Journal of
Mathematical
Inequalities
Volume 19, Number 4 (2025), 1155–1170 doi:10.7153/jmi-2025-19-74

ON A STEVIĆ–SHARMA TYPE OPERATOR FROM DIRICHLET–ZYGMUND–TYPE SPACE TO BLOCH–TYPE SPACE

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(Communicated by M. Krnić)

Abstract. The boundedness, essential norm and compactness of a Stević-Sharma-type operator from Dirichlet-Zygmund-type space into Bloch-type space are investigated in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The class $H(\mathbb{D})$ denotes the set of all analytic functions on $\mathbb{D}$, while $S(\mathbb{D})$ is the family of all analytic self-maps of $\mathbb{D}$. Denote by $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Suppose that $0 < p < \infty$ and $\alpha > -1$. The Dirichlet-type space $\mathscr{D}^p_\alpha$ is defined as the set of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|^p_{\mathscr{D}^p_\alpha} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dxdy$ is the normalized Lebesgue area measure. For $p \geqslant 1$, $\mathscr{D}^p_\alpha$ is a Banach space under the norm $\|\cdot\|_{\mathscr{D}^p_\alpha}$. When $\alpha > p-1$, $\mathscr{D}^p_\alpha$ coincides with the weighted Bergman space $\mathscr{A}^p_{\alpha-p}$. For $\alpha = p-2$ and $p > 1$, $\mathscr{D}^p_{p-2}$ is known as the Besov space.

The Dirichlet-Zygmund-type space $\mathscr{Z}^p_\alpha$, which was introduced in [42], consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{\mathscr{Z}^p_\alpha} = |f(0)|^p + |f'(0)|^p + \int_{\mathbb{D}} |f''(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

If $\alpha > p-1$, $\mathscr{Z}^p_\alpha$ reduces to $\mathscr{D}^p_{\alpha-p}$. For $\alpha = p-2$, $\mathscr{Z}^p_{p-2}$ is called the Besov-Zygmund-type space (studied in [7,45]). Note that $\mathscr{Z}^1_0$ is the minimal Möbius invariant

Mathematics subject classification (2020): Primary 47B38; Secondary 47B33, 30H30.
Keywords and phrases: Stević-Sharma-type operator, Dirichlet-Zygmund-type space, Bloch-type space, essential norm.
This work was supported by the National Natural Science Foundation of China (No. 12301023), the Key Scientific Research Projects of the Higher Education Institutions of Henan Province (No. 26B110004) and the Natural Science Foundation of Henan (No. 242300420631).
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space B_1, which includes functions of the form $f(z)=\sum_{n=1}^{\infty} b_n \sigma_{\lambda_n}(z) \in H(\mathbb{D})$, where $\{b_n\} \in l^1$, $\lambda_n \in \mathbb{D}$ and $\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$. The norm is given by

$$\|f\|_{B_1}=\inf\left\{\sum_{n=1}^{\infty}|b_n| : f(z)=\sum_{n=1}^{\infty} b_n \sigma_{\lambda_n}(z)\right\}.$$

See [5] for more results on the minimal Möbius invariant space. We primarily focus on the case $p-2<\alpha \leqslant p-1$.

Let μ be a radial weight, i.e., a strictly positive continuous function on $\mathbb{D}$ that is radial (meaning $\mu(z)=\mu(|z|)$ for all $z \in \mathbb{D}$). The Bloch-type space $\mathscr{B}_\mu$ is defined as the set of all $f \in H(\mathbb{D})$ satisfying $\sup_{z\in\mathbb{D}} \mu(z)|f'(z)|<\infty$. Equipped with the norm

$$\|f\|_{\mathscr{B}_\mu}=|f(0)|+\sup_{z\in\mathbb{D}} \mu(z)|f'(z)|,$$

$\mathscr{B}_\mu$ becomes a Banach space. When $\mu(z)=(1-|z|^2)^\alpha$ where $\alpha>0$, the induced space $\mathscr{B}_\mu$ becomes the α-Bloch space, which in the case $\alpha=1$ reduces to the classical Bloch space. For further investigations on the classical Bloch space, the α-Bloch space and Bloch-type spaces, as well as some concrete operators on them, see for instance [6, 13, 14, 16, 21, 22, 26, 41, 43].

Let $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. The multiplication and composition operators are defined respectively by $M_\psi f=\psi \cdot f$ and $C_\varphi f=f\circ\varphi$ for $f \in H(\mathbb{D})$. Their product forms the weighted composition operator $W_{\psi,\varphi} f=\psi\cdot f\circ\varphi$, which has been extensively studied. Additionally, the differentiation operator D, defined by $Df(z)=f'(z)$ for $f\in H(\mathbb{D})$, holds significant importance in operator theory and numerous other areas of mathematics.

The first papers on product-type operators including the differentiation operator dealt with the operators DC_φ and $C_\varphi D$, (see, for example, [12, 15, 16, 20, 23–25]). In [32, 33], Stević et al. introduced the following so-called Stević-Sharma operator

$$T_{u,v,\varphi} f(z)=u(z)f(\varphi(z))+v(z)f'(\varphi(z)), \quad f\in H(\mathbb{D}),$$

where $u,v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. By taking some specific choices of the involving symbols, we can obtain the general product-type operators:

$$\begin{gathered} M_u C_\varphi=T_{u,0,\varphi}, \quad C_\varphi M_u=T_{u\circ\varphi,0,\varphi}, \quad M_u D=T_{0,u,id}, \quad DM_u=T_{u',u,id}, \quad C_\varphi D=T_{0,1,\varphi}, \\ DC_\varphi=T_{0,\varphi',\varphi}, \quad M_u C_\varphi D=T_{0,u,\varphi}, \quad M_u DC_\varphi=T_{0,u\varphi',\varphi}, \quad C_\varphi M_u D=T_{0,u\circ\varphi,\varphi}, \\ DM_u C_\varphi=T_{u',u\varphi',\varphi}, \quad C_\varphi DM_u=T_{u'\circ\varphi,u\circ\varphi,\varphi}, \quad DC_\varphi M_u=T_{\varphi'(u'\circ\varphi),\varphi'(u\circ\varphi),\varphi}. \end{gathered}$$

Consequently, the Stević-Sharma operator holds particular significance and has aroused great interest of experts (see, for instance, [4, 19, 37, 38, 40, 43] and the references therein).

In [34], Stević et al. introduced the following product-type operator

$$T^n_{u,v,\varphi} f(z)=u(z)f^{(n)}(\varphi(z))+v(z)f^{(n+1)}(\varphi(z)), \quad n\in\mathbb{N}_0, \tag{1}$$

and investigated its boundedness and compactness from a general space to Bloch-type space. Subsequently, Abbasi in [1] studied the boundedness, compactness and essential norm of $T^n_{u,v,\varphi}$ from Hardy space to nth weighted-type space. Abbasi and Zhu et al. in [3,44] characterized the boundedness, compactness and essential norm of $T^n_{u,v,\varphi}$ from or to Zygmund-type space. The second author et al. investigated the boundedness and compactness of $T^n_{u,v,\varphi}$ from Hardy space [9] and $Q_k(p,q)$ space [11] to Zygmund-type space or Bloch-type space. Since the publication of [34] have also appeared several extensions of operator (1) on the unit disc, as well as on the unit ball (see, for example, [2, 10, 27–31, 35, 36]).

In [10], we investigated the generalized Stević-Sharma type operator defined as:

$$T^{m,n}_{u,v,\varphi}f(z) = u(z)f^{(m)}(\varphi(z)) + v(z)f^{(n)}(\varphi(z)), \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N}. \tag{2}$$

Without loss of generality, we assume $m < n$. Note that when $m = 0$ and $n = 1$, this operator reduces to the classical Stević-Sharma operator. The boundedness, essential norm and compactness of $T^{m,n}_{u,v,\varphi}$ acting from the derivative Hardy spaces into Zygmund-type spaces were investigated in [10]. Furthermore, [39] characterized the boundedness and compactness of $T^{m,n}_{u,v,\varphi}$ from H^∞ space into Bloch-type spaces.

In this study, we focus on analyzing the boundedness, essential norm, and compactness of the Stević-Sharma type operator $T^{m,n}_{u,v,\varphi}$ when mapping from Dirichlet-Zygmund-type spaces $\mathscr{Z}^p_\alpha$ to Bloch-type spaces $\mathscr{B}_\mu$.

Recall that the essential norm of a bounded linear operator $T : X \to Y$ is the distance from T to the compact operators $K : X \to Y$, that is,

$$\|T\|_{e,X\to Y} = \inf\Big\{\|T-K\|_{X\to Y} : K \text{ is compact}\Big\}.$$

Here X and Y are Banach spaces.

Throughout this paper, for nonnegative quantities X and Y, we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ if there exists a positive constant C independent of X and Y such that $X \leqslant CY$. Moreover, we write $X \approx Y$ if $X \lesssim Y \lesssim X$.

2. Preliminary results

In this section, we present several lemmas that will be utilized in the proofs of the main results. The first lemma is established in [42].

LEMMA 1. *Suppose that $1 \leqslant p < \infty$ and $p-2 < \alpha \leqslant p-1$. Then for any $f \in \mathscr{Z}^p_\alpha$, $\|f\|_\infty \lesssim \|f\|_{\mathscr{Z}^p_\alpha}$ and*

$$|f^{(i)}(z)| \lesssim \frac{\|f\|_{\mathscr{Z}^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}+i-2}}, \quad i \in \mathbb{N}.$$

For any $w \in \mathbb{D}$ and $j \in \mathbb{N}$, set

$$f_{j,w}(z) = \frac{(1-|w|^2)^j}{(1-\overline{w}z)^{\frac{\alpha+2}{p}+j-2}}, \quad z \in \mathbb{D}. \tag{3}$$

By [42, Lemma 2.2], we have $f_{j,w} \in \mathscr{Z}_\alpha^p$ and $\sup_{w\in\mathbb{D}} \|f_{j,w}\|_{\mathscr{Z}_\alpha^p} \lesssim 1$ for every $j \in \mathbb{N}$, where $1 \leqslant p < \infty$ and $p-2 < \alpha \leqslant p-1$. Moreover, we easily see that $f_{j,w}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$.

LEMMA 2. *Let $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $m+1 < n$. For any $w \in \mathbb{D}\setminus\{0\}$ and $i,k \in \{m,m+1,n,n+1\}$, there exists a function $g_{i,w} \in \mathscr{Z}_\alpha^p$ such that*

$$g_{i,w}^{(k)}(w) = \frac{\overline{w}^k \delta_{ik}}{(1-|w|^2)^{\frac{\alpha+2}{p}+k-2}},$$

where δ_{ik} is the Kronecker delta.

Proof. For any $w \in \mathbb{D}\setminus\{0\}$ and constants c_1, c_2, c_3, c_4, let

$$g_w(z) = \sum_{j=1}^{4} c_j f_{j,w}(z),$$

where $f_{j,w}$ is defined in (3). For each $i \in \{m,m+1,n,n+1\}$, the system of linear equations

$$\begin{cases} g_w^{(m)}(w) = \frac{\overline{w}^m}{(1-|w|^2)^{\frac{\alpha+2}{p}+m-2}} \sum_{j=1}^{4} c_j \prod_{k=0}^{m-1}(\frac{\alpha+2}{p}+j-2+k) = \frac{\overline{w}^m \delta_{im}}{(1-|w|^2)^{\frac{\alpha+2}{p}+m-2}} \\ g_w^{(m+1)}(w) = \frac{\overline{w}^{m+1}}{(1-|w|^2)^{\frac{\alpha+2}{p}+m-1}} \sum_{j=1}^{4} c_j \prod_{k=0}^{m}(\frac{\alpha+2}{p}+j-2+k) = \frac{\overline{w}^{m+1} \delta_{i(m+1)}}{(1-|w|^2)^{\frac{\alpha+2}{p}+m-1}} \\ g_w^{(n)}(w) = \frac{\overline{w}^n}{(1-|w|^2)^{\frac{\alpha+2}{p}+n-2}} \sum_{j=1}^{4} c_j \prod_{k=0}^{n-1}(\frac{\alpha+2}{p}+j-2+k) = \frac{\overline{w}^n \delta_{in}}{(1-|w|^2)^{\frac{\alpha+2}{p}+n-2}} \\ g_w^{(n+1)}(w) = \frac{\overline{w}^{n+1}}{(1-|w|^2)^{\frac{\alpha+2}{p}+n-1}} \sum_{j=1}^{4} c_j \prod_{k=0}^{n}(\frac{\alpha+2}{p}+j-2+k) = \frac{\overline{w}^{n+1} \delta_{i(n+1)}}{(1-|w|^2)^{\frac{\alpha+2}{p}+n-1}} \end{cases}$$

has a unique solution $c_{i,j}, j \in \{1,2,3,4\}$ that is independent of w, where $\prod_{k=0}^{m-1}(\frac{\alpha+2}{p}+j-2+k) = 1$ if $m=0$. Since the determinant of coefficient matrix M equals

$$\begin{aligned}&\Big(\frac{\alpha+2}{p}-1\Big)\Big(\frac{\alpha+2}{p}\Big)^2\Big(\frac{\alpha+2}{p}+1\Big)^3\Big(\frac{\alpha+2}{p}+2\Big)^4\cdots\Big(\frac{\alpha+2}{p}+m-2\Big)^4\\ &\cdot\Big(\frac{\alpha+2}{p}+m-1\Big)^3\Big(\frac{\alpha+2}{p}+m\Big)^2\cdots\Big(\frac{\alpha+2}{p}+n-2\Big)^2\Big(\frac{\alpha+2}{p}+n-1\Big)\\ &\cdot(n-m)^2(n-m-1)^2 \neq 0.\end{aligned}$$

Meanwhile, M equals

$$\Big(\frac{\alpha+2}{p}+1\Big)\Big(\frac{\alpha+2}{p}+2\Big)^2\cdots\Big(\frac{\alpha+2}{p}+n-2\Big)^2\Big(\frac{\alpha+2}{p}+n-1\Big)n^2(n^2-1) \neq 0,$$

if $m=0$.

For such chosen numbers $c_{i,j}, j \in \{1,2,3,4\}$ the function

$$g_{i,w}(z) := \sum_{j=1}^{4} c_{i,j} f_{j,w}(z)$$

satisfies the desired conditions. □

Similar to [42, Lemma 2.7], we obtain the result below. One can consult [18] for more research.

LEMMA 3. *Let $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and μ be a radial weight such that the operator $T^{m,n}_{u,v,\varphi} : \mathscr{Z}^p_\alpha \to \mathscr{B}_\mu$ is bounded. Then $T^{m,n}_{u,v,\varphi} : \mathscr{Z}^p_\alpha \to \mathscr{B}_\mu$ is compact if and only if $\|T^{m,n}_{u,v,\varphi} f_k\|_{\mathscr{B}_\mu} \to 0$ as $k \to \infty$ for each sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathscr{Z}^p_\alpha$ bounded in norm which converges to zero uniformly in $\overline{\mathbb{D}}$ as $k \to \infty$.*

3. Main results

In this section, we first present necessary and sufficient conditions for the boundedness of the Stević-Sharma type operator $T^{m,n}_{u,v,\varphi} : \mathscr{Z}^p_\alpha \to \mathscr{B}_\mu$ under different cases involving the parameters m and n. To simplify the notation, we adopt the following conventions:

$$\begin{aligned} E_m(z) &= |u'(z)|, \\ E_{m+1}(z) &= |u(z)\varphi'(z)|, \\ E_n(z) &= |v'(z)|, \\ E_{n+1}(z) &= |v(z)\varphi'(z)|. \end{aligned}$$

THEOREM 1. *Let $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$, $m, n \in \mathbb{N}$, $m+1 < n$, μ be a radial weight and I denote the set $\{m, m+1, n, n+1\}$. Then the following statements are equivalent.*

(i) *The operator $T^{m,n}_{u,v,\varphi} : \mathscr{Z}^p_\alpha \to \mathscr{B}_\mu$ is bounded.*

(ii)

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu} < \infty \quad \textit{and} \quad \sum_{i\in I} \sup_{z\in\mathbb{D}} \mu(z) E_i(z) < \infty,$$

where $f_{j,w}$ are defined in (3).

(iii)

$$\sum_{i\in I} \sup_{z\in\mathbb{D}} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} < \infty.$$

Proof. (i)⇒(ii). Assume that $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded. Since for each $w \in \mathbb{D}$ and $j \in \{1,2,3,4\}$, $\|f_{j,w}\|_\infty \lesssim 1$, we have $\sup_{w\in\mathbb{D}} \|T_{u,v,\varphi}^{m,n} f_{j,w}\|_{\mathscr{B}_\mu} < \infty$, and consequently

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T_{u,v,\varphi}^{m,n} f_{j,w}\|_{\mathscr{B}_\mu} < \infty.$$

Taking $f_m(z) = z^m \in \mathscr{Z}_\alpha^p$, from the boundedness of $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$, we get

$$\infty > \|T_{u,v,\varphi}^{m,n} f_m\|_{\mathscr{B}_\mu} \geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T_{u,v,\varphi}^{m,n} f_m)'(z)| = \sup_{z\in\mathbb{D}} \mu(z)E_m(z)m!,$$

which yields

$$\sup_{z\in\mathbb{D}} \mu(z)E_m(z) < \infty. \tag{4}$$

Applying the operator $T_{u,v,\varphi}^{m,n}$ to $f_{m+1}(z) = z^{m+1} \in \mathscr{Z}_\alpha^p$ we have

$$\begin{aligned}\infty > \|T_{u,v,\varphi}^{m,n} f_{m+1}\|_{\mathscr{B}_\mu} &\geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T_{u,v,\varphi}^{m,n} f_{m+1})'(z)|\\ &= \sup_{z\in\mathbb{D}} \mu(z)|E_m(z)\varphi(z)(m+1)! + E_{m+1}(z)(m+1)!|\\ &\geqslant \sup_{z\in\mathbb{D}} \mu(z)E_{m+1}(z)(m+1)! - \sup_{z\in\mathbb{D}} \mu(z)E_m(z)\varphi(z)(m+1)!,\end{aligned}$$

from which along with (4) and the fact that $|\varphi(z)| < 1$ it follows that

$$\sup_{z\in\mathbb{D}} \mu(z)E_{m+1}(z) < \infty. \tag{5}$$

By using the function $f_n(z) = z^n \in \mathscr{Z}_\alpha^p$, we obtain

$$\begin{aligned}\infty >&\|T_{u,v,\varphi}^{m,n} f_n\|_{\mathscr{B}_\mu} \geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T_{u,v,\varphi}^{m,n} f_n)'(z)|\\ =&\sup_{z\in\mathbb{D}} \mu(z)\left|E_m(z)\varphi(z)^{n-m}\frac{n!}{(n-m)!} + E_{m+1}(z)\varphi(z)^{n-m-1}\frac{n!}{(n-m-1)!} + E_n(z)n!\right|,\end{aligned}$$

from which along with (4), (5), the triangle inequality and the fact that $|\varphi(z)| < 1$ gives

$$\sup_{z\in\mathbb{D}} \mu(z)E_n(z) < \infty. \tag{6}$$

Taking $f_{n+1}(z) = z^{n+1} \in \mathscr{Z}_\alpha^p$, similarly we have

$$\begin{aligned}\infty >&\|T_{u,v,\varphi}^{m,n} f_{n+1}\|_{\mathscr{B}_\mu} \geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T_{u,v,\varphi}^{m,n} f_{n+1})'(z)|\\ =&\sup_{z\in\mathbb{D}} \mu(z)\bigg|E_m(z)\varphi(z)^{n-m+1}\frac{(n+1)!}{(n-m+1)!} + E_{m+1}(z)\varphi(z)^{n-m}\frac{(n+1)!}{(n-m)!}\\ &+ E_n(z)\varphi(z)(n+1)! + E_{n+1}(z)(n+1)!\bigg|,\end{aligned}$$

from which along with (4), (5), (6), the triangle inequality and the fact that $|\varphi(z)|<1$ it follows that

$$\sup_{z\in\mathbb{D}}\mu(z)E_{n+1}(z)<\infty. \tag{7}$$

Combining (4)–(7) we can see that

$$\sum_{i\in I}\sup_{z\in\mathbb{D}}\mu(z)E_i(z)<\infty.$$

(ii)$\Rightarrow$(iii). Suppose that (ii) holds. For each $i\in I$ and $\varphi(w)\neq 0$, Lemma 2 says that there exist constants $c_{i,j}, j\in\{1,2,3,4\}$ such that

$$g_{i,\varphi(w)}(z)=\sum_{j=1}^{4}c_{i,j}f_{j,\varphi(w)}(z)\in\mathscr{Z}_\alpha^p, \tag{8}$$

and

$$g_{i,\varphi(w)}^{(k)}(z)=\frac{\overline{\varphi(w)}^k\delta_{ik}}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+k-2}},$$

where $f_{j,\varphi(w)}$ are defined in (3) and $k\in I$. Then we have

$$\begin{aligned}\infty&>\sum_{j=1}^{4}\sup_{w\in\mathbb{D}}\|T_{u,v,\varphi}^{m,n}f_{j,\varphi(w)}\|_{\mathscr{B}_\mu}\gtrsim\sup_{w\in\mathbb{D}}\|T_{u,v,\varphi}^{m,n}g_{i,\varphi(w)}\|_{\mathscr{B}_\mu}\\&\geqslant\mu(w)\Big|(T_{u,v,\varphi}^{m,n}g_{i,\varphi(w)})'(w)\Big|=\frac{\mu(w)E_i(w)|\varphi(w)|^i}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+i-2}}.\end{aligned} \tag{9}$$

From (9) it follows that for each $i\in I$,

$$\begin{aligned}&\sup_{w\in\mathbb{D}}\frac{\mu(w)E_i(w)}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+i-2}}\\&\leqslant\sup_{|\varphi(w)|>\frac12}\frac{\mu(w)E_i(w)}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+i-2}}+\sup_{|\varphi(w)|\leqslant\frac12}\frac{\mu(w)E_i(w)}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+i-2}}\\&\leqslant 2^i\sup_{|\varphi(w)|>\frac12}\frac{\mu(w)E_i(w)|\varphi(w)|^i}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+i-2}}+\Big(\frac43\Big)^i\sup_{|\varphi(w)|\leqslant\frac12}\mu(w)E_i(w)\\&<\infty.\end{aligned}$$

Thus,

$$\sum_{i\in I}\sup_{z\in\mathbb{D}}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}<\infty.$$

(iii)⇒(i). Assume that (iii) holds. For any $f \in \mathscr{Z}_\alpha^p$, using Lemma 1 we have

$$\mu(z)|(T_{u,v,\varphi}^{m,n}f)'(z)| \leqslant \sum_{i\in I}\mu(z)E_i(z)|f^{(i)}(\varphi(z))| \lesssim \|f\|_{\mathscr{Z}_\alpha^p}\sum_{i\in I}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}. \tag{10}$$

In addition,

$$\begin{aligned}|(T_{u,v,\varphi}^{m,n}f)(0)| \leqslant& |u(0)f^{(m)}(\varphi(0))| + |v(0)f^{(n)}(\varphi(0))| \\ \lesssim& \left(\frac{|u(0)|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+m-2}} + \frac{|v(0)|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+n-2}}\right)\|f\|_{\mathscr{Z}_\alpha^p}.\end{aligned}$$

Hence $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded. The proof is completed. □

When $m+1=n$, as demonstrated in the proof of Lemma 2, we similarly conclude that for any $w \in \mathbb{D}\setminus\{0\}$ and $i,k \in \{m,n,n+1\}$, there exist constants $d_{i,j}$ (where $j \in \{1,2,3\}$) such that the function $h_{i,w} = \sum_{j=1}^{3} d_{i,j} f_{j,w}(z) \in \mathscr{Z}_\alpha^p$ satisfies

$$h_{i,w}^{(k)}(w) = \frac{\overline{w}^k\delta_{ik}}{(1-|w|^2)^k}.$$

By this and a slight modification of the proof of Theorem 1, we obtain the following result.

THEOREM 2. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$, $m,n \in \mathbb{N}$, $m+1=n$, μ be a radial weight and J denote the set $\{m,n+1\}$. Then the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded.*

(ii)

$$\sum_{j=1}^{3}\sup_{w\in\mathbb{D}}\|T_{u,v,\varphi}^{m,n}f_{j,w}\|_{\mathscr{B}_\mu} < \infty,$$

and

$$\sum_{i\in J}\sup_{z\in\mathbb{D}}\mu(z)E_i(z) + \sup_{z\in\mathbb{D}}\mu(z)|u(z)\varphi'(z)+v'(z)| < \infty,$$

where $f_{j,w}$ are defined in (3).

(iii)

$$\sum_{i\in J}\sup_{z\in\mathbb{D}}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} + \sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n-2}} < \infty.$$

When $m=0$, we consider two separate cases: $n=1$ and $n>1$. In the same manner as before we have the following theorems.

THEOREM 3. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$ and μ be a radial weight. Then the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded.*

(ii) $u \in \mathscr{B}_\mu$,

$$\sum_{j=1}^{3} \sup_{w\in\mathbb{D}} \|T_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu} < \infty,$$

and

$$\sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)+v'(z)| + \sup_{z\in\mathbb{D}} \mu(z)|v(z)\varphi'(z)| < \infty,$$

where $f_{j,w}$ are defined in (3).

(iii) $u \in \mathscr{B}_\mu$,

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|v(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} < \infty.$$

THEOREM 4. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$, $n \in \mathbb{N}$, $n > 1$, μ be a radial weight, L denote the set $\{1, n, n+1\}$ and $E_1(z) = |u(z)\varphi'(z)|$. Then the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi}^{0,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded.*

(ii) $u \in \mathscr{B}_\mu$,

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T_{u,v,\varphi}^{0,n} f_{j,w}\|_{\mathscr{B}_\mu} < \infty \quad \text{and} \quad \sum_{i\in L} \sup_{z\in\mathbb{D}} \mu(z) E_i(z) < \infty,$$

where $f_{j,w}$ are defined in (3).

(iii) $u \in \mathscr{B}_\mu$,

$$\sum_{i\in L} \sup_{z\in\mathbb{D}} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} < \infty.$$

Now we estimate the essential norm of the operator $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$. Furthermore, we establish several equivalent conditions for the compactness of $T_{u,v,\varphi}^{m,n}$.

THEOREM 5. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$, $m,n \in \mathbb{N}$, $m+1 < n$, μ be a radial weight and I denote the set $\{m, m+1, n, n+1\}$. Suppose that $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded. Then*

$$\|T_{u,v,\varphi}^{m,n}\|_{e,\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu} \approx \sum_{j=1}^{4} \limsup_{|w|\to 1} \|T_{u,v,\varphi}^{m,n} f_{j,w}\|_{\mathscr{B}_\mu} \approx \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}},$$

where $f_{j,w}$ are defined in (3).

Proof. For each $j \in \{1,2,3,4\}$ and $w \in \mathbb{D}$, we have $\|f_{j,w}\|_{\mathscr{Z}_\alpha^p} \lesssim 1$. Moreover, $f_{j,w}$ converge to zero uniformly on the compact subsets of $\mathbb{D}$. For any compact operator K from $\mathscr{Z}_\alpha^p$ into $\mathscr{B}_\mu$, by using some standard arguments (see, e.g., [8, 17]) we obtain

$$\lim_{|w|\to 1} \|Kf_{j,w}\|_{\mathscr{B}_\mu} = 0.$$

It follows that

$$\begin{aligned}\|T_{u,v,\varphi}^{m,n} - K\|_{\mathscr{Z}_\alpha^p \to \mathscr{B}_\mu} &\gtrsim \limsup_{|w|\to 1} \|(T_{u,v,\varphi}^{m,n} - K)f_{j,w}\|_{\mathscr{B}_\mu} \\ &\geqslant \limsup_{|w|\to 1} \|T_{u,v,\varphi}^{m,n} f_{j,w}\|_{\mathscr{B}_\mu} - \limsup_{|w|\to 1} \|Kf_{j,w}\|_{\mathscr{B}_\mu} \\ &= \limsup_{|w|\to 1} \|T_{u,v,\varphi}^{m,n} f_{j,w}\|_{\mathscr{B}_\mu}.\end{aligned}$$

Thus,

$$\|T_{u,v,\varphi}^{m,n}\|_{e,\mathscr{Z}_\alpha^p \to \mathscr{B}_\mu} = \inf_K \|T_{u,v,\varphi}^{m,n} - K\|_{\mathscr{Z}_\alpha^p \to \mathscr{B}_\mu} \gtrsim \sum_{j=1}^{4} \limsup_{|w|\to 1} \|T_{u,v,\varphi}^{m,n} f_{j,w}\|_{\mathscr{B}_\mu}. \tag{11}$$

Next, we prove that

$$\|T_{u,v,\varphi}^{m,n}\|_{e,\mathscr{Z}_\alpha^p \to \mathscr{B}_\mu} \gtrsim \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}. \tag{12}$$

Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Since $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded, for any compact operator $K : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ and $i \in I$, applying Lemma 3 and (9), we obtain

$$\begin{aligned}\|T_{u,v,\varphi}^{m,n} - K\|_{\mathscr{Z}_\alpha^p \to \mathscr{B}_\mu} &\gtrsim \limsup_{j\to\infty} \|T_{u,v,\varphi}^{m,n} g_{i,\varphi(z_j)}\|_{\mathscr{B}_\mu} - \limsup_{j\to\infty} \|Kg_{i,\varphi(z_j)}\|_{\mathscr{B}_\mu} \\ &\gtrsim \limsup_{j\to\infty} \frac{\mu(z_j)E_i(z_j)|\varphi(z_j)|^i}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+i-2}},\end{aligned}$$

where the functions $g_{i,\varphi(z_j)}$ are defined in (8). Consequently,

$$\|T_{u,v,\varphi}^{m,n}\|_{e,\mathscr{Z}_\alpha^p \to \mathscr{B}_\mu} \gtrsim \limsup_{j\to\infty} \frac{\mu(z_j)E_i(z_j)|\varphi(z_j)|^i}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+i-2}} = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}.$$

This establishes the validity of inequality (12).

Now, we show that

$$\|T_{u,v,\varphi}^{m,n}\|_{e,\mathscr{Z}_\alpha^p \to \mathscr{B}_\mu} \lesssim \min\left\{ \sum_{j=1}^{4} \limsup_{|w|\to 1} \|T_{u,v,\varphi}^{m,n} f_{j,w}\|_{\mathscr{B}_\mu}, \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} \right\}.$$

For $r \in [0,1)$, let $K_r f(z) = f_r(z) = f(rz)$. Then K_r is compact on $\mathscr{Z}_\alpha^p$ with $\|K_r\| \leqslant 1$. It is easy to see that $f_r \to f$ uniformly on the compact subsets of $\mathbb{D}$ as $r \to 1$.

Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for each $j \in \mathbb{N}$, $T^{m,n}_{u,v,\varphi} K_{r_j} : \mathscr{Z}^p_\alpha \to \mathscr{B}_\mu$ is compact, and so

$$\|T^{m,n}_{u,v,\varphi}\|_{e,\mathscr{Z}^p_\alpha \to \mathscr{B}_\mu} \leqslant \limsup_{j\to\infty} \|T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}\|_{\mathscr{Z}^p_\alpha \to \mathscr{B}_\mu}.$$

Therefore, it is sufficient to show that

$$\begin{aligned}&\limsup_{j\to\infty} \|T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}\|_{\mathscr{Z}^p_\alpha \to \mathscr{B}_\mu}\\ &\lesssim \min\left\{ \sum_{j=1}^4 \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu}, \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} \right\}. \end{aligned} \tag{13}$$

For every $f \in \mathscr{Z}^p_\alpha$ such that $\|f\|_{\mathscr{Z}^p_\alpha} \leqslant 1$, we have

$$\begin{aligned} &\|(T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}) f\|_{\mathscr{B}_\mu}\\ =&|T^{m,n}_{u,v,\varphi} f(0) - T^{m,n}_{u,v,\varphi} f_{r_j}(0)| + \sup_{z\in\mathbb{D}} \mu(z) |(T^{m,n}_{u,v,\varphi} f - T^{m,n}_{u,v,\varphi} f_{r_j})'(z)|\\ \leqslant& \underbrace{|u(0)(f-f_{r_j})^{(m)}(\varphi(0))| + |v(0)(f-f_{r_j})^{(n)}(\varphi(0))|}_{\Phi_0}\\ &+ \underbrace{\sup_{|\varphi(z)|\leqslant r_N} \mu(z) \sum_{i\in I} |(f-f_{r_j})^{(i)}(\varphi(z))| E_i(z)}_{\Phi_1}\\ &+ \underbrace{\sup_{|\varphi(z)|> r_N} \mu(z) \sum_{i\in I} |(f-f_{r_j})^{(i)}(\varphi(z))| E_i(z)}_{\Phi_2}, \end{aligned} \tag{14}$$

where $N \in \mathbb{N}$ such that $r_j \geqslant \frac{2}{3}$ for all $j \geqslant N$. Furthermore, we have $(f - f_{r_j})^{(t)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$ for any $t \in \mathbb{N}_0$. Now Theorem 1 implies

$$\limsup_{j\to\infty} \Phi_0 = \limsup_{j\to\infty} \Phi_1 = 0. \tag{15}$$

Obviously,

$$\Phi_2 \leqslant \sum_{i\in I} \underbrace{\sup_{|\varphi(z)|>r_N} \mu(z)|f^{(i)}(\varphi(z))|E_i(z)}_{\Psi_i} + \sum_{i\in I} \underbrace{\sup_{|\varphi(z)|>r_N} \mu(z) r_j^i |f^{(i)}(r_j\varphi(z))| E_i(z)}_{\Omega_i} \tag{16}$$

For each $i \in I$, using Lemma 1, (8) and (9) we obtain

$$\begin{aligned} \Psi_i =& \sup_{|\varphi(z)|>r_N} \frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2} |f^{(i)}(\varphi(z))|}{|\varphi(z)|^i} \frac{\mu(z)E_i(z)|\varphi(z)|^i}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}\\ \lesssim& \|f\|_{\mathscr{Z}^p_\alpha} \sup_{|\varphi(z)|>r_N} \|T^{m,n}_{u,v,\varphi} g_{i,\varphi(z)}\|_{\mathscr{B}_\mu}\\ \lesssim& \sum_{j=1}^4 \sup_{|w|>r_N} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu}. \end{aligned} \tag{17}$$

On the other hand,

$$\Psi_i = \sup_{|\varphi(z)|>r_N} (1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}|f^{(i)}(\varphi(z))|\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}$$

$$\lesssim \|f\|_{\mathscr{Z}_\alpha^p} \sup_{|\varphi(z)|>r_N} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}. \tag{18}$$

Taking the limits as $N\to\infty$ in (17) and (18) yields

$$\limsup_{j\to\infty}\Psi_i \lesssim \sum_{j=1}^{4}\limsup_{|w|\to 1}\|T_{u,v,\varphi}^{m,n}f_{j,w}\|_{\mathscr{B}_\mu}, \tag{19}$$

and

$$\limsup_{j\to\infty}\Psi_i \lesssim \limsup_{|\varphi(z)|\to 1}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} \tag{20}$$

Similarly, we have

$$\limsup_{j\to\infty}\Omega_i \lesssim \sum_{j=1}^{4}\limsup_{|w|\to 1}\|T_{u,v,\varphi}^{m,n}f_{j,w}\|_{\mathscr{B}_\mu} \text{ and } \limsup_{j\to\infty}\Omega_i \lesssim \limsup_{|\varphi(z)|\to 1}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}. \tag{21}$$

Therefore, from (14)–(16), (19)–(21), we get

$$\limsup_{j\to\infty}\|T_{u,v,\varphi}^{m,n}-T_{u,v,\varphi}^{m,n}K_{r_j}\|_{\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu} = \limsup_{j\to\infty}\sup_{\|f\|_{\mathscr{Z}_\alpha^p}\leqslant 1}\|(T_{u,v,\varphi}^{m,n}-T_{u,v,\varphi}^{m,n}K_{r_j})f\|_{\mathscr{B}_\mu}$$

$$\lesssim \sum_{j=1}^{4}\limsup_{|w|\to 1}\|T_{u,v,\varphi}^{m,n}f_{j,w}\|_{\mathscr{B}_\mu},$$

and

$$\limsup_{j\to\infty}\|T_{u,v,\varphi}^{m,n}-T_{u,v,\varphi}^{m,n}K_{r_j}\|_{\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu} \lesssim \sum_{i\in I}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}.$$

From the last two inequalities we get (13) and the proof is completed. □

Using the same approach as in the proof of Theorem 5, which along with Theorem 2, the results below can be derived for the case $m+1=n$.

THEOREM 6. *Let $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $1\leqslant p<\infty$, $p-2<\alpha\leqslant p-1$, $m,n\in\mathbb{N}$, $m+1=n$, μ be a radial weight and J denote the set $\{m,n+1\}$. Suppose that $T_{u,v,\varphi}^{m,n}:\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu$ is bounded. Then*

$$\|T_{u,v,\varphi}^{m,n}\|_{e,\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu} \approx \sum_{j=1}^{3}\limsup_{|w|\to 1}\|T_{u,v,\varphi}^{m,n}f_{j,w}\|_{\mathscr{B}_\mu}$$

$$\approx \sum_{i\in J}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} + \limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n-2}},$$

where $f_{j,w}$ are defined in (3).

For the case $m=0$, note that every sequence in $\mathscr{Z}_\alpha^p$ bounded in norm has a subsequence which converges uniformly in $\overline{\mathbb{D}}$ to a function in $\mathscr{Z}_\alpha^p$ (see [42, Lemma 2.5]), which along with the similar arguments as in the proof of Theorem 5 yields the following theorems.

THEOREM 7. *Let $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $1\leqslant p<\infty$, $p-2<\alpha\leqslant p-1$ and μ be a radial weight. Suppose that $T_{u,v,\varphi}:\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu$ is bounded. Then*

$$\|T_{u,v,\varphi}\|_{e,\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu}\approx\sum_{j=1}^{3}\limsup_{|w|\to1}\|T_{u,v,\varphi}f_{j,w}\|_{\mathscr{B}_\mu}$$
$$\approx\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}}+\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|v(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}},$$

where $f_{j,w}$ are defined in (3).

THEOREM 8. *Let $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $1\leqslant p<\infty$, $p-2<\alpha\leqslant p-1$, $n\in\mathbb{N}$, $n>1$, μ be a radial weight, L denote the set $\{1,n,n+1\}$ and $E_1(z)=|u(z)\varphi'(z)|$. Suppose that $T_{u,v,\varphi}^{0,n}:\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu$ is bounded. Then*

$$\|T_{u,v,\varphi}^{0,n}\|_{e,\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu}\approx\sum_{j=1}^{4}\limsup_{|w|\to1}\|T_{u,v,\varphi}^{0,n}f_{j,w}\|_{\mathscr{B}_\mu}\approx\sum_{i\in L}\limsup_{|\varphi(z)|\to1}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}},$$

where $f_{j,w}$ are defined in (3).

It is well known that $\|T\|_{e,X\to Y}=0$ if and only if $T:X\to Y$ is compact. Therefore, from Theorems 5–8 we immediately obtain the following corollaries.

COROLLARY 1. *Let $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $1\leqslant p<\infty$, $p-2<\alpha\leqslant p-1$, $m,n\in\mathbb{N}$, $m+1<n$, μ be a radial weight and I denote the set $\{m,m+1,n,n+1\}$. Suppose that $T_{u,v,\varphi}^{m,n}:\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu$ is bounded. Then the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi}^{m,n}:\mathscr{Z}_\alpha^p\to\mathscr{B}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{4}\limsup_{|w|\to1}\|T_{u,v,\varphi}^{m,n}f_{j,w}\|_{\mathscr{B}_\mu}=0,$$

where $f_{j,w}$ are defined in (3).

(iii)

$$\sum_{i\in I}\limsup_{|\varphi(z)|\to1}\frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}}=0.$$

COROLLARY 2. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$, $m,n \in \mathbb{N}$, $m+1 = n$, μ be a radial weight and J denote the set $\{m, n+1\}$. Suppose that $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded. Then the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi}^{m,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{3} \limsup_{|w|\to 1} \|T_{u,v,\varphi}^{m,n} f_{j,w}\|_{\mathscr{B}_\mu} = 0,$$

where $f_{j,w}$ are defined in (3).

(iii)

$$\sum_{i\in J} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n-2}} = 0.$$

COROLLARY 3. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$ and μ be a radial weight. Suppose that $T_{u,v,\varphi} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded. Then the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{3} \limsup_{|w|\to 1} \|T_{u,v,\varphi} f_{j,w}\|_{\mathscr{B}_\mu} = 0,$$

where $f_{j,w}$ are defined in (3).

(iii)

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|v(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

COROLLARY 4. *Let $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $1 \leqslant p < \infty$, $p-2 < \alpha \leqslant p-1$, $n \in \mathbb{N}$, $n > 1$, μ be a radial weight, L denote the set $\{1, n, n+1\}$ and $E_1(z) = |u(z)\varphi'(z)|$. Suppose that $T_{u,v,\varphi}^{0,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is bounded. Then the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi}^{0,n} : \mathscr{Z}_\alpha^p \to \mathscr{B}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{4} \limsup_{|w|\to 1} \|T_{u,v,\varphi}^{0,n} f_{j,w}\|_{\mathscr{B}_\mu} = 0,$$

where $f_{j,w}$ are defined in (3).

(iii)

$$\sum_{i\in L} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)E_i(z)}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+i-2}} = 0.$$

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(Received May 14, 2025)

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