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CONVERGENCE THEOREMS FOR MAXIMUM WEIGHTED SUMS OF COORDINATEWISE ASYMPTOTICALLY ALMOST NEGATIVELY ASSOCIATED RANDOM VECTORS IN HILBERT SPACE

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Abstract. This paper studies some convergence theorems such as weak law of large numbers, complete f-moment convergence, complete moment convergence, complete convergence and strong law of large numbers for maximum weighted sums of coordinatewise asymptotically almost negatively associated random vectors in Hilbert spaces. The results improve or extend some corresponding results in the literature.

1. Introduction

In the study of classical limit theory, scholars often assumed that random variables are independent of each other. However, in many real-world phenomena the fact is that random variables are actually interconnected and depend on each other. For example, in finance, the prices of stocks or bonds often exhibit dependence structures, and understanding these dependencies is crucial for risk management and investment strategies. Similarly, in epidemiology, the spread of diseases can be modeled using dependent random variables, which helps in predicting and controlling the outbreak. To sum up, the study of dependent random variables is crucial for modeling, analyzing, and understanding complex systems and phenomena in various scientific and engineering disciplines.

As one of the most practicable dependencies, the concept of negatively associated random variables was first raised by Joag-Dev and Proschan [9] as follows.

DEFINITION 1.1. A finite family of random variables $\{X_i, 1 \leqslant i \leqslant n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1,2,\cdots,n\}$ and any real coordinatewise nondecreasing (or nonincreasing) functions f_1 on $\mathbb{R}^A$ and f_2 on $\mathbb{R}^B$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leqslant 0,$$

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whenever the covariance above exists. An infinite family of random variables is NA if every finite subfamily is NA.

Joag-Dev and Proschan [9] gave a lot of examples satisfying the NA structure. Considering the importance of this structure, many limit theorems have been established for NA random variables. For more details, we refer to Shao [15], Kuczmaszewska [13], Chen and Sung [2], and Wu et al. [20] among others.

The concept of NA random variables was also extended to Hilbert spaces. Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$. Define $X^{(j)} = \langle X, e^{(j)}\rangle$ with $\{e^{(j)}, j \geqslant 1\}$ being an orthonormal basis in H, and X is an H-valued random vector. The concept of H-valued NA random vectors was introduced in Ko et al. [12] as follows.

DEFINITION 1.2. A sequence $\{X_n, n \geqslant 1\}$ of H-valued random vectors is said to be NA if there exists an orthonormal basis $\{e^{(j)}, j \geqslant 1\}$ in H such that for any $d \geqslant 1$, the sequence $\{(X_n^{(1)}, X_n^{(2)}, \cdots, X_n^{(d)}), n \geqslant 1\}$ of $\mathbb{R}^d$-valued random vectors is NA.

Ko et al. [12] as well as Thanh [16] established the almost sure convergence for H-valued NA random vectors. Miao [14] proved the Hajeck-Renyi inequality for H-valued NA random vectors.

Huan et al. [7] raised the concept of coordinatewise negatively associated (CNA) random vectors in Hilbert space as follows.

DEFINITION 1.3. If the sequence $\{X_n^{(j)}, n \geqslant 1\}$ of random variables is NA for each $j \geqslant 1$, where $X_n^{(j)} = \langle X_n, e^{(j)}\rangle$, then the sequence $\{X_n, n \geqslant 1\}$ of H-valued random vectors is said to be CNA.

As stated and exemplified in Huan et al. [7], NA random vectors in Hilbert space are CNA but not vice versa. Huan et al. [7] obtained the complete convergence for partial sums of CNA random vectors in Hilbert space. Huan [6] obtained the complete convergence result for H-valued CNA random vectors which extends the corresponding one of Huan et al. [7]. Ko [11] extended the results of Huan et al. [7] from complete convergence to complete moment convergence. Huang and Wu [8] further obtained the complete convergence and complete moment convergence for maximum weighted sums of CNA random vectors in Hilbert space, which improve and extend the results in Huan et al. [7] and Ko [11].

Recently, Ko [10] extended the concept of asymptotically almost negatively associated (AANA) random variables in probability space, which was raised by Chandra and Ghosal [1], to the concept of coordinatewise asymptotically almost negatively associated (CAANA) random vectors in Hilbert space as follows.

DEFINITION 1.4. If the sequence $\{X_n^{(j)}, n \geqslant 1\}$ of random variables is AANA for each $j \geqslant 1$, that is, there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\begin{aligned} &Cov(f(X_n^{(j)}), g(X_{n+1}^{(j)}, X_{n+2}^{(j)}, \cdots, X_{n+k}^{(j)})) \\ &\leqslant q(n)\{Var(f(X_n^{(j)}))Var(g(X_{n+1}^{(j)}, X_{n+2}^{(j)}, \cdots, X_{n+k}^{(j)}))\}^{1/2} \end{aligned}$$

for all $n, k \geqslant 1$ and for all coordinatewise nondecreasing continuous functions f and

g whenever the variances exist, then the sequence $\{X_n, n \geqslant 1\}$ of H-valued random vectors is said to be CAANA.

Ko [10] studied the complete moment convergence for partial sums of CAANA random vectors in Hilbert spaces and obtained the following result.

THEOREM A. *Let $1/2 < \alpha < 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of H-valued CAANA random vectors with mixing coefficients $\{q(n), n \geqslant 1\}$ and zero means. If $\{X_n, n \geqslant 1\}$ is coordinatewise weakly upper bounded by a random vector X satisfying $\sum_{j=1}^{\infty} \mathbb{E}|X^{(j)}|^{1/\alpha} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{-\alpha-1} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} X_i\right\| - \varepsilon n^{\alpha}\right)_+ < \infty,$$

where $x_+ = \max\{x, 0\}$.

However, to the best of our knowledge, there are not so many works based on the CAANA random vectors in Hilbert spaces. The weak law of large numbers, strong law of large numbers and complete convergence have not been found in the literature. The paper will focus on this topic. The weak law of large numbers, complete f-moment convergence, complete moment convergence, complete convergence and strong law of large numbers are established for maximum weighted sums of CAANA random vectors in Hilbert spaces. These results improve or extend Theorem A and some other corresponding results in the literature.

Recall that the concept of complete convergence was introduced by Hsu and Robbins [5] as follows: A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to converge completely to a constant u if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - u| > \varepsilon) < \infty.$$

It follows by the Borel-Cantelli lemma that $\{X_n, n \geqslant 1\}$ converges completely to u implies $X_n \to u$ almost surely (a.s.). Wu et al. [19] introduced the concept of complete f-moment convergence as follows: Let $\{a_n, n \geqslant 1\}$ be a sequence of positive constants and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and continuous function with $f(0) = 0$. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to exhibit complete f-moment convergence if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n E f(\{|X_n| - \varepsilon\}_+) < \infty.$$

Obviously, by taking $f(x) = x^r$ for some $r > 0$, the complete f-moment convergence equals to complete r-th moment convergence, the concept of which first appeared in Chow [3].

At last, recall that if $P(|X_n^{(j)}| > x) \leqslant P(|X^{(j)}| > x)$ for all $j \geqslant 1$, $n \geqslant 1$ and $x \geqslant 0$, then the sequence of random vectors $\{X_n, n \geqslant 1\}$ is said to be coordinatewise stochastically dominated by X. Throughout the paper, let C be a positive constant whose value may be different in different lines. Let $\log x = \ln \max(x, e)$ and $I(\cdot)$ be the indicator function.

2. Preliminaries

In this section, we state some lemmas which will be used in proving our main results.

LEMMA 2.1. (Yuan and An [22]) *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of AANA random variables with mixing coefficients* $\{q(n), n \geqslant 1\}$, *and let* $\{f_n, n \geqslant 1\}$ *be a sequence of all nondecreasing continuous functions, then* $\{f_n(X_n), n \geqslant 1\}$ *is still a sequence of AANA random variables with mixing coefficients* $\{q(n), n \geqslant 1\}$.

LEMMA 2.2. (Yuan and An [22]) *Let* $1 \leqslant p \leqslant 2$. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of AANA random variables with mixing coefficients* $\{q(n), n \geqslant 1\}$ *and zero means. If* $\sum_{n=1}^{\infty} q^2(n) < \infty$, *there exists a constant* $C_p > 0$ *depending only on* p *such that*

$$E\left(\max_{1\leqslant m\leqslant n}\left|\sum_{i=1}^{m} X_i\right|^p\right) \leqslant C_p \sum_{i=1}^{n} E|X_i|^p.$$

The inequality in Lemma 2.2 can be extended to Hilbert space as follows, where as a special case of the lemma, the second inequality was obtained in Ko [10].

LEMMA 2.3. *Let* $1 \leqslant p \leqslant 2$. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of* H*-valued CAANA random vectors with mixing coefficients* $\{q(n), n \geqslant 1\}$ *and zero means. If* $\sum_{n=1}^{\infty} q^2(n) < \infty$, *there exists a constant* $C_p > 0$ *depending only on* p *such that*

$$E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} X_i\right\|^p\right) \leqslant C_p \sum_{j=1}^{\infty}\sum_{i=1}^{n} E|X_i^{(j)}|^p.$$

In particular,

$$E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} X_i\right\|^2\right) \leqslant C \sum_{i=1}^{n} E\|X_i\|^2.$$

Proof. We obtain by C_r inequality and Lemma 2.2 that

$$\begin{aligned}
E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} X_i\right\|^p\right) &= E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} X_i\right\|^2\right)^{p/2} \\
&= E\left(\max_{1\leqslant m\leqslant n}\sum_{j=1}^{\infty}\left(\sum_{i=1}^{m}\langle X_i, e^{(j)}\rangle\right)^2\right)^{p/2} \\
&\leqslant E\left(\sum_{j=1}^{\infty}\max_{1\leqslant m\leqslant n}\left|\sum_{i=1}^{m}\langle X_i, e^{(j)}\rangle\right|^2\right)^{p/2}
\end{aligned}$$

$$\begin{aligned}&\leqslant E\left(\sum_{j=1}^{\infty}\max_{1\leqslant m\leqslant n}\left|\sum_{i=1}^{m}X_i^{(j)}\right|^p\right)\\&=\sum_{j=1}^{\infty}E\left(\max_{1\leqslant m\leqslant n}\left|\sum_{i=1}^{m}X_i^{(j)}\right|^p\right)\\&\leqslant C\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|X_i^{(j)}|^p.\quad\square\end{aligned}$$

LEMMA 2.4. (Wu [17]) *Let* $\{X_n, n\geqslant 1\}$ *be a sequence of random variables stochastically dominated by a random variable* X*, i.e.,* $\sup_{n\geqslant 1}P(|X_n|>x)\leqslant CP(|X|>x)$ *for any* $x\geqslant 0$*. Then for any* $\alpha>0$ *and* $b>0$*,*

$$\begin{aligned}&E|X_n|^{\alpha}I(|X_n|>b)\leqslant CE|X|^{\alpha}I(|X|>b);\\&E|X_n|^{\alpha}I(|X_n|\leqslant b)\leqslant C[E|X|^{\alpha}I(|X|\leqslant b)+b^{\alpha}P(|X|>b)].\end{aligned}$$

LEMMA 2.5. (Wu et al. [21]) *Let* $\{Y_i, 1\leqslant i\leqslant n\}$ *and* $\{Z_i, 1\leqslant i\leqslant n\}$ *be two sequences of random vectors. Then for any* $q>r>0$*,* $\varepsilon>0$*, and* $a>0$*, the following inequality holds:*

$$\begin{aligned}&E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}(Y_i+Z_i)\right\|-\varepsilon a\right)_+^r\\&\leqslant C_r\left(\varepsilon^{-q}+\frac{r}{q-r}\right)a^{r-q}E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}Y_i\right\|^q\right)+C_rE\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}Z_i\right\|^r\right),\end{aligned}$$

where $C_r=1$ *if* $0<r\leqslant 1$ *or* $C_r=2^{r-1}$ *if* $r>1$*.*

LEMMA 2.6. (Wu et al. [18]) *Let* $\alpha>0$*,* $r>0$*,* $0<p<s$ *and* X *be a random variable. Then*

$$\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-1}E|X|^rI(|X|>n^{\alpha})\leqslant\begin{cases}CE|X|^p, & \text{if } r<p,\\ CE|X|^p\log|X|, & \text{if } r=p,\\ CE|X|^r, & \text{if } r>p,\end{cases}$$

and

$$\sum_{n=1}^{\infty}n^{\alpha p-\alpha s-1}E|X|^sI(|X|\leqslant n^{\alpha})\leqslant CE|X|^p.$$

3. Main results

In this section, we will state the main results and their proofs one by one as follows.

THEOREM 3.1. *Let $1 \leqslant q \leqslant 2$. Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of nonnegative real numbers, $\{b_n, n \geqslant 1\}$ be a sequence of positive numbers such that $b_n \to \infty$ and $\{X_n, n \geqslant 1\}$ be a sequence of H-valued CAANA random vectors with mixing coefficients $\{q(n), n \geqslant 1\}$ satisfying $\sum_{n=1}^{\infty} q^2(n) < \infty$. If as $n \to \infty$,*

(i) $\sum_{j=1}^{\infty} \sum_{i=1}^{n} P(|a_{ni}X_i^{(j)}| > b_n) \to 0$,

(ii) $b_n^{-q} \sum_{j=1}^{\infty} \sum_{i=1}^{n} E|a_{ni}X_i^{(j)}|^q I(|a_{ni}X_i^{(j)}| \leqslant b_n) \to 0$,

then we obtain that

$$\frac{\max_{1 \leqslant m \leqslant n} \|\sum_{i=1}^{m}(a_{ni}X_i - EX_{ni})\|}{b_n} \xrightarrow{P} 0 \ \text{as} \ n \to \infty,$$

where $X_{ni} = \sum_{j=1}^{\infty} X_{ni}^{(j)} e^{(j)}$ with

$$X_{ni}^{(j)} = -b_n I(a_{ni}X_i^{(j)} < -b_n) + a_{ni}X_i^{(j)} I(|a_{ni}X_i^{(j)}| \leqslant b_n) + b_n I(a_{ni}X_i^{(j)} > b_n).$$

Proof. Noting that

$$\max_{1 \leqslant m \leqslant n} \left\| \sum_{i=1}^{m} (a_{ni}X_i - EX_{ni}) \right\| \leqslant \max_{1 \leqslant m \leqslant n} \left\| \sum_{i=1}^{m} (a_{ni}X_i - X_{ni}) \right\| + \left\| \sum_{i=1}^{m} (X_{ni} - EX_{ni}) \right\|,$$

it suffices to prove that for any $\varepsilon > 0$,

$$P\left(\max_{1 \leqslant m \leqslant n} \left\| \sum_{i=1}^{m} (a_{ni}X_i - X_{ni}) \right\| > 0 \right) \to 0 \to 0 \ \text{as} \ n \to \infty \tag{1}$$

and

$$P\left(\max_{1 \leqslant m \leqslant n} \left\| \sum_{i=1}^{m} (X_{ni} - EX_{ni}) \right\| > \varepsilon b_n \right) \to 0 \ \text{as} \ n \to \infty. \tag{2}$$

It follows from (i) that

$$\begin{aligned} & P\left(\max_{1 \leqslant m \leqslant n} \left\| \sum_{i=1}^{m} (a_{ni}X_i - X_{ni}) \right\| > 0 \right) \\ & \leqslant P\left(\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{n} \{|a_{ni}X_i^{(j)}| > b_n\} \right) \\ & \leqslant \sum_{j=1}^{\infty} \sum_{i=1}^{n} P(|a_{ni}X_i^{(j)}| > b_n) \to 0 \ \text{as} \ n \to \infty, \end{aligned}$$

which verifies (1). On the other hand, it follows from Lemma 2.1 and Definition 1.4 that $\{X_{ni}, 1 \leqslant i \leqslant n\}$ is still a sequence of CAANA random vectors with the mixing coefficients $\{q(n), n \geqslant 1\}$. Hence, by Markov's inequality, Lemma 2.3, (i) and (ii) we have

$$\begin{aligned}
& P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}(X_{ni}-EX_{ni})\right\|>\varepsilon b_n\right) \\
\leqslant\ & \varepsilon^{-q}b_n^{-q}E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}(X_{ni}-EX_{ni})\right\|^q\right) \\
\leqslant\ & Cb_n^{-q}\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|X_{ni}^{(j)}-EX_{ni}^{(j)}|^q \\
\leqslant\ & Cb_n^{-q}\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|X_{ni}^{(j)}|^q \\
\leqslant\ & Cb_n^{-q}\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|a_{ni}X_i^{(j)}|^qI(|a_{ni}X_i^{(j)}|\leqslant b_n)+C\sum_{j=1}^{\infty}\sum_{i=1}^{n}P(|a_{ni}X_i^{(j)}|>b_n) \\
\to\ & 0 \ \text{ as } \ n\to\infty.
\end{aligned}$$

Hence, (2) has also been proved and then the proof is complete. □

By Theorem 3.1, we obtain the following Marcinkiewicz-Zygmund type weak law of large numbers for maximum weighted sums of H-valued CAANA random vectors.

THEOREM 3.2. *Let $1 \leqslant p < q \leqslant 2$. Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of real numbers such that $\sum_{i=1}^{n}|a_{ni}|^q = O(n)$. Let $\{X_n, n \geqslant 1\}$ be sequence of zero mean H-valued CAANA random vectors with mixing coefficients $\{q(n), n \geqslant 1\}$ satisfying $\sum_{n=1}^{\infty} q^2(n) < \infty$. If $\{X_n, n \geqslant 1\}$ is coordinatewise stochastically dominated by a random vector X with $\sum_{j=1}^{\infty} E|X^{(j)}|^p < \infty$, then*

$$\frac{\max_{1\leqslant m\leqslant n}\|\sum_{i=1}^{m}a_{ni}X_i\|}{n^{1/p}}\xrightarrow{P}0 \ \ as \ \ n\to\infty.$$

Proof. Noting that $a_{ni} = a_{ni+} - a_{ni-}$, where $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\}$, we can assume without loss of generality that $a_{ni} > 0$ for all $1 \leqslant i \leqslant n, n \geqslant 1$. Applying Theorem 3.1 with $b_n = n^{1/p}$, we have by $\sum_{i=1}^{n}|a_{ni}|^q = O(n)$ and Lemma 2.4 that

$$\begin{aligned}
& \sum_{j=1}^{\infty}\sum_{i=1}^{n}P(|a_{ni}X_i^{(j)}|>n^{1/p}) \\
\leqslant\ & C\sum_{j=1}^{\infty}\sum_{i=1}^{n}P(|a_{ni}X^{(j)}|>n^{1/p}) \\
=\ & C\sum_{j=1}^{\infty}\sum_{i=1}^{n}P(|a_{ni}X^{(j)}|>n^{1/p},|X^{(j)}|\leqslant n^{1/p})+C\sum_{j=1}^{\infty}\sum_{i=1}^{n}P(|a_{ni}X^{(j)}|>n^{1/p},|X^{(j)}|>n^{1/p}) \\
\leqslant\ & Cn^{-q/p}\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|a_{ni}X^{(j)}|^qI(|X^{(j)}|\leqslant n^{1/p})+C\sum_{j=1}^{\infty}nP(|X^{(j)}|>n^{1/p})
\end{aligned}$$

$$
\begin{aligned}
&\leqslant Cn^{1-q/p}\sum_{j=1}^{\infty}[E|X^{(j)}|^q I(|X^{(j)}|\leqslant n^{1/p})+n^{q/p}P(|X^{(j)}|>n^{1/p})] \\
&= Cn^{1-q/p}\sum_{j=1}^{\infty}\left[\int_0^{\infty}P\left(|X^{(j)}|^q I(|X^{(j)}|\leqslant n^{1/p})>s\right)ds+n^{q/p}P(|X^{(j)}|>n^{1/p})\right] \\
&= Cn^{1-q/p}\sum_{j=1}^{\infty}\left[\int_0^{n^{q/p}}P\left(|X^{(j)}|^q I(|X^{(j)}|\leqslant n^{1/p})>s\right)ds+n^{q/p}P(|X^{(j)}|>n^{1/p})\right] \\
&= Cn^{1-q/p}\sum_{j=1}^{\infty}\left\{\int_0^{n^{q/p}}\left[P\left(|X^{(j)}|>s^{1/q}\right)-P\left(|X^{(j)}|>n^{1/p}\right)\right]ds+n^{q/p}P(|X^{(j)}|>n^{1/p})\right\} \\
&= Cn^{1-q/p}\sum_{j=1}^{\infty}\int_0^{n^{q/p}}P\left(|X^{(j)}|>s^{1/q}\right)ds \\
&= Cqn^{1-q/p}\sum_{j=1}^{\infty}\int_0^{n^{1/p}}t^{q-1}P(|X^{(j)}|>t)dt \\
&= Cqn^{1-q/p}\sum_{j=1}^{\infty}\int_0^{1}t^{q-1}P(|X^{(j)}|>t)dt+Cqn^{1-q/p}\sum_{j=1}^{\infty}\sum_{k=1}^{n-1}\int_{k^{1/p}}^{(k+1)^{1/p}}t^{q-1}P(|X^{(j)}|>t)dt \\
&\leqslant Cn^{1-q/p}\sum_{j=1}^{\infty}E|X^{(j)}|^p+Cn^{1-q/p}\sum_{j=1}^{\infty}\sum_{k=1}^{n-1}E|X^{(j)}|^p I(|X^{(j)}|>k^{1/p})\int_{k^{1/p}}^{(k+1)^{1/p}}t^{q-p-1}dt \\
&\leqslant Cn^{1-q/p}\sum_{j=1}^{\infty}E|X^{(j)}|^p+Cn^{1-q/p}\sum_{k=1}^{n}\left(k^{q/p-2}\sum_{j=1}^{\infty}E|X^{(j)}|^p I(|X^{(j)}|>k^{1/p})\right)\to 0 \\
&\quad \text{as } n\to\infty,
\end{aligned}
$$

where the convergence of the last inequality follows from the fact that $\sum_{j=1}^{\infty}E|X^{(j)}|^p I(|X^{(j)}|>k^{1/p})=o(1)$ as $k\to\infty$ and $n^{1-q/p}\sum_{k=1}^{n}k^{q/p-2}=O(1)$. This implies that (i) has been checked. Similarly, we also verify (ii) by the statement above and

$$
\begin{aligned}
&n^{-q/p}\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|a_{ni}X_i^{(j)}|^q I(|a_{ni}X_i^{(j)}|\leqslant n^{1/p}) \\
&= n^{-q/p}\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|a_{ni}X_i^{(j)}|^q I(|a_{ni}X_i^{(j)}|\leqslant n^{1/p},|X_i^{(j)}|\leqslant n^{1/p}) \\
&\quad +n^{-q/p}\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|a_{ni}X_i^{(j)}|^q I(|a_{ni}X_i^{(j)}|\leqslant n^{1/p},|X_i^{(j)}|>n^{1/p}) \\
&\leqslant n^{-q/p}\sum_{j=1}^{\infty}\sum_{i=1}^{n}E|a_{ni}X_i^{(j)}|^q I(|X_i^{(j)}|\leqslant n^{1/p})+\sum_{j=1}^{\infty}\sum_{i=1}^{n}P(|X_i^{(j)}|>n^{1/p}) \\
&\leqslant Cn^{1-q/p}\sum_{j=1}^{\infty}E|X^{(j)}|^q I(|X^{(j)}|\leqslant n^{1/p})+C\sum_{j=1}^{\infty}nP(|X^{(j)}|>n^{1/p})\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Hence, it remains to show

$$n^{-1/p} \max_{1\leqslant m\leqslant n} \left\| \sum_{i=1}^{m} EX_{ni} \right\| \to 0 \ \text{ as } \ n \to \infty. \tag{3}$$

Actually, it follows from $\sum_{i=1}^{n} |a_{ni}|^q = O(n)$ that $\max_{1\leqslant i\leqslant n} |a_{ni}| = O(n^{1/q})$ and for all $0 < t < q$,

$$\sum_{i=1}^{n} |a_{ni}|^t \leqslant \left(\sum_{i=1}^{n} |a_{ni}|^q \right)^{t/q} \left(\sum_{i=1}^{n} 1 \right)^{1-t/q} = O(n). \tag{4}$$

Hence, we obtain by $EX_i = 0$, Lemma 2.4, (4) and $\sum_{j=1}^{\infty} E|X^{(j)}|^p < \infty$ that

$$\begin{aligned}
&n^{-1/p} \max_{1\leqslant m\leqslant n} \left\| \sum_{i=1}^{m} EX_{ni} \right\| = n^{-1/p} \max_{1\leqslant m\leqslant n} \left\| \sum_{i=1}^{m} E(a_{ni}X_i - X_{ni}) \right\| \\
&\leqslant n^{-1/p} \sum_{j=1}^{\infty} \sum_{i=1}^{n} E|a_{ni}X_i^{(j)}| I(|a_{ni}X_i^{(j)}| > n^{1/p}) \\
&\leqslant n^{-1} \sum_{j=1}^{\infty} \sum_{i=1}^{n} E|a_{ni}X_i^{(j)}|^p I(|a_{ni}X_i^{(j)}| > n^{1/p}) \\
&\leqslant n^{-1} \sum_{j=1}^{\infty} \sum_{i=1}^{n} |a_{ni}|^p E|X_i^{(j)}|^p I(C|X_i^{(j)}| > n^{1/p-1/q}) \\
&\leqslant C \sum_{j=1}^{\infty} E|X^{(j)}|^p I(C|X^{(j)}| > n^{1/p-1/q}) \to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

The proof is complete. □

REMARK 3.1. He and Pan [4] obtained weak law of large numbers for maximum weighted sums of coordinatewise asymptotically negatively associated (CANA) random vectors in Hilbert space. Taking $b_n = n^{1/p}$ in Theorem 3.1, we get the same result as in He and Pan [4]. Furthermore, taking $p = 1$, Theorem 3.2 reduces to the Kolmogorov weak law of large numbers obtained in He and Pan [4]. Observing that the concepts of CAANA random vectors and CANA random vectors do not imply each other, our results extend and improve the two results in He and Pan [4].

THEOREM 3.3. *Let* $0 < r < 2$, $1 \leqslant p < 2$ *and* $\alpha p \geqslant 1$. *Let* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of real numbers such that* $\sum_{i=1}^{n} |a_{ni}|^q = O(n)$ *for some* $q > \max\{r, p\}$. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of zero mean* H*-valued CAANA random vectors with mixing coefficients* $\{q(n), n \geqslant 1\}$ *satisfying* $\sum_{n=1}^{\infty} q^2(n) < \infty$. *Suppose that* $\{X_n, n \geqslant 1\}$ *is coordinatewise stochastically dominated by a random vector* X. *If*

$$\begin{cases} \sum_{j=1}^{\infty} E|X^{(j)}|^p < \infty, & \text{for } \ r < p, \\ \sum_{j=1}^{\infty} E|X^{(j)}|^p \log|X^{(j)}| < \infty, & \text{for } \ r = p, \\ \sum_{j=1}^{\infty} E|X^{(j)}|^r < \infty, & \text{for } \ p < r < 2, \end{cases} \tag{5}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \varepsilon n^{\alpha}\right)_+^r < \infty.$$

Proof. Without loss of generality, we also assume that $a_{ni} > 0$ for each $1 \leqslant i \leqslant n, n \geqslant 1$. Define for $1 \leqslant i \leqslant n, n \geqslant 1$ and $j \geqslant 1$ that

$$\begin{aligned}
&Y_{ni}^{(j)} = -n^{\alpha} I(X_i^{(j)} < -n^{\alpha}) + X_i^{(j)} I(|X_i^{(j)}| \leqslant n^{\alpha}) + n^{\alpha} I(X_i^{(j)} > n^{\alpha}),\\
&Z_{ni}^{(j)} = X_i^{(j)} - Y_{ni}^{(j)} = (X_i^{(j)} + n^{\alpha}) I(X_i^{(j)} < -n^{\alpha}) + (X_i^{(j)} - n^{\alpha}) I(X_i^{(j)} > n^{\alpha}),\\
&Y_{ni} = \sum_{j=1}^{\infty} Y_{ni}^{(j)} e^{(j)} \quad \text{and} \quad Z_{ni} = \sum_{j=1}^{\infty} Z_{ni}^{(j)} e^{(j)}.
\end{aligned}$$

It follows from Lemma 2.1 and Definition 1.4 that both $\{Y_{ni}, 1 \leqslant i \leqslant n\}$ and $\{Z_{ni}, 1 \leqslant i \leqslant n\}$ are CAANA random vectors with the same mixing coefficients $\{q(n), n \geqslant 1\}$ and $Y_{ni} + Z_{ni} = X_i$ for each $1 \leqslant i \leqslant n, n \geqslant 1$. Furthermore, by $EX_i = 0$ for each $1 \leqslant i \leqslant n, n \geqslant 1$, Lemma 2.4, (4), $\sum_{j=1}^{\infty} E|X^{(j)}|^p < \infty$, and $\alpha p \geqslant 1$ we have

$$\begin{aligned}
n^{-\alpha} \max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}EY_{ni}\right\| &= n^{-\alpha} \max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}EZ_{ni}\right\|\\
&\leqslant n^{-\alpha} \sum_{j=1}^{\infty}\sum_{i=1}^{n} |a_{ni}| E|X_i^{(j)}| I(|X_i^{(j)}| > n^{\alpha})\\
&\leqslant Cn^{1-\alpha} \sum_{j=1}^{\infty} E|X^{(j)}| I(|X^{(j)}| > n^{\alpha})\\
&\leqslant Cn^{1-\alpha p} \sum_{j=1}^{\infty} E|X^{(j)}|^p I(|X^{(j)}| > n^{\alpha})\\
&\leqslant C \sum_{j=1}^{\infty} E|X^{(j)}|^p I(|X^{(j)}| > n^{\alpha}) \to 0, \text{ as } n \to \infty,
\end{aligned}$$

which implies that

$$\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}EY_{ni}\right\| \leqslant \frac{1}{3}\varepsilon n^{\alpha} \tag{6}$$

for all n large enough. Take $\beta = \min\{2, q\}$. It follows from (6) and Lemma 2.5 that if $0 < r < 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \varepsilon n^{\alpha}\right)_+^r\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} [a_{ni}(Y_{ni} - EY_{ni}) + a_{ni}Z_{ni}]\right\| - \frac{2}{3}\varepsilon n^{\alpha}\right)_+^r
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}(Y_{ni}-EY_{ni})\right\|^{\beta}\right)\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}Z_{ni}\right\|^{r}\right)\\
&\doteq I_1+I_2.
\end{aligned}
$$

By Lemma 2.3, Lemma 2.4, (4) and Lemma 2.6, we have

$$
\begin{aligned}
I_1 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-2}\sum_{j=1}^{\infty}\sum_{i=1}^{n} |a_{ni}|^{\beta} E|Y_{ni}^{(j)}|^{\beta}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-1}\sum_{j=1}^{\infty} E|X^{(j)}|^{\beta} I(|X^{(j)}|\leqslant n^{\alpha})\\
&\quad +\sum_{n=1}^{\infty} n^{\alpha p-1}\sum_{j=1}^{\infty} P(|X^{(j)}|>n^{\alpha})\\
&= C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} n^{\alpha p-\alpha\beta-1} E|X^{(j)}|^{\beta} I(|X^{(j)}|\leqslant n^{\alpha})\\
&\quad +\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} n^{\alpha p-1} P(|X^{(j)}|>n^{\alpha})\\
&\leqslant C\sum_{j=1}^{\infty} E|X^{(j)}|^{p}<\infty.
\end{aligned}
$$

Furthermore, by C_r inequality, Lemma 2.4, (4) and Lemma 2.6, we have

$$
\begin{aligned}
I_2 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2}\sum_{j=1}^{\infty}\sum_{i=1}^{n} |a_{ni}|^{r} E|Z_{ni}^{(j)}|^{r}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1}\sum_{j=1}^{\infty} E|X^{(j)}|^{r} I(|X^{(j)}|>n^{\alpha})\\
&= C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-1} E|X^{(j)}|^{r} I(|X^{(j)}|>n^{\alpha})\\
&\leqslant C\sum_{j=1}^{\infty} E|X^{(j)}|^{p}<\infty.
\end{aligned}
$$

Similarly, if $1\leqslant r<2$, we also obtain by $EX_i=EY_{ni}+EZ_{ni}=0$ for each $1\leqslant i\leqslant n, n\geqslant 1$ and Lemma 2.5 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\|-\varepsilon n^{\alpha}\right)_{+}^{r}\\
&=\sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}[a_{ni}(Y_{ni}-EY_{ni})+a_{ni}(Z_{ni}-EZ_{ni})]\right\|-\varepsilon n^{\alpha}\right)_{+}^{r}
\end{aligned}
$$

$$\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-\alpha\beta-2}E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}a_{ni}(Y_{ni}-EY_{ni})\right\|^{\beta}\right)$$
$$+C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m}a_{ni}(Z_{ni}-EZ_{ni})\right\|^{r}\right)$$
$$\doteq I_1+I_2'.$$

By the proof for the case $0<r<1$, we also have $I_1<\infty$ for $1\leqslant r<2$. Now we prove $I_2'<\infty$. It follows from Lemma 2.3, Lemma 2.4, (4) and Lemma 2.6 that

$$\begin{aligned}
I_2' &\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}\sum_{j=1}^{\infty}\sum_{i=1}^{n}|a_{ni}|^rE|Z_{ni}^{(j)}-EZ_{ni}^{(j)}|^r\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-2}\sum_{j=1}^{\infty}\sum_{i=1}^{n}|a_{ni}|^rE|Z_{ni}^{(j)}|^r\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-1}\sum_{j=1}^{\infty}E|X^{(j)}|^rI(|X^{(j)}|>n^{\alpha})\\
&= C\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}n^{\alpha p-\alpha r-1}E|X^{(j)}|^rI(|X^{(j)}|>n^{\alpha})\\
&\leqslant \begin{cases} C\sum_{j=1}^{\infty}E|X^{(j)}|^p<\infty, & \text{for } r<p,\\ C\sum_{j=1}^{\infty}E|X^{(j)}|^p\log|X^{(j)}|<\infty, & \text{for } r=p,\\ C\sum_{j=1}^{\infty}E|X^{(j)}|^r<\infty, & \text{for } p<r<2.\end{cases}
\end{aligned}$$

The proof is complete. □

REMARK 3.2. It is easy to check that if we take $a_{ni}=1$ for each $1\leqslant i\leqslant n$, $n\geqslant 1$, $r=1$, $p=1/\alpha$ with $1/2<\alpha<1$, Theorem 3.3 equals to Theorem A. Hence, Theorem 3.3 improves Theorem A to a more general situation. Furthermore, Huang and Wu [8] also obtained the similar result for CNA random vectors in Hilbert spaces under the condition $\sum_{i=1}^{n}a_{ni}^2=O(n)$. Noting that $\sum_{i=1}^{n}a_{ni}^2=O(n)$ is released to $\sum_{i=1}^{n}|a_{ni}|^q=O(n)$ in Theorem 3.3 and CNA implies CAANA, our result also improves and extends the corresponding result of Huang and Wu [8].

By Theorem 3.3, we can further obtain the following results. The first one is the complete convergence for maximum weighted sums of H-valued CAANA random vectors as follows.

THEOREM 3.4. *Let $1\leqslant p<2$ and $\alpha p\geqslant 1$. Let $\{a_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ be an array of real numbers such that $\sum_{i=1}^{n}|a_{ni}|^q=O(n)$ for some $q>p$. Let $\{X_n,n\geqslant 1\}$ be a sequence of zero mean H-valued CAANA random vectors with mixing coefficients $\{q(n),n\geqslant 1\}$ satisfying $\sum_{n=1}^{\infty}q^2(n)<\infty$. Suppose that $\{X_n,n\geqslant 1\}$ is coordinatewise stochastically dominated by a random vector X. If $\sum_{j=1}^{\infty}E|X^{(j)}|^p<\infty$, then for any*

$\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| > \varepsilon n^{\alpha}\right) < \infty$$

Proof. Taking $0 < r < p$, we obtain by Theorem 3.3 that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \frac{1}{2}\varepsilon n^{\alpha}\right)_+^r \\
&\geqslant \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} \int_0^{\left(\frac{1}{2}\varepsilon n^{\alpha}\right)^r} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \frac{1}{2}\varepsilon n^{\alpha} > x^{1/r}\right) dx \\
&\geqslant \left(\frac{\varepsilon}{2}\right)^r \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| > \varepsilon n^{\alpha}\right).
\end{aligned}$$

The proof is complete. □

Another corollary of Theorem 3.3 is the complete f-moment convergence for maximum weighted sums of H-valued CAANA random vectors.

To obtain the complete f-moment convergence, define $f : \mathbb{R}^+ \to \mathbb{R}^+$ as an increasing and continuous function with $f(0) = 0$. Let $g(x)$ be the inverse function of $f(x)$, scilicet, $g(f(x)) = x$, $x \geqslant 0$. Assume that for some positive constant δ, the function f satisfies

$$\int_{f(\delta)}^{\infty} g^{-r}(t)dt < \infty, \tag{7}$$

where $0 < r < 2$ is defined in Theorem 3.3.

THEOREM 3.5. *Suppose that the conditions of Theorem 3.3 hold. Then for any function f satisfies* (7)*, we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} Ef\left(\left\{n^{-\alpha}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \varepsilon\right\}_+\right) < \infty.$$

Proof. Note that the result in Theorem 3.4 holds under the conditions of Theorem 3.3. It follows from Theorem 3.3, Theorem 3.4 and (7) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} Ef\left(\left\{n^{-\alpha}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \varepsilon\right\}_+\right) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2} \int_0^{\infty} P\left(f\left(\left\{n^{-\alpha}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \varepsilon\right\}_+\right) > t\right) dt \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2} \int_0^{\infty} P\left(n^{-\alpha}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| > \varepsilon + g(t)\right) dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\alpha p-2} \int_0^{f(\delta)} P\left(n^{-\alpha} \max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| > \varepsilon + g(t)\right) dt \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} \int_{f(\delta)}^{\infty} P\left(n^{-\alpha} \max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| > \varepsilon + g(t)\right) dt \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} \int_0^{f(\delta)} P\left(n^{-\alpha} \max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| > \varepsilon\right) dt \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} \int_{f(\delta)}^{\infty} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \varepsilon n^{\alpha} > g(t)n^{\alpha}\right) dt \\
&\leqslant f(\delta) \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| > \varepsilon n^{\alpha}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2} E\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_{ni}X_i\right\| - \varepsilon n^{\alpha}\right)_+^r \int_{f(\delta)}^{\infty} g^{-r}(t)dt \\
&< \infty.
\end{aligned}
$$

The proof is complete. □

THEOREM 3.6. *Suppose that the conditions of Theorem 3.5 hold with* $\alpha p > 1$ *and* a_{ni} *being replaced by* a_i *for each* $1 \leqslant i \leqslant n$*, we have that*

$$
\sum_{n=1}^{\infty} n^{\alpha p-2} Ef\left(\left\{\sup_{m\geqslant n} m^{-\alpha}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right) < \infty.
$$

Proof. It is evident that $\sum_{j=1}^{l} 2^{j(\alpha p-1)} \leqslant C 2^{l(\alpha p-1)}$. Noting that f is increasing, we obtain that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} Ef\left(\left\{\sup_{m\geqslant n} m^{-\alpha}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right) \\
&= \sum_{j=1}^{\infty} \sum_{n=2^{j-1}}^{2^j-1} n^{\alpha p-2} Ef\left(\left\{\sup_{m\geqslant n} m^{-\alpha}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right) \\
&\leqslant C \sum_{j=1}^{\infty} 2^{j(\alpha p-1)} Ef\left(\left\{\sup_{m\geqslant 2^{j-1}} m^{-\alpha}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right) \\
&\leqslant C \sum_{l=1}^{\infty} Ef\left(\left\{\max_{2^{l-1}\leqslant m<2^l} m^{-\alpha}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right) \sum_{j=1}^{l} 2^{j(\alpha p-1)} \\
&\leqslant C \sum_{l=1}^{\infty} 2^{l(\alpha p-1)} Ef\left(\left\{2^{-\alpha(l-1)} \max_{2^{l-1}\leqslant m<2^l}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right) \\
&\leqslant C \sum_{l=1}^{\infty} 2^{l(\alpha p-1)} Ef\left(\left\{2^{-\alpha(l-1)} \max_{1\leqslant m<2^l}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right). \qquad (8)
\end{aligned}
$$

On the other hand, we obtain by Theorem 3.5 (taking the weight $2^{2\alpha/q}a_i$ for each $1 \leqslant i \leqslant n$) and the assumption f is increasing that

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2} Ef\left(\left\{2^{2\alpha} n^{-\alpha} \max_{1\leqslant m<n}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right)\\ &= \sum_{l=0}^{\infty}\sum_{n=2^l}^{2^{l+1}-1} n^{\alpha p-2} Ef\left(\left\{2^{2\alpha} n^{-\alpha} \max_{1\leqslant m<n}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right)\\ &\geqslant \frac{1}{2}\sum_{l=1}^{\infty} 2^{l(\alpha p-1)} Ef\left(\left\{2^{-\alpha(l-1)} \max_{1\leqslant m<2^l}\left\|\sum_{i=1}^{m} a_i X_i\right\| - \varepsilon\right\}_+\right). \end{aligned} \tag{9}$$

Hence, a combination of (8) and (9) completes the proof of the theorem. □

At the end of this section, we further give the Marcinkiewicz-Zygmund strong law of large numbers for maximum weighted sums of H-valued CAANA random vectors.

THEOREM 3.7. *Let $1 \leqslant p < 2$. Let $\{a_n, n \geqslant 1\}$ be a sequence of real numbers such that $\sum_{i=1}^{n}|a_i|^q = O(n)$ for some $q > p$. Let $\{X_n, n \geqslant 1\}$ be a sequence of zero mean H-valued CAANA random vectors with mixing coefficients $\{q(n), n \geqslant 1\}$ satisfying $\sum_{n=1}^{\infty} q^2(n) < \infty$. Suppose that $\{X_n, n \geqslant 1\}$ is coordinatewise stochastically dominated by a random vector X. If $\sum_{j=1}^{\infty} E|X^{(j)}|^p < \infty$, then*

$$\frac{1}{n^{1/p}}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_i X_i\right\| \to 0 \ \ a.s.$$

Proof. Applying Theorem 3.4 with $a_{ni} = a_i$ for each $1 \leqslant i \leqslant n$, $n \geqslant 1$ and taking $\alpha = 1/p$, we have that for any $\varepsilon > 0$,

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_i X_i\right\| > \varepsilon n^{1/p}\right)\\ &= \sum_{k=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1} n^{-1} P\left(\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_i X_i\right\| > \varepsilon n^{1/p}\right)\\ &\geqslant \frac{1}{2}\sum_{j=0}^{\infty} P\left(\max_{1\leqslant m\leqslant 2^j}\left\|\sum_{i=1}^{m} a_i X_i\right\| > \varepsilon (2^{j+1})^{1/p}\right).\end{aligned}$$

By the Borel-Cantelli lemma, this implies that as $j \to \infty$,

$$\frac{1}{(2^j)^{1/p}}\max_{1\leqslant m\leqslant 2^{j+1}}\left\|\sum_{i=1}^{m} a_i X_i\right\| = 4^{1/p}\frac{1}{(2^{j+2})^{1/p}}\max_{1\leqslant m\leqslant 2^{j+1}}\left\|\sum_{i=1}^{m} a_i X_i\right\| \to 0 \ \text{ a.s.}$$

On the other hand, for any fixed n, we notice that there always exists positive integer j such that $2^j \leqslant n < 2^{j+1}$. Hence, we can obtain that

$$\frac{1}{n^{1/p}}\max_{1\leqslant m\leqslant n}\left\|\sum_{i=1}^{m} a_i X_i\right\| \leqslant \frac{1}{(2^j)^{1/p}}\max_{1\leqslant m\leqslant 2^{j+1}}\left\|\sum_{i=1}^{m} a_i X_i\right\| \to 0 \ \text{ a.s.}$$

The proof is complete. □

Conclusions

In this paper, we investigate some convergence theorems such as weak law of large numbers, complete f-moment convergence, complete moment convergence, complete convergence and strong law of large numbers for maximum weighted sums of CAANA random vectors in Hilbert spaces. The results in this paper improve and extend the corresponding theorems of He and Pan [Open Mathematics 21, (2023), 20220556, doi:10.1515/math-2022-0556], Huang and Wu [Journal of Inequalities and Applications, 86 (2018), doi:10.1186/s13660-018-1678-y], and Ko [Statistics and Probability Letters, 138, (2018), 104–110, doi:10.1016/j.spl.2018.02.068].

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Authors' contributions. Both the two authors contributed equally and read and approved the final manuscript.

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REFERENCES

[1] T. K. CHANDRA, S. GHOSAL, *Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables*, Acta Mathematica Hungarica, **71** (4), (1996), 327–336.

[2] P. Y. CHEN, S. H. SUNG, *On the strong convergence for weighted sums of negatively associated random variables*, Statistics and Probability Letters, **92**, (2014), 45–52.

[3] Y. S. CHOW, *On the rate of moment convergence of sample sums and extremes*, Bulletin of the Institute of Mathematics, Academia Sinica, **16** (3), (1988), 177–201.

[4] Q. H. HE, L. PAN, *Convergence properties for coordinatewise asymptotically negatively associated random vectors in Hilbert space*, Open Mathematics **21**, (2023), 20220556, doi:10.1515/math-2022-0556.

[5] P. L. HSU AND H. ROBBINS, *Complete convergence and the law of large numbers*, Proceedings of the national academy of sciences, U.S.A., **33**, (1947), 25–31.

[6] N. V. HUAN, *On complete convergence for sequences of random vectors in Hilbert spaces*, Acta Mathematica Hungarica, **147** (1), (2015), 205–219.

[7] N. V. HUAN, N. V. QUANG, N. T. THUAN, *Baum-Katz type theorems for coordinatewise negatively associated random vectors in Hilbert spaces*, Acta Mathematica Hungarica, **144** (1), (2014), 132–149.

[8] X. HUANG, Y. F. WU, *Strong convergence theorems for coordinatewise negatively associated random vectors in Hilbert space*, Journal of Inequalities and Applications, **86** (2018), 13 pages, doi:10.1186/s13660-018-1678-y.

[9] K. JOAG-DEV, F. PROSCHAN, *Negative association of random variables with applications*, The Annals of Statistics, **11**, (1983), 286–295.
[10] M. H. KO, *On complete moment convergence for CAANA random vectors in Hilbert spaces*, Statistics and Probability Letters, **138**, (2018), 104–110.
[11] M. H. KO, *The complete moment convergence for CNA random vectors in Hilbert spaces*, Journal of Inequalities and Applications, **290**, (2017), doi:10.1186/s13660-017-1566-x.
[12] M. H. KO, T. S. KIM, K. H. HAN, *A note on the almost sure convergence for dependent random variables in a Hilbert space*, Journal of Theoretical Probability, **22**, (2009), 506–513.
[13] A. KUCZMASZEWSKA, *On complete convergence in Marcinkiewicz-Zygmund type SLLN for negatively associated random variables*, Acta Mathematica Hungarica, **128**, (2010), 116–130.
[14] Y. MIAO, *Hajeck-Renyi inequality for dependent random variables in Hilbert space and applications*, Rev. Unión Mat. Argentina., **53** (1), (2012), 101–112.
[15] Q. M. SHAO, *A comparison theorem on moment inequalities between negatively associated and independent random variables*, Journal of Theoretical Probability, **13**, (2000), 343–356.
[16] L. V. THANH, *On the almost sure convergence for dependent random vectors in Hilbert spaces*, Acta Mathematica Hungarica, **139** (3), (2013), 276–285.
[17] Q. Y. WU, *Probability limit theory for mixing sequences*, Science Press, Beijing, 2006.
[18] Y. WU, X. J. WANG, S. H. HU, *Complete moment convergence for weighted sums of weakly dependent random variables and its application in nonparametric regression model*, Statistics and Probability Letters, **127**, (2017), 56–66.
[19] Y. WU, X. J. WANG, T.-C. HU, A. VOLODIN, *Complete f-moment convergence for extended negatively dependent random variables*, RACSAM, **113**, (2019), 333–351.
[20] Y. WU, X. J. WANG, S. H. SUNG, *Complete moment convergence for arrays of rowwise negatively associated random variables and its application in nonparametric regression model*, Probability in the Engineering and Informational Sciences, **32**, (2018), 37–57.
[21] Y. WU, F. ZHANG, X. J. WANG, *Convergence properties for weighted sums of weakly dependent random vectors in Hilbert spaces*, Stochastics, **92** (5), (2020), 716–731.
[22] D. M. YUAN, J. AN, *Rosenthal type inequalities for asymptotically almost negatively associated random variables and applications*, Science in China Series A: Mathematics, **52** (9), (2009), 1887–1904.

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