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BERNSTEIN—DOETSCH THEOREM FOR GEOMETRIC CONVEX FUNCTIONS

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Abstract. The celebrated Bernstein-Doetsch Theorem was under consideration of many researchers for last ten decades. In the present paper we give this Theorem in the geometric convexity. The characterization of the geometric convexity is given as well. Finally, we show that the Jensen geometric convex function can be extended to geometric convex.

1. Introduction

The celebrated Bernstein–Doetsch [1] Theorem is very useful to characterize the continuity of Jensen-convex functions. It states that if a Jensen-convex function is bounded at a point then it is continuous. This characterization of continuity is very strong because boundedness at a single point implies the continuity. Many authors generalized this Theorem to many notions of convexity (see c.f. [2–5, 8, 10, 11]). For more details bout the convexity and their application (see [12], [14] and [15]) In the present paper our first result we employ the locally boundedness of G-Jensen convex function to prove the Bernstein–Doetsch Theorem in the context of G-convexity in Section 2. In Section 3 firstly, we show that the G-Jensen convexity and G-convexity are equivalent when $t \in [0,1] \cap \mathbb{Q}$. Secondly we give two characterizations of G-convexity. In Section 4 extended the G-Jensen convexity to G-convexity.

Throughout the present paper $\mathbb{R}_+$ and I denote the set of positive real numbers and an open subinterval of $\mathbb{R}_+$, respectively.

Montel (see Niculescu [9]) gave the definition of the notion of geometric convexity. This notion of convexity has many applications in branches of Mathematics, for instance, functional equations, inequalities, statistics and optimization. Recall that a function $f : I \to \mathbb{R}_+$ is geometrically convex, shortly G-convex, if the inequality

$$f(x^{1-t}y^t) \leqslant f(x)^{1-t} f(y)^t \qquad (t \in [0,1]) \tag{1}$$

holds for $x, y \in I$.

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A function $f : I \to \mathbb{R}_+$ is said to be geometrically Jensen convex, shortly G-Jensen convex if the inequality

$$f(\sqrt{xy}) \leqslant \sqrt{f(x)f(y)} \qquad (x,y \in I) \tag{2}$$

holds. Let I_a and I_b be two subintervals of $\mathbb{R}_+$, we define the interval

$$I_a \cdot I_b := \{a \cdot b : a \in I_a, b \in I_b\}.$$

2. Bernstein-Doetsch Theorem

In this section we prove a counterpart of the celebrated Bernstein–Doetsch Theorem [1] in the setting of geometric convexity. The next theorem is the our first result that shows a function is a continuous G-convex if and only if it is a continuous G-Jensen convex.

The next theorem shows that if G-Jensen convex is locally bounded above at a point in the domain then it is locally bounded in all the domain. This Theorem is a counterpart of [6, Theorem 6.2.1]

THEOREM 2.1. *Let a function $f : I \to \mathbb{R}_+$ be a G-Jensen convex. If f is a locally bounded from above at an arbitrary point of I then it is locally bounded on I.*

Proof. Assume that f is locally bounded from above at a point call it $p \in I$. First, we show that f is locally bounded from above on I.

Define a sequence of intervals I_n by

$$I_0 := \{p\}, \qquad I_{n+1} := \sqrt{I_n \cdot I}.$$

For $n = 0$ we have that

$$I_1 = \sqrt{I_0 \cdot I}.$$

For $n = 1$ we have that

$$I_2 = \sqrt{I_1 \cdot I} = \sqrt{\sqrt{I_0 I}I}.$$

For $n = 2$ we have that

$$I_3 = \sqrt{I_2 \cdot I} = \sqrt{\sqrt{\sqrt{I_0 I}I}I} = (I_0 I)^{\frac{1}{8}} I^{\frac{1}{4}} I^{\frac{1}{2}}.$$

It follows by induction that

$$I_n = I_0^{\frac{1}{2^n}} I^{\sum_{i=1}^{n} \frac{1}{2^i}} = I_0^{\frac{1}{2^n}} I^{\frac{1}{2}\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}} = p^{\frac{1}{2^n}} I^{1-\frac{1}{2^n}}.$$

By induction we prove that f is locally bounded from above at each point of I_n. By assumption we have that f is locally bounded from above at p. Assume that f is locally bounded from above at each point of I_n. For $x \in I_{n+1}$ and $x_0 \in I_n$, there exists

$y_0 \in I$ such that $x = \sqrt{x_0 y_0}$. By induction we have that there exists a constant $M_0 \geqslant 0$ such that $f(x_0) \leqslant M_0$. In front of the inequality (2) we have that

$$f(x) = f(\sqrt{x_0 y_0}) \leqslant \sqrt{f(x_0) f(y_0)} \leqslant \sqrt{M_0 f(y_0)} := M \qquad (x \in I_{n+1}).$$

Therefore f is locally bounded from above on I_{n+1}. Now we show that

$$I = \bigcup_{n=1}^{\infty} I_n. \tag{3}$$

Assume that $x \in I$ is an arbitrary point, define the sequence x_n by

$$x_n = p^{\frac{-1}{2^n}} x^{\frac{2^n}{2^n - 1}}.$$

Note that $\lim_{n\to\infty} x_n = x$.

The interval I is open, therefore $x_n \in I$ for some n. Hence

$$x = p^{\frac{1}{2^n}} x_n^{1-\frac{1}{2^n}} \in p^{\frac{1}{2^n}} I^{1-\frac{1}{2^n}} = I_n.$$

This proves (3). This shows that f is bounded from above on I.

Now we show that f is locally bounded from below. Let $q \in I$ be arbitrary. For $x \in I$ assume that $y := \frac{q^2}{x}$, this implies that $q = \sqrt{xy}$. Now apply inequality (2) for the point q we have that

$$f(q) \leqslant \sqrt{f(x) f(y)}.$$

Since f is locally bounded above therefore there exists $K > 0$ such that $f(y) \leqslant K$. Hence the above inequality yields that

$$f(x) \geqslant \frac{f(q)^2}{f(y)} \geqslant \frac{f(q)^2}{K} =: M.$$

Thus f is locally bounded below at any point of I. $\square$

Next result is a counterpart of the celebrated Theorem of Bernstein–Doetsch [1] in the G-convexity setting.

THEOREM 2.2. *Let a function $f : I \to \mathbb{R}_+$ be a G-Jensen convex. If f is a locally bounded above at an arbitrary point of I then it is continuous on I.*

Proof. Regarding to Theorem 2.1 If f is locally bounded from above at a point on I then it is locally bounded on I. Let M_f and m_f be defined by

$$m_f(x) = \lim_{r_+ \to 0} \inf_{U(x,r)} f(x)$$

and

$$M_f(x) = \lim_{r_+ \to 0} \sup_{U(x,r)} f(x),$$

where $U(x,r)$ is an open ball of x with radius r.

Clearly, we have that

$$m_f(x) \leqslant f(x) \leqslant M_f(x) \qquad (x \in I). \tag{4}$$

Regarding to locally boundedness of f on I, m_f and M_f take finite numbers for every $x \in I$.

Take an arbitrary point $x \in I$. There exists a nonzero sequence of points x_n in I for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} x_n = x \qquad \text{and} \qquad \lim_{n\to\infty} f(x_n) = m_f(x), \tag{5}$$

and a sequence z_n in I such that

$$\lim_{n\to\infty} z_n = x \qquad \text{and} \qquad \lim_{n\to\infty} f(z_n) = M_f(x), \tag{6}$$

Put

$$y_n = \frac{z_n^2}{x_n}$$

Since $x_n \neq 0$ for all $n \in \mathbb{N}$, therefore also $x \neq 0$. Using (5) and (6) we have that

$$\lim_{n\to\infty} y_n = \frac{\lim_{n\to\infty} z_n^2}{\lim_{n\to\infty} x_n} = \frac{x^2}{x} = x.$$

Furthermore, apply the inequality (2) for the points of sequence

$$z_n = \sqrt{x_n y_n} \tag{7}$$

we have that

$$f(z_n)^2 \leqslant f(x_n) f(y_n)$$

Take limsup to this inequality. In view of the positivity of $m_f(x)$ and $M_f(x)$ we can apply the second inequality in (4) therefore we have that

$$M_f^2(x) \leqslant m_f(x) \limsup_{n\to\infty} f(y_n) \leqslant m_f(x) M_f(x)$$

This inequality implies that

$$M_f(x) \leqslant m_f(x) \qquad (x \in I).$$

Regarding to the two inequalities in (4), we have that

$$M_f(x) = m_f(x) \qquad (x \in I).$$

This equality is sufficient and necessary condition for continuity of f for all $x \in I$.

If $x_n = 0$ for some $n \in \mathbb{N}$ then the equality (7) shows that $z_n = 0$. Thus the equations in (5) and (6) yield that $x = 0$ and $M(0) = m(0)$, this proves that f is continuous at 0. □

It is important to mention that Theorem 2.1 is a consequence of the remark given by Niculescu [9, p. 156] and also combining this remark with the standard Bernstein-Doetsch theorem implies Theorem 2.2.

3. Characterization of G-convexity

In this section we characterize a G-convex function by some inequalities. We need the next lemma in the sequel.

LEMMA 3.1. *If a function* $f:\mathbb{R}_+\to\mathbb{R}_+$ *is a* G*-Jensen convex then for* $n\in\mathbb{N}$ *and* $x_1,\dots,x_n\in\mathbb{R}_+$ *one has*

$$f\big((x_1\dots x_n)^{\frac{1}{n}}\big)\leqslant\big(f(x_1)\dots f(x_n)\big)^{\frac{1}{n}}. \tag{8}$$

Proof. By induction in $p\in\mathbb{N}$ the inequality (2) implies that

$$f((x_1\dots x_{2^p})^{\frac{1}{2^p}})\leqslant(f(x_1)\dots f(x_{2^p}))^{\frac{1}{2^p}}. \tag{9}$$

Indeed, this inequality is valid for $p=1$, assume that it is also valid for p. Now we prove it is valid for $p+1$.

$$\begin{aligned}
f((x_1\dots x_{2^{p+1}})^{\frac{1}{2^{p+1}}}) &= f((x_1\dots x_{2^p}\cdot x_{2^p+1}\dots x_{2^{p+1}})^{\frac{1}{2^{p+1}}})\\
&\leqslant f((x_1\dots x_{2^p})^{\frac{1}{2^p}})^{\frac{1}{2}}\cdot f((x_{2^p+1}\dots x_{2^{p+1}})^{\frac{1}{2^p}})^{\frac{1}{2}}\\
&\leqslant (f(x_1)^{\frac{1}{2^p}}\dots f(x_{2^p})^{\frac{1}{2^p}})^{\frac{1}{2}}\cdot(f(x_{2^p+1})^{\frac{1}{2^p}}\dots f(x_{2^{p+1}})^{\frac{1}{2^p}})^{\frac{1}{2}}\\
&= f(x_1)^{\frac{1}{2^{p+1}}}\dots f(x_{2^p})^{\frac{1}{2^{p+1}}}\cdot f(x_{2^p+1})^{\frac{1}{2^{p+1}}}\dots f(x_{2^{p+1}})^{\frac{1}{2^{p+1}}}\\
&= f(x_1)^{\frac{1}{2^{p+1}}}\dots f(x_{2^{p+1}})^{\frac{1}{2^{p+1}}}
\end{aligned}$$

Let $n\in\mathbb{N}$ and choose p such that $n<2^p$. Pick points $x_1,\dots,x_n$ from $\mathbb{R}_+$ and put

$$x_k=(x_1\dots x_n)^{\frac{1}{n}}\qquad(k=n+1,\dots,2^p). \tag{10}$$

We claim that

$$(x_1\dots x_{2^p})^{\frac{1}{2^p}}=(x_1\dots x_n)^{\frac{1}{n}}. \tag{11}$$

Indeed, using (10), it follows that

$$\begin{aligned}
(x_1\dots x_{2^p})^{\frac{1}{2^p}} &= (x_1\dots x_n\cdot x_{n+1}\dots x_{2^p})^{\frac{1}{2^p}}\\
&= (x_1\dots x_n\cdot x_k^{2^p-n})^{\frac{1}{2^p}}\\
&= (x_1\dots x_n\cdot(x_1\dots x_n)^{\frac{2^p-n}{n}})^{\frac{1}{2^p}}\\
&= (x_1\dots x_n)^{\frac{1}{2^p}}\cdot(x_1\dots x_n)^{\frac{1}{n}-\frac{1}{2^p}}\\
&= (x_1\dots x_n)^{\frac{1}{n}}.
\end{aligned}$$

Now applying (11) and (9), yields that

$$\begin{aligned} f\Big((x_1\dots x_n)^{\frac{1}{n}}\Big) &= f\Big((x_1\dots x_{2^p})^{\frac{1}{2^p}}\Big) \\ &\leqslant \big(f(x_1)\dots f(x_{2^p})\big)^{\frac{1}{2^p}} \\ &= \big(f(x_1)\dots f(x_n)f(x_{n+1})\dots f(x_{2^p})\big)^{\frac{1}{2^p}} \\ &= \big(f(x_1)\dots f(x_n)f((x_1\dots x_n)^{\frac{1}{n}})\dots f((x_1\dots x_n)^{\frac{1}{n}})\big)^{\frac{1}{2^p}} \\ &= \big(f(x_1)\dots f(x_n)f((x_1\dots x_n)^{\frac{1}{n}})^{2^p-n}\big)^{\frac{1}{2^p}} \end{aligned}$$

This inequality implies that

$$f\Big((x_1\dots x_n)^{\frac{1}{n}}\Big)^{2^p} \leqslant f(x_1)\dots f(x_n)f\big((x_1\dots x_n)^{\frac{1}{n}}\big)^{2^p-n}.$$

This inequality follows that

$$1 \leqslant f(x_1)\dots f(x_n)f\big((x_1\dots x_n)^{\frac{1}{n}}\big)^{-n}.$$

This shows that

$$f\big((x_1\dots x_n)^{\frac{1}{n}}\big)^n \leqslant f(x_1)\dots f(x_n).$$

This proves (8). □

The next theorem shows that a G-Jensen convex function satisfies the inequality (1) in the rational t. This result is a counterpart of the result given by Kuczma [6, Theorem 5.3.5].

THEOREM 3.2. *If a function $f:\mathbb{R}_+\to\mathbb{R}_+$ is G-Jensen convex then for $x,y\in\mathbb{R}_+$ one has*

$$f(x^{1-t}y^t) \leqslant f(x)^{1-t}f(y)^t \qquad (t\in[0,1]\cap\mathbb{Q}). \tag{12}$$

Proof. Since f is G-Jensen convex therefore Lemma 3.1 implies that the inequality

$$f\big((x_1\dots x_n)^{\frac{1}{n}}\big) \leqslant \big(f(x_1)\dots f(x_n)\big)^{\frac{1}{n}}.$$

Let $t=\frac{k}{n}$, where $k\in\mathbb{N}$ and $n\in\mathbb{N}$ with $k<n$. Put $x_1=\ldots=x_k=y$ and $x_{k+1}=\ldots=x_n=x$ in the above inequality one has

$$f\big((x^{n-k}y^k)^{\frac{1}{n}}\big) \leqslant \big(f(x)^{n-k}f(y)^k\big)^{\frac{1}{n}}.$$

which shows that the inequality (12) is valid. □

Now we give the characterization theorem of G-convexity. This theorem is a counterpart of the result given by Makó and Páles [7].

THEOREM 3.3. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be an arbitrary function. Then the following assertions are equivalent:*

(i) f *is a* G*-convex,*

(ii) for $x,y,u \in \mathbb{R}_+$ *with* $x<u<y$ *the inequality*

$$\left(\frac{f(u)}{f(x)}\right)^{\frac{1}{\log(u)-\log(x)}} \leqslant \left(\frac{f(y)}{f(u)}\right)^{\frac{1}{\log(y)-\log(u)}} \tag{13}$$

holds.

(iii) There exists a function $a:\mathbb{R}_+\to\mathbb{R}$ *such that*

$$\frac{f(x)}{f(u)} \geqslant a(u)^{\log(x)-\log(u)} \qquad (x,u\in I). \tag{14}$$

Proof. $(i) \Longrightarrow (ii)$ Let f is G-convex and $x<u<y$ be arbitrary in $\mathbb{R}_+$. Choose $t\in[0,1]$ such that $u:=x^t y^{1-t}$ therefore we have that

$$\log(u) = t\log(x) + (1-t)\log(y).$$

Let

$$t := \frac{\log(y)-\log(u)}{\log(y)-\log(x)}, \qquad \text{this yields that} \qquad 1-t = \frac{\log(u)-\log(x)}{\log(y)-\log(x)}.$$

Since f is G-convex therefore applying the inequality (1), we have that

$$f(u) \leqslant f(x)^t f(y)^{1-t} = f(x)^{\frac{\log(y)-\log(u)}{\log(y)-\log(x)}} f(y)^{\frac{\log(u)-\log(x)}{\log(y)-\log(x)}}.$$

This inequality implies that

$$f(u)^{\log(y)-\log(x)} \frac{f(u)^{-\log(u)}}{f(u)^{-\log(u)}} \leqslant f(x)^{\log(y)-\log(u)} f(y)^{log(u)-\log(x)}$$

After some calculations for this inequality, we get

$$\left(\frac{f(u)}{f(x)}\right)^{\log(y)-\log(u)} \leqslant \left(\frac{f(y)}{f(u)}\right)^{\log(u)-\log(x)}.$$

Raise this inequality to power $\frac{1}{(\log(y)-\log(u))(\log(u)-\log(x))}$ we arrive at the inequality (13).

$(ii) \Longrightarrow (iii)$ For $u\in\mathbb{R}_+$ define

$$a(u) := \inf_{y\in I, u<y} \left(\frac{f(y)}{f(u)}\right)^{\frac{1}{\log(y)-\log(u)}}.$$

Therefore by (13), we obtain that

$$\left(\frac{f(u)}{f(x)}\right)^{\frac{1}{\log(u)-\log(x)}} \leqslant a(u) \leqslant \left(\frac{f(y)}{f(u)}\right)^{\frac{1}{\log(y)-\log(u)}} \tag{15}$$

for all $x<u<y$. Clearly, for $x=u$ the inequality (14) is valid. For $x<u$ raise the left hand side of this inequality to power $\log(u)-\log(x)$ implies that

$$\frac{f(u)}{f(x)} \leqslant a(u)^{\log(u)-\log(x)}.$$

This inequality yields that

$$\frac{f(x)}{f(u)} \geqslant a(u)^{\log(x)-\log(u)}.$$

This proves (14) for $x<u$. For $u<x$ raise the right hand of the inequality (15) to power $\log(y)-\log(u)$ and put $y=x$ we get (14).

$(iii) \Longrightarrow (i)$ Let $xy \in I$ and $t \in [0,1]$. The inequality (14) with $x=y$ yields that

$$\frac{f(y)}{f(u)} \geqslant a(u)^{\log(y)-\log(u)} \qquad (y,u \in I). \tag{16}$$

Raise (14) to power t and (16) to power $1-t$ and multiply side by side the results so obtained we conclude that

$$\left(\frac{f(y)}{f(u)}\right)^{1-t}\left(\frac{f(x)}{f(u)}\right)^{t} \geqslant a(u)^{(1-t)(\log(y)-\log(u))}a(u)^{t(\log(x)-\log(u))} \qquad (y,u \in I).$$

This inequality is equivalent to

$$\frac{f(y)}{f(u)}\left(\frac{f(u)}{f(y)}\right)^{t}\left(\frac{f(x)}{f(u)}\right)^{t} \geqslant a(u)^{(1-t)(\log(y)-\log(u))}a(u)^{t(\log(x)-\log(u))} \qquad (y,u \in I).$$

Put $u := x^t y^{1-t}$ in this inequality, we obtain that

$$\begin{aligned}\frac{f(x)^t f(y)^{1-t}}{f(x^t y^{1-t})} &\geqslant a(x^t y^{1-t})^{(1-t)(\log(y)-\log(x^t y^{1-t}))}a(x^t y^{1-t})^{t(\log(x)-\log(x^t y^{1-t}))}\\ &= a(x^t y^{1-t})^{-t(1-t)(\log(x)-\log(y))}a(x^t y^{1-t})^{t(1-t)(\log(x)-\log(y))} = 1.\end{aligned}$$

Therefore (1) holds, hence f is G-convex. □

4. Extended G-Jensen convexity to G-convexity

In this section we prove that a G-Jensen convex function can be admit G-convex function in its domain. This result is a counterpart of the result given by Páles [13]. To prove the mention result we need the next lemma.

LEMMA 4.1. *Let D be a dense subset of I and $f : D \to \mathbb{R}_+$ be an arbitrary function. Then for any $[a,b] \subset I$, there exists $L>0$ such that*

$$\frac{f(x)}{f(y)} \leqslant L^{\log\left(\frac{x}{y}\right)} \qquad (x,y \in [a,b] \cap D). \tag{17}$$

Then f admits an extension $g : I \to \mathbb{R}_+$ such that g locally satisfies (17).

Proof. Choose an arbitrary point $x \in I$ and let $\{x_n\} \subset D$ be a sequence converging to x. Then there exists a compact subinterval $[a,b]$ of I contains all points of $\{x_n\}$. We choose a monotone subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Now applying (17) for $\{x_{n_k}\}$, we have that

$$\frac{f(x_{n_k})}{f(x_{m_k})} \leqslant L^{\log\left(\frac{x_{n_k}}{y_{m_k}}\right)} \qquad (n,m \in \mathbb{N}).$$

Upon taking $\lim_{k\to\infty}$ we obtain that the sequence $\{f(x_{n_k})\}$ is bounded, since it is monotone, therefore it converges to some function say it is $g : I \to \mathbb{R}_+$. The limit g does not depend on $\{x_n\}$ and it is easy one can see that $f(x) = g(x)$ (because we can choose $x_n = x$).

Now we show that g satisfies (17). Let $x,y \in [a,b]$ be arbitrary and let $\{x_n\}$ and $\{y_n\}$ be convergent sequences in $[a,b] \cap D$ such that $x_n \to x$ and $y_n \to y$. Apply (17) we get that

$$\frac{f(x_n)}{f(y_n)} \leqslant L^{\log\left(\frac{x}{y}\right)}.$$

Let n goes to infinity we get that

$$\frac{g(x)}{g(y)} \leqslant L^{\log\left(\frac{x}{y}\right)}.$$

This proves that g satisfies (17) locally on I. $\square$

Now we give the main result of this section.

THEOREM 4.2. *If $f : \mathbb{R}_+ \cap \mathbb{Q} \to \mathbb{R}_+$ is a G-Jensen convex then there exists a G-convex $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g|_{\mathbb{R}_+ \cap \mathbb{Q}} = f$.*

Proof. Since f is G-Jensen convex, therefore in view of Theorem 3.2, it follows that

$$f(x^t y^{1-t}) \leqslant f(x)^t f(y)^{1-t} \qquad (x,y \in \mathbb{R}_+ \cap \mathbb{Q}, t \in [0,1] \cap \mathbb{Q}). \tag{18}$$

Apply (13), we get that

$$\left(\frac{f(u)}{f(x)}\right)^{\frac{1}{\log(u)-\log(x)}} \leqslant \left(\frac{f(y)}{f(u)}\right)^{\frac{1}{\log(y)-\log(u)}} \qquad (x,u,y \in \mathbb{R}_+ \cap \mathbb{Q}, x < u < y). \tag{19}$$

Let $[a,b] \subset \mathbb{R}_+$ be arbitrary and choose $b', b'' \in \mathbb{R}_+ \cap \mathbb{Q}$ such that $a < b < b' < b''$. Now using the inequality (19) several times for $x,y \in [a,b] \cap \mathbb{Q}$ with $x < y$ we obtain that

$$\left(\frac{f(y)}{f(x)}\right)^{\frac{1}{\log(y)-\log(x)}} \leqslant \left(\frac{f(b')}{f(y)}\right)^{\frac{1}{\log(b')-\log(y)}} \leqslant \left(\frac{f(b'')}{f(b')}\right)^{\frac{1}{\log(b'')-\log(b')}} := \alpha.$$

This implies that

$$\frac{f(y)}{f(x)} \leqslant \alpha^{\log(y)-\log(x)} = \alpha^{\log(\frac{y}{x})}.$$

Now apply Lemma 4.1 with $D = \mathbb{R}_+ \cap \mathbb{Q}$ and $L = \alpha$ we get that there exists a continuous extension $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g = f|_{\mathbb{R}_+ \cap \mathbb{Q}}$. Using the density of $\mathbb{R}_+ \cap \mathbb{Q}$ in $\mathbb{R}_+$, $[0,1] \cap \mathbb{Q}$ in $[0,1]$ and continuity of g, it follows that g satisfies the inequality (18) for all $x, y \in \mathbb{R}_+$ and $t \in [0,1]$. Therefore f is G-convex. □

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