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ON THE GENERALIZED JORDAN–VON NEUMANN TYPE CONSTANT AND THE FIXED POINT PROPERTY FOR MULTIVALUED NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we show some sufficient conditions on a Banach space X concerning the generalized von Neumann-Jordan type constant $C^{(p)}_{-\infty}(a,X)$, the coefficient $R(1,X)$ and the coefficient of weak orthogonality, which imply the existence of fixed points for multivalued nonexpansive mappings.

1. Introduction

In 1969, Nadler [17] established the multivalued version of Banach contraction principle. A key technology to get fixed point property for multivalued nonexpansive mapping is Edelstein's method of asymptotic centers. For instance, using it T. C. Lim [16] proved that every multivalued nonexpansive self-mapping $T : E \to K(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X. W. A. Kirk and S. Massa [15] proved that if a nonempty bounded closed convex subset E of a Banach space X has a property that the asymptotic center in E of each bounded sequence of X is nonempty and compact, then every multivalued nonexpansive self-mapping $T : E \to KC(E)$ has a fixed point. In 2004, Domínguez and Lorenzo [4] proved that every multivalued nonexpansive mapping $T : E \to KC(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a nearly uniformly convex Banach space X.

In 2006, S. Dhompongsa et al. [8, 9] introduced the Domínguez-Lorenzo condition and property (D) which imply the fixed point property for multivalued nonexpansive mappings. In 2007, T. D. Benavides and Gavira [2] had established the fixed point property for multivalued nonexpansive mappings in terms of the modulus of squareness, universal infinite-dimensional modulus, and Opial modulus. A. Kaewkhao [13] has established the fixed point property for multivalued nonexpansive mappings in terms of the James constant, the Jordan-von Neumann constant, weak orthogonality. In 2010, T. D. Benavides and Gavira [3] had given a survey of this subject and presented the

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main known results and current research directions. For more details about recent work on fixed point property for multivalued nonexpansive mapping, one can refer to [21, 22, 23, 25].

Let X be a Banach space with unit ball $B_X = \{x \in X : \|x\| \leqslant 1\}$. The following constant of a Banach space

$$C_{NJ}(X) = \sup\left\{\frac{\| x+y \|^2 + \| x-y \|^2}{2(\| x \|^2 + \| y \|^2)} : x,y \in X, (x,y) \neq (0,0)\right\},$$

is called the von Neumann-Jordan constant [5], which is widely studied by many authors [2, 3, 8, 17].

In order to promote the results of $C_{NJ}(X)$, Dhompangsa in [7] introduced the constant $C_{NJ}(a,X)$, for $a \geqslant 0$.

$$C_{NJ}(a,X) = \sup\left\{\frac{\| x+y \|^2 + \| x-z \|^2}{2 \| x \|^2 + \| y \|^2 + \|z\|^2} : x,y,z \in X\right\},$$

where $(x,y,z) \neq (0,0,0)$, and $\|y-z\| \leqslant a\|x\|$. It is clear that $C_{NJ}(0,X) = C_{NJ}(X)$.

Cui [6] and Dinarvand [10] introduced the constant $C_{NJ}^{(p)}(X)$ and $C_{NJ}^{(p)}(a,X)$, respectively, and gave some sufficient conditions for the normal structure, where $a \geqslant 0$, $1 \leqslant p < \infty$.

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\| x+y \|^p + \| x-y \|^p}{2^{p-1}(\| x \|^p + \| y \|^p)} : x,y \in X, (x,y) \neq (0,0)\right\}.$$

$$C_{NJ}^{(p)}(a,X) = \sup\left\{\frac{\| x+y \|^p + \| x-z \|^p}{2^{p-2} \| x \|^p + 2^{p-3}(\| y \|^p + \|z\|^p)} : x,y,z \in X\right\},$$

where $(x,y,z) \neq (0,0,0)$, and $\|y-z\| \leqslant a\|x\|$. It was proved that the generalized von Neumann-Jordan constant satisfies the inequality $C_{NJ}^{(p)}(X) \leqslant 2$, and that Banach space X is uniformly non-square if and only if $C_{NJ}^{(p)}(X) < 2$ (see [6]). If $C_{NJ}^{(p)}(X) < 1 + \frac{1}{\mu(X)^p}$, then the Banach space X has normal structure (see [20]).

To further describe the geometric properties of Banach space, such as uniform non-square and normal structure, the constants $C_{-\infty}(X), C_{-\infty}^{(p)}(X)$ and $C_{-\infty}(a,X)$ were introduced, respectively in [18, 24, 25], where $a \geqslant 0$, $1 \leqslant p < \infty$.

$$C_{-\infty}(X) = \sup\left\{\frac{\min\left\{ \| x+y \|^2, \| x-y \|^2 \right\}}{2(\| x \|^2 + \| y \|^2)} : x,y \in X, (x,y) \neq (0,0)\right\}.$$

$$C_{-\infty}^{(p)}(X) = \sup\left\{\frac{\min\left\{ \| x+y \|^p, \| x-y \|^p \right\}}{2^{p-1}(\| x \|^p + \| y \|^p)} : x,y \in X, (x,y) \neq (0,0)\right\}.$$

$$C_{-\infty}(a,X) = \sup\left\{\frac{\min\left\{ \| x,y \|^2, \| x-z \|^2 \right\}}{2 \| x \|^2 + \| y \|^2 + \|z\|^2} : x,y,z \in X\right\},$$

where $(x,y,z) \neq (0,0,0)$, and $\|y-z\| \leqslant a\|x\|$.

Inspired by the above results, in 2024, Tang et al. [19] introduced a new generalized Jordan-von Neumann type constant $C_{-\infty}^{(p)}(a,X)$,

$$C_{-\infty}^{(p)}(a,X) = \sup\left\{ \frac{\min\left\{ \| x+y \|^p, \| x-z \|^p \right\}}{2^{p-2} \| x \|^p + 2^{p-3}(\| y \|^p + \|z\|^p)} : x,y,z \in X \right\},$$

where $(x,y,z) \neq (0,0,0)$, and $\|y-z\| \leqslant a\|x\|$. It is clear this definition is equivalent to

$$C_{-\infty}^{(p)}(a,X) = \sup\left\{ \frac{\min\left\{ \| x+y \|^p, \| x-z \|^p \right\}}{2^{p-2} \| x \|^p + 2^{p-3}(\| y \|^p + \|z\|^p)} : x,y,z \in B_X \right\},$$

where $(x,y,z) \neq (0,0,0)$, and $\|y-z\| \leqslant a\|x\|$. They analyzed some properties of this constant, and gave some sufficient conditions for normal structure.

2. Preliminaries

The following coefficient is defined by T. D. Benavides [1] as

$$R(1,X) = \sup\{\liminf_{n\to\infty} \| x_n + x \|\},$$

where the supremum is taken over all $x \in X$ with $\| x \| \leqslant 1$ and all weakly null sequences (x_n) in the unit ball B_X such that

$$D[(x_n)] := \limsup_{n\to\infty}(\limsup_{m\to\infty} \| x_n - x_m \|) \leqslant 1.$$

It is clear that $1 \leqslant R(1,X) \leqslant 2$. Some geometric condition sufficient for normal structure in term of this coefficient have been studied in [11, 20].

The coefficient of weak orthogonality $\mu(X)$, defined by the infimum of the set of real numbers $\lambda > 0$ such that

$$\limsup_{n\to\infty} \| x + x_n \| \leqslant \lambda \limsup_{n\to\infty} \| x - x_n \|$$

for all $x \in X$ and all weakly null sequences (x_n) in X [12].

Let C be a nonempty subset of a Banach space X. We shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : C \to CB(X)$ is said to be nonexpansive if

$$H(Tx,Ty) \leqslant \| x-y \|,$$

for all $x,y \in C$, where $H(.,.)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,\ B) := \max\{\sup_{x\in A}\inf_{y\in B} \| x-y \|, \sup_{y\in B}\inf_{x\in A} \| x-y \|\},\ A,\ B \in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r(C,\{x_n\})$ and the asymptotic center $A(C,\{x_n\})$ of $\{x_n\}$ in C are defined by

$$r(C,\{x_n\}) = \inf\{\limsup_n \| x_n - x \| \, x \in C\},$$

and

$$A(C,\{x_n\}) = \{x \in C : \limsup_n \| x_n - x \| = r(C,\{x_n\})\},$$

respectively. It is known that $A(C,\{x_n\})$ is a nonempty weakly compact convex set whenever C is. The sequence $\{x_n\}$ is called regular with respect to C if $r(C,\{x_n\}) = r(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C,\{x_n\}) = A(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) = \inf_{x\in C}\sup_{y\in D} \| x - y \| .$$

S. Dhompongsa et al. [9] introduced the property (D) if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset C of X, any sequence $\{x_n\} \subset C$ which is regular asymptotically uniform relative to C, and any sequence $\{y_n\} \subset A(C,\{x_n\})$ which is regular asymptotically uniform relative to X we have

$$r(C,\{y_n\}) \leqslant \lambda r(C,\{x_n\}).$$

The Domínguez-Lorenzo condition ((DL)-condition,in short) introduced in [8] is defined as follows: if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C,

$$r_C(A(C,\{x_n\})) \leqslant \lambda r(C,\{x_n\}).$$

It is clear from the definition that property (D) is weaker than the (DL)-condition. The next results shows that property (D) is stronger than weak normal structure and also implies the existence of fixed points for multivalued nonexpansive mappings [9]: Let X be a Banach space satisfying ((DL)-condition) property (D), then X has weak normal structure; Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies ((DL)-condition) the property (D). Let $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.

3. The generalized Jordan-von Neumann type constant and the coefficient $R(1,X)$

In this section, we show a sufficient condition concerning the generalized von Neumann-Jordan constant, and the coefficient $R(1,X)$, which implies the existence of fixed points for multivalued nonexpansive mappings.

First recall some basic facts about ultrapowers. Let $\mathscr{F}$ be a filter on $\mathbb{N}$. A sequence $\{x_n\}$ in X converges to x with respect to $\mathscr{F}$, denoted by $\lim_{\mathscr{F}} x_n = x$ if for each neighborhood U of x, $\{n\in\mathbb{N}\}\in\mathscr{F}$. A filter U on $\mathbb{N}$ is called to be an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $A : A\in\mathbb{N},\ n_0\in A$ for some fixed $n_0\in\mathbb{N}$, otherwise, it is called nontrivial. Let $l_\infty(X)$ denotes that the subspace of the product space $\Pi_{n\in\mathbb{N}}X$ equipped with the norm $\|(x_n)\|:=\sup_{n\in\mathbb{N}}\|x_n\|<\infty$. Let $\mathscr{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathscr{U}}=\{(x_n)\in l_\infty(X):\lim_{\mathscr{U}}\|x_n\|=0\}.$$

The ultrapower of X, denoted by $\tilde{X}$, is the quotient space $l_\infty(X)/N_{\mathscr{U}}$ equipped with the quotient norm, and $(x_n)_{\mathscr{U}}$ denotes the elements of the ultrapower. Note that if $\mathscr{U}$ is non-trivial, then X can be embedded into $\tilde{X}$ isometrically. It was shown that if the space X is super-reflexive, then X has uniformly structure if and if $\tilde{X}$ has normal structure (see [14]).

THEOREM 1. (Main) *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ is a bounded sequence in C regular with respect to C. Then for every $a\in[0,2]$, we have*

$$r_C(A(C,\{x_n\}))\leqslant\frac{2^{\frac{p-1}{p}}R(1,X)(C_{-\infty}^{(p)}(a,X))^{\frac{1}{p}}}{R(1,X)+1}r(C,\{x_n\}).$$

Proof. Denote $r(C,\{x_n\})$ as r and $A(C,\{x_n\})$ as A. We should assume that $r>0$, by passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x\in C$ and $d=\lim_{n\neq m}\|x_n-x_m\|$ exists. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Observe that the norm is weakly lower semicontinuous, we have

$$\liminf_n\|x_n-x\|\leqslant\liminf_n\liminf_m\|x_n-x_m\|=\lim_{n\neq m}\|x_n-x_m\|=d.$$

Let $\varepsilon>0$, taking a subsequence if necessary, we can assume that $\|x_n-x\|<d+\varepsilon$ for all n. Let $z\in A$, then we have $\limsup_n\|x_n-z\|=r$ and $\|x-z\|\leqslant\liminf_n\|x_n-z\|\leqslant r$. Denote $R=R(1,X)$, then by definition we have

$$R\geqslant\liminf_n\left\|\frac{x_n-x}{d+\varepsilon}+\frac{z-x}{r}\right\|=\liminf_n\left\|\frac{x_n-x}{d+\varepsilon}-\frac{z-x}{r}\right\|.$$

By the convexity of C, we have $\frac{R-1}{R+1}x+\frac{2}{R+1}z\in C$, since the norm is weakly lower semicontinuous, we get

$$\begin{aligned}
&\liminf_n \Big\| \frac{x_n - z}{r} + \frac{1}{R}\Big(\frac{x_n - x}{d+\varepsilon} - \frac{x-z}{r}\Big)\Big\| \\
&= \liminf_n \Big\| \Big(\frac{1}{r} + \frac{1}{R(d+\varepsilon)}\Big)(x_n - x) + \Big(\frac{1}{r} - \frac{1}{Rr}\Big)x - \Big(\frac{1}{r} - \frac{1}{Rr}\Big)z\Big\| \\
&\geqslant \Big\| \frac{R-1}{Rr}x + \frac{2}{Rr}z - \frac{R+1}{Rr}z\Big\| \\
&= \frac{R+1}{Rr}\Big\| \frac{R-1}{R+1}x + \frac{2}{R+1}z - z\Big\| \\
&\geqslant \Big(1 + \frac{1}{R}\Big)\frac{r_C(A)}{r},
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_n \Big\| \frac{x_n - z}{r} - \frac{1-a}{R}\Big(\frac{x_n - x}{d+\varepsilon} - \frac{x-z}{r}\Big)\Big\| \\
&\geqslant \Big\| \Big(\frac{1}{r} - \frac{1-a}{R(d+\varepsilon)}\Big)(x_n - x) + \Big(1 + \frac{1-a}{R}\Big)\frac{x-z}{r}\Big\| \\
&\geqslant \Big(1 + \frac{1-a}{R}\Big)\frac{r_C(A)}{r}.
\end{aligned}$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

1. $\| x_N - z \| \leqslant r + \varepsilon$;
2. $\| \frac{(x_N - x)}{d+\varepsilon} - \frac{x-z}{r} \| \leqslant R\Big(\frac{r+\varepsilon}{r}\Big)$;
3. $\| R(x_N - z) + \frac{r(x_N - x)}{d+\varepsilon} - (x-z) \| \geqslant (R+1) r_C(A)(\frac{r-\varepsilon}{r})$;
4. $\Big\| R(x_N - z) - (1-a)\Big(\frac{r(x_N - x)}{d+\varepsilon} - (x-z)\Big)\Big\| \geqslant (R+1-a) r_C(A)(\frac{r-\varepsilon}{r})$.

Now, let $\tilde{u} = (\frac{x_N - z}{r+\varepsilon})_{\mathscr{U}}$, $\tilde{v} = \frac{1}{R(r+\varepsilon)}(\frac{r(x_N - x)}{d+\varepsilon} - (x-z))_{\mathscr{U}}$ and $\tilde{w} = \frac{1-a}{R(r+\varepsilon)}(\frac{r(x_N - x)}{d+\varepsilon} - (x-z))_{\mathscr{U}}$. Using the above estimates, we obtain that all of $\tilde{u}, \tilde{v}$ and $\tilde{w}$ belong to B_X, and $\|\tilde{v} - \tilde{w}\| \leqslant a\|\tilde{u}\|$. Then,

$$\begin{aligned}
\| \tilde{u} + \tilde{v} \| &= \frac{1}{R(r+\varepsilon)}\Big\| R(x_N - z) + \frac{r(x_N - x)}{d+\varepsilon} - (x-z)\Big\| \\
&\geqslant \Big(1 + \frac{1}{R}\Big)\frac{r_C(A)}{r}\Big(\frac{r-\varepsilon}{r+\varepsilon}\Big),
\end{aligned}$$

$$\begin{aligned}
\| \tilde{u} - \tilde{w} \| &= \frac{1}{R(r+\varepsilon)}\Big\| R(x_N - z) - (1-a)\Big(\frac{r(x_N - x)}{d+\varepsilon} - (x-z)\Big)\Big\| \\
&\geqslant \Big(1 + \frac{1-a}{R}\Big)\frac{r_C(A)}{r}\Big(\frac{r-\varepsilon}{r+\varepsilon}\Big).
\end{aligned}$$

Notice that $a \in [0,2]$ and $1 \leqslant R \leqslant 2$, then $\left(1+\frac{1-a}{R}\right) \leqslant 1+\frac{1}{R}$. By the definition of $C_{-\infty}^{(p)}(a,\tilde{X})$, we have

$$C_{-\infty}^{(p)}(a,\tilde{X}) \geqslant \frac{\| \tilde{u}+\tilde{v} \|^p \wedge \| \tilde{u}-\tilde{v} \|^p}{2^{p-2} \| \tilde{u} \|^p + 2^{p-3}(\| \tilde{v} \|^p + \| \tilde{w} \|^p)}$$
$$\geqslant \left(\frac{1}{2^{p-1}}\left(\frac{r_C(A)}{r}\right)^p\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p\right)\left(1+\frac{1-a}{R}\right)^p.$$

Since the above inequality is true for every $\varepsilon > 0$ and $C_{-\infty}^{(p)}(a,X) = C_{-\infty}^{(p)}(a,\tilde{X})$ (see Lemma 2 in [19]), we obtain that

$$r_C(A(C,\{x_n\})) \leqslant \frac{2^{\frac{p-1}{p}} R(1,X)(C_{-\infty}^{(p)}(a,X))^{\frac{1}{p}}}{(R(1,X)+1-a)} r(C,\{x_n\}). \quad \square$$

COROLLARY 1. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{-\infty}^{(p)}(a,X) < \frac{1}{2^{p-1}}\left(1+\frac{1-a}{R(1,X)}\right)^p$ and $T: C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

Proof. If $C_{-\infty}^{(p)}(a,X) < \frac{1}{2^{p-1}}\left(1+\frac{1-a}{R(1,X)}\right)^p$, then X satisfy the (DL)-condition by Theorem 1, so T has a fixed point. $\square$

COROLLARY 2. *Let X be a Banach space such that $C_{-\infty}^{(p)}(a,X) < \frac{1}{2^{p-1}}\left(1+\frac{1-a}{R(1,X)}\right)^p$. Then X has normal structure.*

Proof. By Theorem 1, it is easy to prove that X has weak normal structure. Since $1 \leqslant R(1,X) \leqslant 2$, we obtain $C_{-\infty}^{(p)}(a,X) < \frac{1}{2^{p-1}}\left(1+\frac{1-a}{R(1,X)}\right)^p < 2$. This implies that X is uniformly nonsquare, then X is reflexive, therefore weakly normal structure coincide with normal structure. $\square$

4. The generalized Jordan-von Neumann type constant and the coefficient of weak orthogonality

In this section, we show a sufficient condition concerning the generalized von Neumann-Jordan constant, and the coefficient of weak orthogonality, which implies the existence of fixed points for multivalued nonexpansive mappings.

THEOREM 2. *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ is a bounded sequence in C regular with respect to C. Then for every $a \in [0,2]$, we have*

$$r_C(A(C,\{x_n\})) \leqslant \frac{2^{\frac{p-3}{p}} \mu [C_{-\infty}^{(p)}(a,X)(2\mu^p + 1 + |1-a|^p)]^{\frac{1}{p}}}{\mu^2+1} r(C,\{x_n\}).$$

Proof. Denote $r(C,\{x_n\})$ as r, $A(C,\{x_n\})$ as A and $\mu(X)$ as μ, respectively. We can assume that $r>0$, by passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x\in C$. Let $z\in A$, then,

$$\limsup_n \| x_n - z \| = r, \qquad \|x-z\| \leqslant r.$$

By the definition of r, we have

$$\begin{aligned}\limsup_n \|x_n - 2x + z\| &= \limsup_n \|(x_n - x) + (z-x)\|\\ &\leqslant \limsup_n \|(x_n - x) - (z-x)\|\\ &= \mu r.\end{aligned}$$

Convexity of C implies that $\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z \in C$, and by the definition of r, we obtain that

$$\limsup_n \left\| x_n - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\| \geqslant r.$$

On the other hand, by the weakly lower semicontinuity of the norm, we get

$$\liminf_n \| (\mu^2 - 1 + a)(x_n - x) - (\mu^2 + 1 - a)(z-x) \| \geqslant |\mu^2 + 1 - a| \, \| z - x \| .$$

For every $\varepsilon > 0$, there exists $N\in\mathbb{N}$ such that

1. $\| x_N - z \| \leqslant r + \varepsilon$;
2. $\| x_N - 2x + z \| \leqslant \mu(r+\varepsilon)$;
3. $\left\| x_N - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\| \geqslant r - \varepsilon$;
4. $\| (\mu^2 - 1 + a)(x_N - x) - (\mu^2 + 1 - a)(z - x) \| \geqslant (\mu^2+1) \, \| z - x \| \, (\frac{r-\varepsilon}{r})$.

Now, let $u = \mu^2(x_N - z)$, $v = (x_N - 2x + z)$ and $w = (1-a)(x_N - 2x + z)$, respectively, using the above estimates, we obtain that $\| u \| \leqslant \mu^2(r+\varepsilon)$, $\| v \| \leqslant \mu(r+\varepsilon)$, $\| v \| \leqslant \mu(1-a)(r+\varepsilon)$ and $\|v-w\| \leqslant a\|u\|$. Thus,

$$\begin{aligned}\| u + v \| &= \| \mu^2((x_N - x) - (z-x)) + (x_N - x) + (z - x) \|\\ &= (\mu^2+1)\left\|(x_N - x) - \frac{\mu^2-1}{\mu^2+1}(z-x)\right\|\\ &\geqslant (\mu^2+1)\left\|x_N - \left(\frac{2}{\mu^2+1}x + \frac{\mu^2-1}{\mu^2+1}z\right)\right\|\\ &\geqslant (\mu^2+1)(r-\varepsilon),\end{aligned}$$

$$\begin{aligned}\| u - v \| &= \left\|\mu^2((x_N - x) - (z-x)) - (1-a)\Big((x_N - x) + (z-x)\Big)\right\|\\ &= \| (\mu^2 - 1 + a)(x_N - x) - (\mu^2 + 1 - a)(z - x) \|\\ &\geqslant (\mu^2+1) \, \| z - x \| \left(\frac{r-\varepsilon}{r}\right).\end{aligned}$$

Since $\|z-x\| \leqslant r$, we have $(\mu^2+1)\parallel z-x\parallel \left(\frac{r-\varepsilon}{r}\right) \leqslant (\mu^2+1)(r-\varepsilon)$. By the definition of $C^{(p)}_{-\infty}(a,X)$, we get

$$C^{(p)}_{-\infty}(a,X) \geqslant \frac{\min\{\parallel u+v\parallel^p, \parallel u-v\parallel^p\}}{2^{p-2}\parallel u\parallel^p + 2^{p-3}(\parallel v\parallel^p + \|w\|^p)}$$
$$\geqslant \left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p \left(\frac{\|z-x\|}{r}\right)^p \frac{(\mu^2+1)^p}{2^{p-3}\mu^p(2\mu^p+1+|1-a|^p)}.$$

Let $\varepsilon \to 0^+$, we obtain

$$\parallel z-x\parallel \leqslant \frac{2^{\frac{p-3}{p}}\mu[C^{(p)}_{-\infty}(a,X)(2\mu^p+1+|1-a|^p)]^{\frac{1}{p}}}{\mu^2+1} r.$$

Since this inequality holds for arbitrary $z \in A$, we obtain that

$$r_C(A) \leqslant \frac{2^{\frac{p-3}{p}}\mu[C^{(p)}_{-\infty}(a,X)(2\mu^p+1+|1-a|^p)]^{\frac{1}{p}}}{\mu^2+1} r. \quad \square$$

COROLLARY 3. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C^{(p)}_{-\infty}(a,X) < \frac{(\mu^2+1)^p}{2^{p-3}\mu^p(2\mu^p+1+|1-a|^p)}$ and let $T : C \to KC(C)$ be a multivalued nonexpansive mapping. Then T has a fixed point.*

Proof. If $C^{(p)}_{-\infty}(a,X) < \frac{(\mu^2+1)^p}{2^{p-3}\mu^p(2\mu^p+1+|1-a|^p)}$, then by Theorem 2, X satisfies the (DL)-condition, then T has a fixed point. $\square$

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