} a_{m,k} f_k\right)\right].$$

Therefore (15) and Minkowski's inequality imply

$$\overline{S} \leqslant h^{(q-1)p'} \sum_{m=1}^{\infty} v_m^{-p'} \left[\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n^q\right)^{\frac{1}{q-1}} \left(\sum_{k=1}^{m} f_k\right) + \left(\sum_{n=m}^{\infty} a_{n,m} u_n^q\right)^{\frac{1}{q-1}} \left(\sum_{k=1}^{m} a_{m,k} f_k\right)\right]^{(q-1)p'}$$

$$= h^{(q-1)p'} \left[\left(\sum_{m=1}^{\infty} v_m^{-p'} \left[\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n^q\right)^{\frac{1}{q-1}} \left(\sum_{k=1}^{k} f_k\right) + \left(\sum_{n=m}^{\infty} a_{n,m} u_n^q\right)^{\frac{1}{q-1}} \left(\sum_{k=1}^{m} a_{m,k} f_k\right)\right]^{(q-1)p'}\right)^{\frac{1}{(q-1)p'}}\right]^{(q-1)p'}$$

$$\leqslant h^{(q-1)p'} \left[\left(\sum_{m=1}^{\infty} v_m^{-p'} \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n^q\right)^{p'} \left(\sum_{k=1}^{m} f_k\right)^{(q-1)p'}\right)^{\frac{1}{(q-1)p'}} + \left(\sum_{m=1}^{\infty} v_m^{-p'} \left(\sum_{n=m}^{\infty} a_{n,m} u_n^q\right)^{p'} \left(\sum_{k=1}^{m} a_{m,k} f_k\right)^{(q-1)p'}\right)^{\frac{1}{(q-1)p'}}\right]^{(q-1)p'}$$

$$= h^{(q-1)p'} \left[S_1^{\frac{1}{(q-1)p'}} + S_2^{\frac{1}{(q-1)p'}}\right]^{(q-1)p'},$$

i.e.,

$$\overline{S} \leqslant h^{(q-1)p'} \left\{S_1^{\frac{1}{(q-1)p'}} + S_2^{\frac{1}{(q-1)p'}}\right\}^{(q-1)p'}, \tag{17}$$

where

$$S_1 = \sum_{m=1}^{\infty} v_m^{-p'} \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n^q\right)^{p'} \left(\sum_{k=1}^{m} f_k\right)^{(q-1)p'},$$

$$S_2 = \sum_{m=1}^{\infty} v_m^{-p'} \left(\sum_{n=m}^{\infty} a_{n,m} u_n^q\right)^{p'} \left(\sum_{k=1}^{m} a_{m,k} f_k\right)^{(q-1)p'}.$$

From (16) and (17) we have

$$S^{\frac{1}{q-1}} \leqslant q^{\frac{1}{q-1}} \|f\|_{l_{p,v}}^{\frac{1}{q-1}} h\left(S_1^{\frac{1}{(q-1)p'}} + S_2^{\frac{1}{(q-1)p'}}\right). \tag{18}$$

Further, we estimate S_1 and S_2 separately. To estimate S_1, we use discrete Hardy inequality (2) with the exponents $\overline{p} := p$, $\overline{q} := (q-1)\,p'$ and weight sequences $\overline{u}_m^{\overline{q}} := v_m^{-p'}\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n^q\right)^{p'}$, $\overline{v}_m := v_m$, i.e.,

$$S_1^{\frac{1}{(q-1)p'}} = \left(\sum_{m=1}^{\infty} \overline{u}_m^{\overline{q}} \left(\sum_{k=1}^{m} f_k\right)^{\overline{q}}\right)^{\frac{1}{\overline{q}}} \leqslant C_{\overline{p},\overline{q}} \left(\sum_{m=1}^{\infty} v_m^{\overline{p}} f_m^{\overline{p}}\right)^{\frac{1}{\overline{p}}}, \tag{19}$$

since the condition (3) is satisfied, i.e.,

$$\begin{aligned} A_1 :&= \sup_{k\in\mathbb{N}} \left(\sum_{m=k}^{\infty} \overline{u}_m^{\overline{q}}\right)^{\frac{1}{\overline{q}}} \left(\sum_{m=1}^{k} \overline{v}_m^{-\overline{p}'}\right)^{\frac{1}{\overline{p}'}} \\ &= \sup_{k\in\mathbb{N}} \left(\sum_{m=k}^{\infty} v_m^{-p'} \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n^q\right)^{p'}\right)^{\frac{1}{(q-1)p'}} \left(\sum_{m=1}^{k} v_m^{-p'}\right)^{\frac{1}{p'}} = B_1^{q'} < \infty. \end{aligned}$$

By (4) we get the upper estimate for the best constant $C_{\overline{p},\overline{q}}$ in (19)

$$C_{\overline{p},\overline{q}} \leqslant q^{\frac{1}{p'(q-1)}} \left(q'\right)^{\frac{1}{p'}} B_1^{q'}.$$

Therefore,

$$S_1^{\frac{1}{(q-1)p'}} \leqslant q^{\frac{1}{p'(q-1)}} \left(q'\right)^{\frac{1}{p'}} B_1^{q'} \|f\|_{l_{p,v}}. \tag{20}$$

Now we estimate S_2. Consequently using Fubini's theorem and Minkowski's inequality we obtain

$$\begin{aligned} S_2 &= \sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q\right)^{p'} \left(\sum_{m=1}^{k} a_{k,m} f_m\right)^{(q-1)p'} \\ &= \sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q\right)^{p'} \left\{\sum_{l=1}^{k} \left[\left(\sum_{m=1}^{l} a_{l,m} f_m\right)^{(q-1)p'} - \left(\sum_{m=1}^{l-1} a_{l-1,m} f_m\right)^{(q-1)p'}\right]\right\} \\ &= \sum_{l=1}^{\infty} \left[\left(\sum_{m=1}^{l} a_{l,m} f_m\right)^{(q-1)p'} - \left(\sum_{m=1}^{l-1} a_{l-1,m} f_m\right)^{(q-1)p'}\right] \left(\sum_{k=l}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q\right)^{p'}\right) \\ &= \sum_{l=1}^{\infty} \left[\left(\sum_{m=1}^{l} a_{l,m} f_m\right)^{(q-1)p'} - \left(\sum_{m=1}^{l-1} a_{l-1,m} f_m\right)^{(q-1)p'}\right] \left(\sum_{k=l}^{\infty} u_k^q\right)^{\frac{p'}{q'}} B_2^{p'}(l) \end{aligned}$$

$$\leqslant B_2^{p'}\left(\left[\sum_{l=1}^{\infty}\left[\left(\sum_{m=1}^{l}a_{l,m}f_m\right)^{(q-1)p'}-\left(\sum_{m=1}^{l-1}a_{l-1,m}f_m\right)^{(q-1)p'}\right]\left(\sum_{k=l}^{\infty}u_k^q\right)^{\frac{p'}{q'}}\right]^{\frac{q'}{p'}}\right)^{\frac{p'}{q'}}$$

$$\leqslant B_2^{p'}\left(\sum_{k=1}^{\infty}u_k^q\left[\sum_{l=1}^{k}\left[\left(\sum_{m=1}^{l}a_{l,m}f_m\right)^{(q-1)p'}-\left(\sum_{m=1}^{l-1}a_{l-1,m}f_m\right)^{(q-1)p'}\right]\right]^{\frac{q'}{p'}}\right)^{\frac{p'}{q'}}$$

$$=B_2^{p'}\left[\sum_{k=1}^{\infty}u_k^q\left(\sum_{m=1}^{k}a_{k,m}f_m\right)^q\right]^{\frac{p'}{q'}}=B_2^{p'}S^{\frac{p'}{q'}}.$$

From this we get

$$S_2^{\frac{1}{(q-1)p'}}\leqslant B_2^{\frac{1}{q-1}}S^{\frac{1}{q}}. \tag{21}$$

Taking into account the above estimates in (18) we get

$$\begin{aligned} S^{\frac{1}{q-1}} &\leqslant q^{\frac{1}{q-1}}\|f\|_{l_{p,v}}^{\frac{1}{q-1}}h\left(q^{\frac{1}{(q-1)p'}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}\|f\|_{l_{p,v}}+B_2^{\frac{1}{q-1}}S^{\frac{1}{q}}\right)\\ &=hq^{\frac{p'+1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}\|f\|_{l_{p,v}}^{q'}+h\left(qB_2\|f\|_{l_{p,v}}\right)^{\frac{1}{q-1}}S^{\frac{1}{q}}. \end{aligned} \tag{22}$$

Now applying Young's inequality to the second term on the right hand side of (22) as

$$h\left(qB_2\|f\|_{l_{p,v}}\right)^{\frac{1}{q-1}}S^{\frac{1}{q}}\leqslant\frac{h^q\left(qB_2\|f\|_{l_{p,v}}\right)^{q'}}{q}+\frac{S^{\frac{1}{q-1}}}{q'}$$

we obtain that

$$S^{\frac{1}{q-1}}\leqslant hq^{\frac{p'+1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}\|f\|_{l_{p,v}}^{q'}+h^qq^{q'-1}B_2^{q'}\|f\|_{l_{p,v}}^{q'}+\frac{S^{\frac{1}{q-1}}}{q'},$$

and then

$$S^{\frac{1}{q-1}}\leqslant hq^{\frac{1}{p'(q-1)}+q'}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}\|f\|_{l_{p,v}}^{q'}+h^qq^{q'}B_2^{q'}\|f\|_{l_{p,v}}^{q'},$$

i.e.,

$$S^{\frac{1}{q}}\leqslant q\left(hq^{\frac{1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}+h^q\right)^{\frac{1}{q'}}B\|f\|_{l_{p,v}},$$

where $B=\max\{B_1,B_2\}$. This means that the conditions $B_1<\infty$ and $B_2<\infty$ are sufficient for the inequality (1) to hold. We also get the upper estimate for the best constant C

$$C\leqslant q\left(hq^{\frac{1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}+h^q\right)^{\frac{1}{q'}}B.$$

Using $S^{\frac{1}{q}} \leqslant C\|f\|_{l_{p,v}}$, we rewrite (22) in the form

$$S^{\frac{1}{q-1}} \leqslant hq^{\frac{1}{q-1}}\|f\|_{l_{p,v}}^{q'}\left(q^{\frac{1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}+B_2^{\frac{1}{q-1}}C\right),$$

i.e,

$$\left(\frac{S^{\frac{1}{q}}}{\|f\|_{l_{p,v}}}\right)^{q'} \leqslant hq^{\frac{1}{q-1}}\left(q^{\frac{1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}+B_2^{\frac{1}{q-1}}C\right).$$

From the arbitrary of f in the last estimate we have

$$C^{q'} \leqslant hq^{\frac{1}{q-1}}\left(q^{\frac{1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}+B_2^{\frac{1}{q-1}}C\right),$$

i.e.,

$$\frac{C^{q'}}{q^{\frac{1}{q-1}}\left(q^{\frac{1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}+B_2^{\frac{1}{q-1}}C\right)} \leqslant h.$$

Now we consider the function

$$f(x)=\frac{x^{q'}}{q^{\frac{p'+1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}+\left(qB_2\right)^{\frac{1}{q-1}}x}$$

corresponding to the left-hand side of the estimate. It is easy to see that this function is monotone increasing and continuous in $(0,\infty)$, $f(0)=0$ and $f(\infty)=\infty$, which implies that the equation $f(x)=h$ has exactly one positive solution in $(0,\infty)$. If X is a solution of the equation, which means that

$$X^{q'}=h\left(q^{\frac{p'+1}{p'(q-1)}}\left(q'\right)^{\frac{1}{p'}}B_1^{q'}+\left(qB_2\right)^{\frac{1}{q-1}}X\right),$$

then, $C \leqslant X$. This proves the upper estimate (8) in the case $q \geqslant 2$.

Case 2. Let $1<q \leqslant 2$. Using (4) we have that

$$\begin{aligned}
&\sum_{n=k}^{\infty} a_{n,k}u_n^q\left(\sum_{m=1}^{k} a_{n,m}f_m\right)^{q-1}\\
&\leqslant h^{q-1}\sum_{n=k}^{\infty} a_{n,k}u_n^q\left(a_{n,k}\sum_{m=1}^{k} f_m+\sum_{m=1}^{k} a_{k,m}f_m\right)^{q-1}\\
&\leqslant h^{q-1}\sum_{n=k}^{\infty} a_{n,k}u_n^q\left(\left(a_{n,k}\sum_{m=1}^{k} f_m\right)^{q-1}+\left(\sum_{m=1}^{k} a_{k,m}f_m\right)^{q-1}\right)\\
&=h^{q-1}\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\left(\sum_{m=1}^{k} f_m\right)^{q-1}+\sum_{n=k}^{\infty} a_{n,k}u_n^q\left(\sum_{m=1}^{k} a_{k,m}f_m\right)^{q-1}\right).
\end{aligned}$$

Putting this expression in (15) and using Minkowski's inequality we obtain

$$\overline{S} = \sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{n,m} f_m \right)^{q-1} \right)^{p'}$$

$$\leqslant h^{(q-1)p'} \left(\left[\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q \left(\sum_{m=1}^{k} f_m \right)^{q-1} + \sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{k,m} f_m \right)^{q-1} \right)^{p'} \right]^{\frac{1}{p'}} \right)^{p'}$$

$$\leqslant h^{(q-1)p'} \left(\left[\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q \right)^{p'} \left(\sum_{m=1}^{k} f_m \right)^{(q-1)p'} \right]^{\frac{1}{p'}} \right.$$

$$\left. + \left[\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \right)^{p'} \left(\sum_{m=1}^{k} a_{k,m} f_m \right)^{(q-1)p'} \right]^{\frac{1}{p'}} \right)^{p'}$$

$$= h^{(q-1)p'} \left(S_1^{\frac{1}{p'}} + S_2^{\frac{1}{p'}} \right)^{p'},$$

i.e,

$$\overline{S} \leqslant h^{(q-1)p'} \left(S_1^{\frac{1}{p'}} + S_2^{\frac{1}{p'}} \right)^{p'}.$$

From (20) and (21) and we have, respectively, the following

$$S_1^{\frac{1}{p'}} \leqslant q^{\frac{1}{p'}} \left(q' \right)^{\frac{q-1}{p'}} B_1^q \|f\|_{p,v}^{q-1}$$

and

$$S_2^{\frac{1}{p'}} \leqslant B_2 S^{\frac{1}{q'}}.$$

According to (16) we get the following estimate:

$$S \leqslant q \|f\|_{l_{p,v}} h^{q-1} \left(q^{\frac{1}{p'}} \left(q' \right)^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^{q-1} + B_2 S^{\frac{1}{q'}} \right), \tag{23}$$

i.e.,

$$S \leqslant h^{q-1} q^{\frac{1}{p'}+1} \left(q' \right)^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^q + h^{q-1} q B_2 \|f\|_{l_{p,v}} S^{\frac{1}{q'}}.$$

Applying Young's inequality to the last term of the above sum, we get the following estimate

$$S \leqslant h^{q-1} q^{\frac{p'+1}{p'}} \left(q' \right)^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^q + \frac{\left(h^{q-1} q B_2 \|f\|_{l_{p,v}} \right)^q}{q} + \frac{S}{q'},$$

i.e,

$$S \leqslant \left(h^{q-1} q^{\frac{2p'+1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} + h^{(q-1)q} q^q\right) B^q \|f\|_{l_{p,v}}^q,$$

where $B = \max\{B_1, B_2\}$. From this we obtain

$$S^{\frac{1}{q}} \leqslant \left(h^{q-1} q^{\frac{2p'+1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} + h^{(q-1)q} q^q\right)^{\frac{1}{q}} B \|f\|_{l_{p,v}} \tag{24}$$

and

$$C \leqslant \left(h^{q-1} q^{\frac{2p'+1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} + h^{(q-1)q} q^q\right)^{\frac{1}{q}} B.$$

This means that the conditions $B_1 < \infty$ and $B_2 < \infty$ are sufficient for the inequality (1) to hold. Using $S^{\frac{1}{q}} \leqslant C\|f\|_{l_{p,v}}$ we rewrite (23) in the following form

$$S \leqslant q\|f\|_{l_{p,v}} h^{q-1} \left(q^{\frac{1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^{q-1} + B_2 \left(C\|f\|_{l_{p,v}}\right)^{q-1}\right),$$

i.e,

$$\frac{S}{\|f\|_{l_{p,v}}^q} \leqslant h^{q-1} \left(q^{\frac{p'+1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} B_1^q + qB_2 C^{q-1}\right),$$

which implies the estimate for the best constant

$$C^q \leqslant h^{q-1} \left(q^{\frac{p'+1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} B_1^q + qB_2 C^{q-1}\right).$$

Consequently, we obtain

$$\frac{C^q}{q^{\frac{p'+1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} B_1^q + qB_2 C^{q-1}} \leqslant h^{q-1}.$$

Let us now consider the function

$$f(x) = \frac{x^q}{q^{\frac{p'+1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} B_1^q + qB_2 x^{q-1}}$$

according to the left side of the estimate. It is to see that this function is monotone increasing and continuous in $(0, \infty)$ and the equation $f(x) = h^{q-1}$ has one positive solution in $(0, \infty)$. If X is a solution of the equation, which means that

$$X^q = h^{q-1} \left(q^{\frac{p'+1}{p'}} \left(q'\right)^{\frac{q-1}{p'}} B_1^q + qB_2 X^{q-1}\right)$$

then $C \leqslant X$. The proof is complete. □

3.2. Proof of theorem 2

Proof. Necessity and lower estimate. Let the Hardy-type inequality (1) hold and C be its best constant. Then choosing the test sequence f_{n,k_0} in (1) as

$$f_{n,k_0} = v_n^{-p'} \chi_{[1,k_0]}(n),$$

we get

$$\left(\sum_{n=1}^{\infty} v_n^p f_{n,k_0}^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{k_0} v_n^p \left(v_n^{-p'}\right)^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{k_0} v_n^{-p'}\right)^{\frac{1}{p}}$$

and

$$\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} a_{n,k} f_{k,k_0}\right)^q\right)^{\frac{1}{q}} = \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} a_{n,k} v_k^{-p'} \chi_{[1,k_0]}(k)\right)^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{n=1}^{k_0} u_n^q \left(\sum_{k=1}^{n} a_{n,k} v_k^{-p'}\right)^q + \sum_{n=k_0+1}^{\infty} u_n^q \left(\sum_{k=1}^{k_0} a_{n,k} v_k^{-p'}\right)^q\right)^{\frac{1}{q}}.$$

Using these in (1) we obtain

$$\left[\sum_{n=1}^{k_0} u_n^q \left(\sum_{k=1}^{n} a_{n,k} v_k^{-p'}\right)^q\right] \left(\sum_{n=1}^{k_0} v_n^{-p'}\right)^{-\frac{q}{p}}$$
$$+ \left[\sum_{n=k_0+1}^{\infty} u_n^q \left(\sum_{k=1}^{k_0} a_{n,k} v_k^{-p'}\right)^q\right] \left(\sum_{n=1}^{k_0} v_n^{-p'}\right)^{-\frac{q}{p}} \leqslant C^q,$$

i.e.,

$$B_3^q(k_0) + \left[\sum_{n=k_0+1}^{\infty} u_n^q \left(\sum_{k=1}^{k_0} a_{n,k} v_k^{-p'}\right)^q\right] \left(\sum_{n=1}^{k_0} v_n^{-p'}\right)^{-\frac{q}{p}} \leqslant C^q.$$

Using the monotonicity of $a_{n,k} - a_{n,k} \geqslant a_{n,k_0}$ for $k \leqslant k_0$ – we have

$$C^q \geqslant B_3^q(k_0) + \left(\sum_{n=k_0+1}^{\infty} a_{n,k_0}^q u_n^q\right) \left(\sum_{n=1}^{k_0} v_n^{-p'}\right)^{\frac{q}{p'}}$$

which implies

$$\sup_{k_0 \geqslant 1} \left(B_3^q(k_0) + \tilde{A}_1^q(k_0+1)\right) \leqslant C^q .$$

Necessity of $B_3 < \infty$ and the first part of (10) have been proved.

Now we prove the rest part. By choosing the test sequence f_{n,k_0} in (1) as

$$f_{n,k_0} = a_{k_0,n}^{p'-1} v_n^{-p'} \chi_{[1,k_0]}(n),$$

we get

$$\left(\sum_{n=1}^{\infty} v_n^p f_{n,k_0}^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{k_0} a_{k_0,n}^{p'} v_n^{-p'}\right)^{\frac{1}{p}} \tag{25}$$

and

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} a_{n,k} f_{k,k_0}\right)^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} a_{n,k} a_{k_0,k}^{p'-1} v_k^{-p'} \chi_{[1,k_0]}(k)\right)^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{n=1}^{k_0} u_n^q \left(\sum_{k=1}^{n} a_{n,k} a_{k_0,k}^{p'-1} v_k^{-p'}\right)^q + \sum_{n=k_0+1}^{\infty} u_n^q \left(\sum_{k=1}^{k_0} a_{n,k} a_{k_0,k}^{p'-1} v_k^{-p'}\right)^q\right)^{\frac{1}{q}}
\end{aligned}$$

[using the monotonicity of $a_{n,k}$ in the first sum as $a_{k_0,k} \geqslant a_{n,k}$ for $k_0 \geqslant n$ and in the second sum as $a_{n,k} \geqslant a_{k_0,k}$ for $n > k_0$ we get]

$$\geqslant \left(\sum_{n=1}^{k_0} u_n^q \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{-p'}\right)^q + \left(\sum_{n=k_0+1}^{\infty} u_n^q\right) \left(\sum_{k=1}^{k_0} a_{k_0,k}^{p'} v_k^{-p'}\right)^q\right)^{\frac{1}{q}}. \tag{26}$$

Taking into account (25) and (26) in (1) we obtain

$$\sum_{n=1}^{k_0} u_n^q \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{-p'}\right)^q + \left(\sum_{n=k_0+1}^{\infty} u_n^q\right) \left(\sum_{k=1}^{k_0} a_{k_0,k}^{p'} v_k^{-p'}\right)^q \leqslant C^q \left(\sum_{n=1}^{k_0} a_{k_0,n}^{p'} v_n^{-p'}\right)^{\frac{q}{p}}.$$

Multiplaying both sides of the estimate by $\left(\sum_{n=k_0}^{\infty} u_n^q\right)^{p'-1}$ we have

$$\begin{aligned}
&\left(\sum_{n=1}^{k_0} u_n^q \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{-p'}\right)^q\right) \left(\sum_{n=k_0}^{\infty} u_n^q\right)^{p'-1} \\
&+ \left(\sum_{n=k_0+1}^{\infty} u_n^q\right) \left(\sum_{n=k_0}^{\infty} u_n^q\right)^{p'-1} \left(\sum_{k=1}^{k_0} a_{k_0,k}^{p'} v_k^{-p'}\right)^q \\
&\leqslant C^q \left(\sum_{n=1}^{k_0} a_{k_0,n}^{p'} v_n^{-p'}\right)^{\frac{q}{p}} \left(\sum_{n=k_0}^{\infty} u_n^q\right)^{p'-1},
\end{aligned}$$

and then

$$\begin{aligned}
&\left(\sum_{n=1}^{k_0} u_n^q \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{-p'}\right)^q\right) \left(\sum_{n=k_0}^{\infty} u_n^q\right)^{p'-1} + \left(\sum_{n=k_0+1}^{\infty} u_n^q\right)^{p'} \left(\sum_{k=1}^{k_0} a_{k_0,k}^{p'} v_k^{-p'}\right)^q \\
&\leqslant C^q \left(\sum_{n=1}^{k_0} a_{k_0,n}^{p'} v_n^{-p'}\right)^{\frac{q}{p}} \left(\sum_{n=k_0}^{\infty} u_n^q\right)^{p'-1},
\end{aligned}$$

i.e.,

$$\tilde{A}_2^{p'q}(k_0+1)+B_4^{p'q}(k_0)\leqslant C^q\overline{A}^{(p'-1)q}(k_0).$$

Using the estimate $\overline{A}\leqslant C$, which was proved in [4] we get

$$B_4^{p'q}(k_0)+\tilde{A}_2^{p'q}(k_0+1)\leqslant C^{p'q}$$

and then

$$\sup_{k_0\in\mathbb{N}}\left(\tilde{A}_2^{p'q}(k_0+1)+B_4^{p'q}(k_0)\right)\leqslant C^{p'q},$$

which proves the second part of (10).

Sufficiency and upper estimate. Let us denote

$$S=\sum_{n=1}^{\infty}v_n^{-p'}\left(\sum_{k=n}^{\infty}a_{k,n}f_k\right)^{p'},$$

then consequently using Lagrange's mean value theorem, Fubini's theorem and Hölder's inequality we get

$$\begin{aligned}
S&=\sum_{n=1}^{\infty}v_n^{-p'}\left(\sum_{m=n}^{\infty}\left[\left(\sum_{k=m}^{\infty}a_{k,n}f_k\right)^{p'}-\left(\sum_{k=m+1}^{\infty}a_{k,n}f_k\right)^{p'}\right]\right)\\
&=p'\sum_{n=1}^{\infty}v_n^{-p'}\left(\sum_{m=n}^{\infty}a_{m,n}f_m\left[\sum_{k=m+1}^{\infty}a_{k,n}f_k+\xi_m a_{m,n}f_m\right]^{p'-1}\right)\\
&\leqslant p'\sum_{n=1}^{\infty}v_n^{-p'}\left(\sum_{m=n}^{\infty}a_{m,n}f_m\left[\sum_{k=m}^{\infty}a_{k,n}f_k\right]^{p'-1}\right)\\
&=p'\sum_{m=1}^{\infty}f_m\left(\sum_{n=1}^{m}a_{m,n}v_n^{-p'}\left[\sum_{k=m}^{\infty}a_{k,n}f_k\right]^{p'-1}\right)\\
&=p'\sum_{m=1}^{\infty}\left(u_m^{-1}f_m\right)\left(u_m\sum_{n=1}^{m}a_{m,n}v_n^{-p'}\left[\sum_{k=m}^{\infty}a_{k,n}f_k\right]^{p'-1}\right)\\
&\leqslant p'\left[\sum_{m=1}^{\infty}u_m^{-q'}f_m^{q'}\right]^{\frac{1}{q'}}\left[\sum_{m=1}^{\infty}u_m^{q}\left(\sum_{n=1}^{m}a_{m,n}v_n^{-p'}\left(\sum_{k=m}^{\infty}a_{k,n}f_k\right)^{p'-1}\right)^{q}\right]^{\frac{1}{q}}\\
&=p'\|f\|_{l_{q',u^{-1}}}\left[\sum_{m=1}^{\infty}u_m^{q}\left(\sum_{n=1}^{m}a_{m,n}v_n^{-p'}\left(\sum_{k=m}^{\infty}a_{k,n}f_k\right)^{p'-1}\right)^{q}\right]^{\frac{1}{q}}\\
&=p'\|f\|_{l_{q',u^{-1}}}\overline{S}^{\frac{1}{q}},
\end{aligned}$$

where $\xi_m \in (0,1)$, $k = 1,\ldots,n$ and

$$\overline{S} = \sum_{m=1}^{\infty} u_m^q \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(\sum_{k=m}^{\infty} a_{k,n} f_k \right)^{p'-1} \right)^q . \tag{27}$$

So we get the following estimate

$$S \leqslant p' \|f\|_{l_{q',u^{-1}}} \overline{S}^{\frac{1}{q}} . \tag{28}$$

Now we estimate $\overline{S}$. For this aim we divide proof into two cases.

Case 1. Let $p' \geqslant 2$. Using Oinarov's condition (5) and Minkowski's inequality we obtain the following estimate for the inner sum of (28)

$$\begin{aligned}
&\left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(\sum_{k=m}^{\infty} a_{k,n} f_k \right)^{p'-1} \right)^{\frac{1}{p'-1}} \\
&\leqslant h \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(a_{m,n} \sum_{k=m}^{\infty} f_k + \sum_{k=m}^{\infty} a_{k,m} f_k \right)^{p'-1} \right)^{\frac{1}{p'-1}} \\
&\leqslant h \left(\left[\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(a_{m,n} \sum_{k=m}^{\infty} f_k \right)^{p'-1} \right]^{\frac{1}{p'-1}} + \left[\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(\sum_{k=m}^{\infty} a_{k,m} f_k \right)^{p'-1} \right]^{\frac{1}{p'-1}} \right) \\
&= h \left(\left(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'} \right)^{\frac{1}{p'-1}} \sum_{k=m}^{\infty} f_k + \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \right)^{\frac{1}{p'-1}} \sum_{k=m}^{\infty} a_{k,m} f_k \right) .
\end{aligned}$$

Using these in (27) and then Minkowski's inequality we have

$$\begin{aligned}
\overline{S} &\leqslant h^{q(p'-1)} \sum_{m=1}^{\infty} u_m^q \left[\left(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'} \right)^{\frac{1}{p'-1}} \sum_{k=m}^{\infty} f_k + \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \right)^{\frac{1}{p'-1}} \sum_{k=m}^{\infty} a_{k,m} f_k \right]^{q(p'-1)} \\
&= h^{q(p'-1)} \Bigg\{ \Bigg(\sum_{m=1}^{\infty} u_m^q \Bigg[\left(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'} \right)^{\frac{1}{p'-1}} \sum_{k=m}^{\infty} f_k \\
&\qquad + \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \right)^{\frac{1}{p'-1}} \sum_{k=m}^{\infty} a_{k,m} f_k \Bigg]^{q(p'-1)} \Bigg)^{\frac{1}{q(p'-1)}} \Bigg\}^{q(p'-1)}
\end{aligned}$$

$$\leqslant h^{q(p'-1)}\left\{\left[\sum_{m=1}^{\infty} u_m^q\left(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'}\right)^q\left(\sum_{k=m}^{\infty} f_k\right)^{q(p'-1)}\right]^{\frac{1}{q(p'-1)}}\right.$$

$$\left.+\left[\sum_{m=1}^{\infty} u_m^q\left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'}\right)^q\left(\sum_{k=m}^{\infty} a_{k,m} f_k\right)^{q(p'-1)}\right]^{\frac{1}{q(p'-1)}}\right\}^{q(p'-1)}$$

$$= h^{q(p'-1)}\left\{S_1^{\frac{1}{q(p'-1)}} + S_2^{\frac{1}{q(p'-1)}}\right\}^{q(p'-1)},$$

i.e.,

$$\overline{S} \leqslant h^{q(p'-1)}\left\{S_1^{\frac{1}{q(p'-1)}} + S_2^{\frac{1}{q(p'-1)}}\right\}^{q(p'-1)}, \tag{29}$$

where

$$S_1 = \sum_{m=1}^{\infty} u_m^q\left(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'}\right)^q\left(\sum_{k=m}^{\infty} f_k\right)^{q(p'-1)},$$

$$S_2 = \sum_{m=1}^{\infty} u_m^q\left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'}\right)^q\left(\sum_{k=m}^{\infty} a_{k,m} f_k\right)^{q(p'-1)}.$$

From (27) and (29) we have

$$S \leqslant p'\|f\|_{l_{q',u^{-1}}}\overline{S}^{\frac{1}{q}} \leqslant p'\|f\|_{l_{q',u^{-1}}} h^{p'-1}\left(S_1^{\frac{1}{q(p'-1)}} + S_2^{\frac{1}{q(p'-1)}}\right)^{p'-1}. \tag{30}$$

Further, we estimate S_1 and S_2 separately. To estimate S_1, we used discrete Hardy inequality (2) of the exponents $\overline{p} := q'$, $\overline{q} := q(p'-1)$ and weights $\overline{u}_m^{\overline{q}} := u_m^q\left(\sum_{n=1}^m a_{m,n}^{p'} v_n^{-p'}\right)^q$, $\overline{v}_m := u_m^{-1}$, i.e,

$$S_1^{\frac{1}{q(p'-1)}} = \left[\sum_{m=1}^{\infty} \overline{u}_m^{\overline{q}}\left(\sum_{k=m}^{\infty} f_k\right)^{\overline{q}}\right]^{\frac{1}{\overline{q}}} \leqslant C_{\overline{p},\overline{q}}\left(\sum_{m=1}^{\infty} v_m^{\overline{p}} f_m^{\overline{p}}\right)^{\frac{1}{\overline{p}}}, \tag{31}$$

since the equivalent condition (3) is satisfied, i.e.,

$$A_1 = \sup_{m\in\mathbb{N}}\left(\sum_{k=1}^{m} u_k^q\left(\sum_{n=1}^{k} a_{k,n}^{p'} v_n^{-p'}\right)^q\right)^{\frac{1}{q(p'-1)}}\left(\sum_{k=m}^{\infty} u_k^{-q}\right)^{\frac{1}{q}} = B_4^p < \infty.$$

Moreover, using (4) we get the following upper estimate for the best constant $C_{\overline{p},\overline{q}}$ in (31):

$$C_{\overline{p},\overline{q}} \leqslant (p')^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p.$$

Therefore,

$$S_1^{\frac{1}{q(p'-1)}} \leqslant (p')^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p \|f\|_{l_{q',u^{-1}}}. \tag{32}$$

Now we estimate S_2. Using consequently Fubini's theorem and Minkowski's inequality, we obtain

$$\begin{aligned}
S_2 &= \sum_{m=1}^{\infty} u_m^q \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'}\right)^q \sum_{s=m}^{\infty} \left[\left(\sum_{k=s}^{\infty} a_{k,s} f_k\right)^{q(p'-1)} - \left(\sum_{k=s+1}^{\infty} a_{k,s+1} f_k\right)^{q(p'-1)}\right] \\
&= \sum_{s=1}^{\infty} \left[\left(\sum_{k=s}^{\infty} a_{k,s} f_k\right)^{q(p'-1)} - \left(\sum_{k=s+1}^{\infty} a_{k,s+1} f_k\right)^{q(p'-1)}\right] \left(\sum_{m=1}^{s} u_m^q \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'}\right)^q\right) \\
&= \sum_{s=1}^{\infty} \left[\left(\sum_{k=s}^{\infty} a_{k,s} f_k\right)^{q(p'-1)} - \left(\sum_{k=s+1}^{\infty} a_{k,s+1} f_k\right)^{q(p'-1)}\right] \left(\sum_{m=1}^{s} v_m^{-p'}\right)^{\frac{q}{p}} B_3^q(s) \\
&\leqslant B_3^q \left(\left[\sum_{s=1}^{\infty} \left[\left(\sum_{k=s}^{\infty} a_{k,s} f_k\right)^{q(p'-1)} - \left(\sum_{k=s+1}^{\infty} a_{k,s+1} f_k\right)^{q(p'-1)}\right] \left(\sum_{m=1}^{s} v_m^{-p'}\right)^{\frac{q}{p}}\right]^{\frac{p}{q}}\right)^{\frac{q}{p}} \\
&\leqslant B_3^q \left(\sum_{m=1}^{\infty} v_m^{-p'} \left(\sum_{s=m}^{\infty} \left[\left(\sum_{k=s}^{\infty} a_{k,s} f_k\right)^{q(p'-1)} - \left(\sum_{k=s+1}^{\infty} a_{k,s+1} f_k\right)^{q(p'-1)}\right]\right)^{\frac{p}{q}}\right)^{\frac{q}{p}} \\
&= B_3^q \left(\sum_{m=1}^{\infty} v_m^{-p'} \left(\sum_{k=m}^{\infty} a_{k,m} f_k\right)^{p'}\right)^{\frac{q}{p}} = B_3^q S^{\frac{q}{p}}.
\end{aligned}$$

From this we have

$$S_2^{\frac{1}{q(p'-1)}} \leqslant B_3^{\frac{1}{p'-1}} S^{\frac{1}{p'}}. \tag{33}$$

Taking into account (32) and (33) in (30) we get

$$S^{\frac{1}{p'-1}} \leqslant (p')^{\frac{1}{p'-1}} \|f\|_{l_{q',u^{-1}}}^{\frac{1}{p'-1}} h\left((p')^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p \|f\|_{l_{q',u^{-1}}} + B_3^{\frac{1}{p'-1}} S^{\frac{1}{p'}}\right),$$

i.e,

$$S^{\frac{1}{p'-1}} \leqslant h(p')^{\frac{q+1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p \|f\|_{l_{q',u^{-1}}}^p + h(p')^{\frac{1}{p'-1}} \|f\|_{l_{q',u^{-1}}}^{\frac{1}{p'-1}} B_3^{\frac{1}{p'-1}} S^{\frac{1}{p'}}. \tag{34}$$

Applying Young's inequality to the second term on the right side as

$$h(p')^{\frac{1}{p'-1}} \|f\|_{l_{q',u^{-1}}}^{\frac{1}{p'-1}} B_3^{\frac{1}{p'-1}} S^{\frac{1}{p'}} \leqslant \frac{\left(h(p')^{\frac{1}{p'-1}} \|f\|_{l_{q',u^{-1}}}^{\frac{1}{p'-1}} B_3^{\frac{1}{p'-1}}\right)^{p'}}{p'} + \frac{S^{\frac{p}{p'}}}{p}$$

we obtain from (34) the following estimate

$$S^{\frac{1}{p'-1}} \leqslant h\left(p'\right)^{\frac{1}{q(p'-1)}+p} p^{\frac{1}{q}} B_4^p \|f\|_{l_{q',u^{-1}}}^p + h^{p'}\left(p'\right)^p B_3^p \|f\|_{l_{q',u^{-1}}}^p,$$

and then

$$S^{\frac{1}{p'}} \leqslant \left(h\left(p'\right)^{\frac{1}{q(p'-1)}+p} p^{\frac{1}{q}} + h^{p'}\left(p'\right)^p\right)^{\frac{1}{p}} B\|f\|_{l_{q',u^{-1}}},$$

where $B = \max\{B_3, B_4\}$.

This means that the conditions $B_3 < \infty$ and $B_4 < \infty$ are sufficient for the inequality (1) to hold. We also get upper estimate for the best constant

$$C \leqslant \left(h\left(p'\right)^{\frac{1}{q(p'-1)}+p} p^{\frac{1}{q}} + h^{p'}\left(p'\right)^p\right)^{\frac{1}{p}} B.$$

Using $S^{\frac{1}{p'}} \leqslant C\|f\|_{l_{q',u^{-1}}}$, we rewrite (34) in the following form

$$S^{\frac{1}{p'-1}} \leqslant h\left(p'\right)^{\frac{1}{p'-1}} \|f\|_{l_{q',u^{-1}}}^p \left(\left(p'\right)^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p + B_3^{\frac{1}{p'-1}} C\right),$$

i.e,

$$\left(\frac{S^{\frac{1}{p'}}}{\|f\|_{l_{q',u^{-1}}}}\right)^p \leqslant h\left(p'\right)^{\frac{1}{p'-1}} \left(\left(p'\right)^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p + B_3^{\frac{1}{p'-1}} C\right).$$

Then from the arbitrary of f we have

$$C^p \leqslant h\left(p'\right)^{\frac{1}{p'-1}} \left(\left(p'\right)^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p + B_3^{\frac{1}{p'-1}} C\right),$$

i.e.,

$$\frac{C^p}{\left(p'\right)^{\frac{1}{p'-1}} \left(\left(p'\right)^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p + B_3^{\frac{1}{p'-1}} C\right)} \leqslant h.$$

Let us consider the function

$$f(x) = \frac{x^p}{\left(p'\right)^{\frac{1}{p'-1}} \left(\left(p'\right)^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p + B_3^{\frac{1}{p'-1}} x\right)}$$

corresponding to the left-hand side of the estimate. It is easy to see that this function is monotone increasing and continuous in $(0,\infty)$, $f(0) = 0$ and $f(\infty) = \infty$, which implies that the equation $f(x) = h$ has exactly one positive solution in $(0,\infty)$. If X is a solution of the equation, i.e.,

$$X^p = h\left(p'\right)^{p-1} \left(\left(p'\right)^{\frac{1}{q(p'-1)}} p^{\frac{1}{q}} B_4^p + B_3^{p-1} X\right)$$

then $C \leqslant X$.

Case 2. Let $1 < p' < 2$. Using Oinarov's condition (5) and the inequality $(X+Y)^{\alpha} \leqslant X^{\alpha} + Y^{\alpha}$, for $0 < \alpha < 1$ we estimate the inner sum of (27) as

$$\begin{aligned}
&\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(\sum_{k=m}^{\infty} a_{k,n} f_k \right)^{p'-1} \\
&\leqslant h^{p'-1} \sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(a_{m,n} \sum_{k=m}^{\infty} f_k + \sum_{k=m}^{\infty} a_{k,m} f_k \right)^{p'-1} \\
&\leqslant h^{p'-1} \left(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'} \left(\sum_{k=m}^{\infty} f_k \right)^{p'-1} + \sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(\sum_{k=m}^{\infty} a_{k,m} f_k \right)^{p'-1} \right).
\end{aligned}$$

Using this estimate in (27), and then applying Minkowski's inequality we get

$$\begin{aligned}
\overline{S} &= \sum_{m=1}^{\infty} u_m^q \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(\sum_{k=m}^{\infty} a_{k,an} f_k \right)^{p'-1} \right)^q \\
&\leqslant h^{q(p'-1)} \sum_{m=1}^{\infty} u_m^q \Bigg(\left(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'} \right) \left(\sum_{k=m}^{\infty} f_k \right)^{p'-1} \\
&\quad + \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \right) \left(\sum_{k=m}^{\infty} a_{k,m} f_k \right)^{p'-1} \Bigg)^q \\
&= h^{q(p'-1)} \Bigg\{ \Bigg[\sum_{m=1}^{\infty} u_m^q \Bigg(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'} \left(\sum_{k=m}^{\infty} f_k \right)^{p'-1} \\
&\quad + \sum_{n=1}^{m} a_{m,n} v_n^{-p'} \left(\sum_{k=m}^{\infty} a_{k,m} f_k \right)^{p'-1} \Bigg)^q \Bigg]^{\frac{1}{q}} \Bigg\}^q \\
&= h^{q(p'-1)} \Bigg\{ \left[\sum_{m=1}^{\infty} u_m^q \left(\sum_{n=1}^{m} a_{m,n}^{p'} v_n^{-p'} \right)^q \left(\sum_{k=m}^{\infty} f_k \right)^{q(p'-1)} \right]^{\frac{1}{q}} \\
&\quad + \left[\sum_{m=1}^{\infty} u_m^q \left(\sum_{n=1}^{m} a_{m,n} v_n^{-p'} \right)^q \left(\sum_{k=m}^{\infty} a_{k,m} f_k \right)^{q(p'-1)} \right]^{\frac{1}{q}} \Bigg\}^q \\
&= h^{q(p'-1)} \left\{ S_1^{\frac{1}{q}} + S_2^{\frac{1}{q}} \right\}^q,
\end{aligned}$$

i.e,

$$\overline{S} \leqslant h^{q(p'-1)} \left\{ S_1^{\frac{1}{q}} + S_2^{\frac{1}{q}} \right\}^q. \tag{35}$$

From (32) and (33) we obtain

$$S_1^{\frac{1}{q}} \leqslant (p')^{\frac{1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} \|f\|_{l_{q',u^{-1}}}^{p'-1}$$

and

$$S_2^{\frac{1}{q}} \leqslant B_3 S^{\frac{1}{p}}.$$

According to (28) and (35) we get the following estimate:

$$S \leqslant p' \|f\|_{l_{q',u^{-1}}} h^{p'-1} \left((p')^{\frac{1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} \|f\|_{q,u}^{p'-1} + B_3 S^{\frac{1}{p}} \right), \tag{36}$$

i.e.,

$$S \leqslant h^{p'-1} (p')^{\frac{1}{q}+1} p^{\frac{p'-1}{q}} B_4^{p'} \|f\|_{l_{q',u^{-1}}}^{p'} + h^{p'-1} p' B_3 \|f\|_{l_{q',u^{-1}}} S^{\frac{1}{p}}.$$

Applying Young's inequality to the second term of the right hand side, we get

$$S \leqslant h^{p'-1} (p')^{\frac{1}{q}+1} p^{\frac{p'-1}{q}} B_4^{p'} \|f\|_{l_{q',u^{-1}}}^{p'} + \frac{\left(h^{p'-1} p' B_3 \|f\|_{l_{q',u^{-1}}}\right)^{p'}}{p'} + \frac{S}{p}$$

and then

$$S \leqslant \left(h^{p'-1} (p')^{\frac{1}{q}+2} p^{\frac{p'-1}{q}} + h^{p'(p'-1)} (p')^{p'} \right) B^{p'} \|f\|_{l_{q',u^{-1}}}^{p'},$$

where $B = \max\{B_3,\ B_4\}$. From this we have

$$S^{\frac{1}{p'}} \leqslant \left(h^{p'-1} (p')^{\frac{1}{q}+2} p^{\frac{p'-1}{q}} + h^{p'(p'-1)} (p')^{p'} \right)^{\frac{1}{p'}} B \|f\|_{l_{q',u^{-1}}},$$

which implies the following estimate for the best constant

$$C \leqslant \left(h^{p'-1} (p')^{\frac{1}{q}+2} p^{\frac{p'-1}{q}} + h^{p'(p'-1)} (p')^{p'} \right)^{\frac{1}{p'}} B.$$

This means that the conditions $B_3 < \infty$ and $B_4 < \infty$ are sufficient for the inequality (1) to hold. Now using $S^{\frac{1}{p'}} \leqslant C \|f\|_{l_{q',u^{-1}}}$ we obtain from (36) the following estimate

$$S \leqslant p' \|f\|_{l_{q',u^{-1}}} h^{p'-1} \left((p')^{\frac{1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} \|f\|_{l_{q',u^{-1}}}^{p'-1} + B_3 C^{p'-1} \|f\|_{l_{q',u^{-1}}}^{p'-1} \right),$$

i.e,

$$\frac{S}{\|f\|_{l_{q',u^{-1}}}^{p'}} \leqslant h^{p'-1} \left((p')^{\frac{q+1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} + p' B_3 C^{p'-1} \right).$$

From the arbitrary of f we have

$$C^{p'} \leqslant h^{p'-1} \left((p')^{\frac{q+1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} + p' B_3 C^{p'-1} \right),$$

which implies

$$\frac{C^{p'}}{(p')^{\frac{q+1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} + p'B_3C^{p'-1}} \leqslant h^{p'-1}.$$

Let us consider the function

$$f(x) = \frac{x^{p'}}{(p')^{\frac{q+1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} + p'B_3x^{p'-1}}.$$

It is easy to see that this function is monotone increasing and continuous in $(0,\infty)$ and the equation $f(x) = h^{p'-1}$ has one positive solution in $(0,\infty)$. If X is a solution of the equation, i.e.,

$$X^{p'} = h^{p'-1}\left((p')^{\frac{q+1}{q}} p^{\frac{p'-1}{q}} B_4^{p'} + p'B_3X^{p'-1}\right)$$

then $C \leqslant X$. The proof is complete. □

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