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ADVANCED NUMERICAL RADIUS INEQUALITIES

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Abstract. In this article, we present some new inequalities for the numerical radius of products of Hilbert space operators. In particular, we show that if $S,T \in \mathbb{B}(\mathbb{H})$, then for any $\alpha,\beta > 0$

$$\omega^r(S^*T) \leqslant \frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{(T^*T)^r}{\alpha} + i\frac{(S^*S)^r}{\beta}\right),$$

for any $r \geqslant 1$. Some consequences that generalize some results from the literature are discussed.

1. Introduction

Given a complex Hilbert space $\mathbb{H}$, endowed with the inner product $\langle\cdot,\cdot\rangle$, we utilize $\mathbb{B}(\mathbb{H})$ to denote the C^*-algebra of all bounded linear operators on $\mathbb{H}$. For $T \in \mathbb{B}(\mathbb{H})$, the numerical radius and the operator norm of T are defined respectively by

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx,x\rangle| \text{ and } \|T\| = \sup_{\|x\|=1} \|Tx\|.$$

It is well known that if T is normal, in the sense that $T^*T = TT^*$, then $\|T\| = \omega(T)$. However, for non-normal operators, this equality fails. In general, the following holds for any $T \in \mathbb{B}(\mathbb{H})$:

$$\frac{1}{2}\|T\| \leqslant \omega(T) \leqslant \|T\|. \tag{1.1}$$

This inequality is significant because it approximates $\omega(T)$ in terms of $\|T\|$, a more manageable quantity to compute than $\omega(T)$. Sharpening (1.1) and other inequalities for the numerical radius has interested numerous researchers in the past few years; see [1, 4, 5, 6, 12] for example.

Among the most interesting bounds, we have the following:

$$\omega(S^*T) \leqslant \frac{1}{2}\left\||S|^2 + |T|^2\right\|, \tag{1.2}$$

$$\omega^2(T) \leqslant \frac{1}{2}\| \, |T|^2 + |T^*|^2 \, \|, \tag{1.3}$$

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$$\omega(T) \leqslant \frac{1}{2} \| |T| + |T^*| \|, \tag{1.4}$$

$$\omega(T) \leqslant \frac{1}{2} \left(\|T\| + \|T^2\|^{\frac{1}{2}} \right), \tag{1.5}$$

for any $S,T \in \mathbb{B}(\mathbb{H})$. Notice that inequality (1.2) is given in [10, Ineq. (17)], inequality (1.3) is shown in [10, Theorem 1], inequality (1.4) is proved in [8, Ineq. (8)], and inequality (1.5) is established in [8, Theorem 1].

It is shown in [11, Corollary 2.2] that

$$\omega(T) \leqslant \frac{\sqrt{2}}{2} \omega(|T| + i|T^*|). \tag{1.6}$$

It has been proven in [11] that this inequality improves upon (1.3).

Another upper estimate for the numerical radius is given in [7, Corollary 3.3] in the following form

$$\omega^2(T) \leqslant \frac{1}{4} \left\| |T|^2 + |T^*|^2 \right\| + \frac{1}{2} \omega(|T||T^*|). \tag{1.7}$$

In the same paper, the authors showed that this inequality strengthens (1.3) and (1.5).

In this paper, we extend inequality (1.6). Indeed, we will show that if $T \in \mathbb{B}(\mathbb{H})$, $0 \leqslant t \leqslant 1$, and $r \geqslant 1$, then for any $\alpha, \beta > 0$

$$\omega^r(T) \leqslant \frac{\sqrt{2\alpha\beta}}{2} \omega \left(\frac{(T^*T)^{rt}}{\alpha} + i \frac{(TT^*)^{r(1-t)}}{\beta} \right).$$

We also present a refinement and extension of inequality (1.7) by proving that

$$\omega^2(T) \leqslant \frac{1}{4} \omega \left(|T|^2 + |T^*|^2 + 2|T||T^*| \right).$$

To achieve our goal, we need the following lemmas.

LEMMA 1.1. [14, Ineq. (2.3)] *Let $T \in \mathbb{B}(\mathbb{H})$. Then*

$$\|\Re T\| \leqslant \omega(T).$$

LEMMA 1.2. [2, Theorem 2.3] *Let f be a non-negative non-decreasing convex function on $[0,\infty)$ and let $A,B \in \mathbb{B}(\mathbb{H})$ be positive operators. Then for any $0 \leqslant t \leqslant 1$*

$$\|f((1-t)A + tB)\| \leqslant \|(1-t)f(A) + tf(B)\|.$$

LEMMA 1.3. [9] *Let $A,B,C \in \mathbb{B}(\mathbb{H})$ be such that A,B are positive. Then $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator in $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$ if and only if*

$$|\langle Cx, y \rangle|^2 \leqslant \langle Ax, x \rangle \langle By, y \rangle$$

for all $x, y \in \mathbb{H}$.

2. Results

We begin this section with a useful lemma.

LEMMA 2.1. *Let $S,T \in \mathbb{B}(\mathbb{H})$. If h is an increasing convex function on $[0,\infty)$, then*

$$h(\omega(S^*T)) \leqslant \left\| \frac{h(T^*T)+h(S^*S)}{2} \right\|.$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. Then

$$\begin{aligned}
|\langle S^*Tx,x\rangle| &= |\langle Tx,Sx\rangle| \\
&\leqslant \|Tx\|\,\|Sx\| \\
&= \sqrt{\langle T^*Tx,x\rangle\,\langle S^*Sx,x\rangle} \\
&\leqslant \frac{1}{2}\left(\langle T^*Tx,x\rangle + \langle S^*Sx,x\rangle\right) \\
&= \frac{1}{2}\langle (T^*T+S^*S)x,x\rangle,
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality, and the second inequality is obtained from the arithmetic-geometric mean inequality. After taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$, we have

$$\omega(S^*T) \leqslant \frac{1}{2}\|T^*T+S^*S\|.$$

Now, since h is an increasing convex function on $[0,\infty)$, by Lemma 1.2, we have

$$\begin{aligned}
h(\omega(S^*T)) &\leqslant h\left(\frac{1}{2}\|T^*T+S^*S\|\right) \\
&= \left\| h\left(\frac{T^*T+S^*S}{2}\right)\right\| \\
&\leqslant \left\| \frac{h(T^*T)+h(S^*S)}{2} \right\|,
\end{aligned}$$

as required. □

We start this section with the following generalization and modification of (1.7). The advantages of this result will be discussed later.

THEOREM 2.1. *Let $S,T \in \mathbb{B}(\mathbb{H})$. If h is an increasing convex function on $[0,\infty)$, then*

$$h^2(\omega(S^*T)) \leqslant \frac{1}{4}\omega\left(h^2(T^*T)+h^2(S^*S)+2h(T^*T)h(S^*S)\right).$$

In particular, for any $r \geqslant 1$

$$\omega^{2r}(S^*T) \leqslant \frac{1}{4}\omega\left((T^*T)^{2r}+(S^*S)^{2r}+2(T^*T)^r(S^*S)^r\right). \tag{2.1}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
&h^2(\omega(S^*T)) \\
&\leqslant \left\| \frac{h(T^*T)+h(S^*S)}{2} \right\|^2 \\
&= \left\| \left(\frac{h(T^*T)+h(S^*S)}{2} \right)^2 \right\| \\
&= \frac{1}{4} \left\| h^2(T^*T)+h^2(S^*S)+h(T^*T)h(S^*S)+h(S^*S)h(T^*T) \right\| \\
&= \frac{1}{4} \left\| h^2(T^*T)+h^2(S^*S)+2\Re(h(T^*T)h(S^*S)) \right\| \\
&= \frac{1}{4} \left\| \Re\left(h^2(T^*T)+h^2(S^*S)+2h(T^*T)h(S^*S)\right) \right\| \\
&\leqslant \frac{1}{4}\omega\left(h^2(T^*T)+h^2(S^*S)+2h(T^*T)h(S^*S)\right) \quad \text{(by Lemma 1.1)},
\end{aligned}$$

as required. □

REMARK 2.1. If we put $r=1$, in (2.1), we obtain

$$\omega^2(S^*T) \leqslant \frac{1}{4}\omega\left(|T|^4+|S|^4+2|T|^2|S|^2\right).$$

The following result is a direct consequence of Theorem 2.1.

COROLLARY 2.1. *Let $A \in \mathbb{B}(\mathbb{H})$. If h is an increasing convex function on $[0,\infty)$, then for any $0 \leqslant t \leqslant 1$*

$$h^2(\omega(A)) \leqslant \frac{1}{4}\omega\left(h^2\left(|A|^{2t}\right)+h^2\left(|A^*|^{2(1-t)}\right)+2h\left(|A|^{2t}\right)h\left(|A^*|^{2(1-t)}\right)\right).$$

In particular, for any $r \geqslant 1$

$$\omega^{2r}(A) \leqslant \frac{1}{4}\omega\left(|A|^{4rt}+|A^*|^{4r(1-t)}+2|A|^{2rt}|A^*|^{2r(1-t)}\right). \tag{2.2}$$

Proof. Let $A=U|A|$ be the polar decomposition of A. Put $S^*=U|A|^{1-t}$ and $T=|A|^t$, we infer that

$$\begin{aligned}
&h^2(\omega(A)) \\
&\leqslant \frac{1}{4}\omega\left(h^2\left(|A|^{2t}\right)+h^2\left(U|A|^{2(1-t)}U^*\right)+2h\left(|A|^{2t}\right)h\left(U|A|^{2(1-t)}U^*\right)\right) \\
&= \frac{1}{4}\omega\left(h^2\left(|A|^{2t}\right)+h^2\left(|A^*|^{2(1-t)}\right)+2h\left(|A|^{2t}\right)h\left(|A^*|^{2(1-t)}\right)\right),
\end{aligned}$$

as required. □

REMARK 2.2. If we put $t=\frac{1}{2}$, in (2.2), we infer

$$\omega^{2r}(T)\leqslant\frac{1}{4}\omega\left(|T|^{2r}+|T^*|^{2r}+2|T|^r|T^*|^r\right),$$

for any $T\in\mathbb{B}(\mathbb{H})$.

REMARK 2.3. Let $T\in\mathbb{B}(\mathbb{H})$. If we put $r=1$, in (2.2), we obtain

$$\omega^{2}(T)\leqslant\frac{1}{4}\omega\left(|T|^{4t}+|T^*|^{4(1-t)}+2|T|^{2t}|T^*|^{2(1-t)}\right),$$

for any $0\leqslant t\leqslant 1$. In particular,

$$\omega^{2}(T)\leqslant\frac{1}{4}\omega\left(|T|^{2}+|T^*|^{2}+2|T||T^*|\right). \tag{2.3}$$

REMARK 2.4. It follows from (2.3) that

$$\begin{aligned}\omega^{2}(T)&\leqslant\frac{1}{4}\omega\left(|T|^{2}+|T^*|^{2}+2|T||T^*|\right)\\&\leqslant\frac{1}{4}\omega\left(|T|^{2}+|T^*|^{2}\right)+\frac{1}{2}\omega\left(|T||T^*|\right)\\&\quad\text{(by the triangle inequality)}\\&=\frac{1}{4}\left\||T|^{2}+|T^*|^{2}\right\|+\frac{1}{2}\omega\left(|T||T^*|\right)\\&\quad\text{(since } |T|^{2}+|T^*|^{2}\text{ is normal).}\end{aligned}$$

Therefore, our result improves (1.7).

As mentioned in the introduction, one of our main goals in this paper is to present an extension of (1.6). This goal is addressed in the following result. To do this, we need the definition of doubly convex functions. Recall that $f:[0,\infty)\to[0,\infty)$ is called geometrically convex [13] if

$$f\left(\sqrt{ab}\right)\leqslant\sqrt{f(a)f(b)},$$

for any $a,b\geqslant 0$. We will say that a function f is doubly convex if f is convex in the usual sense [3].

THEOREM 2.2. *Let $S,T\in\mathbb{B}(\mathbb{H})$. If h is an increasing doubly convex function on $[0,\infty)$, then*

$$h(\omega(S^*T))\leqslant\frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{h(T^*T)}{\alpha}+i\frac{h(S^*S)}{\beta}\right),$$

for any $\alpha,\beta>0$. In particular, for any $r\geqslant 1$

$$\omega^{r}(S^*T)\leqslant\frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{(T^*T)^r}{\alpha}+i\frac{(S^*S)^r}{\beta}\right).$$

Proof. We know that for any $a,b>0$

$$\sqrt{ab}\leqslant\frac{a+b}{2},$$

holds. Now, if we replace a and b by $\frac{1}{\alpha}a$ and $\frac{1}{\beta}b$, $\alpha,\beta>0$, we get

$$\sqrt{ab}\leqslant\frac{\sqrt{\alpha\beta}}{2}\left(\frac{a}{\alpha}+\frac{b}{\beta}\right). \tag{2.4}$$

On the other hand, it has been shown in [11] that

$$\|A+B\|\leqslant\sqrt{2}\omega\left(A+iB\right), \tag{2.5}$$

when $A,B\in\mathbb{B}(\mathbb{H})$ are self-adjoint operators. Hence,

$$\begin{aligned}
h\left(\left|\left\langle S^{*}Tx,x\right\rangle\right|\right) &= h\left(\left|\left\langle Tx,Sx\right\rangle\right|\right)\\
&\leqslant h\left(\|Tx\|\,\|Sx\|\right)\\
&\leqslant h\left(\sqrt{\left\langle T^{*}Tx,x\right\rangle\left\langle S^{*}Sx,x\right\rangle}\right)\\
&\leqslant \sqrt{h\left(\left\langle T^{*}Tx,x\right\rangle\right)h\left(\left\langle S^{*}Sx,x\right\rangle\right)}\\
&\leqslant \sqrt{\left\langle h\left(T^{*}T\right)x,x\right\rangle\left\langle h\left(S^{*}S\right)x,x\right\rangle}\\
&\leqslant \frac{\sqrt{\alpha\beta}}{2}\left(\frac{\left\langle h\left(T^{*}T\right)x,x\right\rangle}{\alpha}+\frac{\left\langle h\left(S^{*}S\right)x,x\right\rangle}{\beta}\right)\\
&= \frac{\sqrt{\alpha\beta}}{2}\left\langle\left(\frac{h\left(T^{*}T\right)}{\alpha}+\frac{h\left(S^{*}S\right)}{\beta}\right)x,x\right\rangle.
\end{aligned}$$

So,

$$h\left(\omega\left(S^{*}T\right)\right)\leqslant\frac{\sqrt{\alpha\beta}}{2}\left\|\frac{h\left(T^{*}T\right)}{\alpha}+\frac{h\left(S^{*}S\right)}{\beta}\right\|.$$

Now, by (2.5), we can write

$$\begin{aligned}
h\left(\omega\left(S^{*}T\right)\right) &\leqslant \frac{\sqrt{\alpha\beta}}{2}\left\|\frac{h\left(T^{*}T\right)}{\alpha}+\frac{h\left(S^{*}S\right)}{\beta}\right\|\\
&\leqslant \frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{h\left(T^{*}T\right)}{\alpha}+i\frac{h\left(S^{*}S\right)}{\beta}\right).
\end{aligned}$$

Indeed, we have shown that

$$h\left(\omega\left(S^{*}T\right)\right)\leqslant\frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{h\left(T^{*}T\right)}{\alpha}+i\frac{h\left(S^{*}S\right)}{\beta}\right),$$

as required. □

The following result is a direct consequence of Theorem 2.2.

COROLLARY 2.2. *Let $A \in \mathbb{B}(\mathbb{H})$. If h is an increasing doubly convex function on $[0,\infty)$, then*

$$h(\omega(A)) \leqslant \frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{h\left((A^*A)^t\right)}{\alpha} + i\frac{h\left((AA^*)^{1-t}\right)}{\beta}\right),$$

for any $\alpha,\beta > 0$. In particular, for any $r \geqslant 1$

$$\omega^r(A) \leqslant \frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{(A^*A)^{rt}}{\alpha} + i\frac{(AA^*)^{r(1-t)}}{\beta}\right). \tag{2.6}$$

Proof. Let $A = U|A|$ be the polar decomposition of A. Put $S^* = U|A|^{1-t}$ and $T = |A|^t$, in Theorem 2.2, we infer that

$$\begin{aligned} h(\omega(A)) &\leqslant \frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{h(T^*T)}{\alpha} + i\frac{h(S^*S)}{\beta}\right) \\ &= \frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{h\left(|A|^{2t}\right)}{\alpha} + i\frac{h\left(U|A|^{2(1-t)}U^*\right)}{\beta}\right) \\ &= \frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{h\left(|A|^{2t}\right)}{\alpha} + i\frac{h\left(|A^*|^{2(1-t)}\right)}{\beta}\right), \end{aligned}$$

as required. □

REMARK 2.5. If we put $\alpha = \beta = 1$, $r = 1$, and $t = \frac{1}{2}$, in (2.6), then we reobtain (1.6). Namely, our result extends (1.6).

The next two results extend two Theorems 2.1 and 2.2 in the above. We remind here that $\begin{bmatrix} T^*T & T^*S \\ S^*T & S^*S \end{bmatrix}$ is a positive operator in $\mathbb{B}(\mathbb{H}\oplus\mathbb{H})$.

THEOREM 2.3. *Let $A,B,C \in \mathbb{B}(\mathbb{H})$ be such that A,B are positive and let h be an increasing convex function on $[0,\infty)$. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator in $\mathbb{B}(\mathbb{H}\oplus\mathbb{H})$, then*

$$h^2(\omega(C)) \leqslant \frac{1}{4}\omega\left(h^2(A) + h^2(B) + 2h(A)h(B)\right).$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. Since $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator, then from Lemma 1.3, we have

$$\begin{aligned} |\langle Cx, x\rangle| &\leqslant \sqrt{\langle Ax, x\rangle \langle Bx, x\rangle} \\ &\leqslant \frac{1}{2}\left(\langle Ax, x\rangle + \langle Bx, x\rangle\right) \\ &\quad \text{(by the arithmetic-geometric mean inequality)} \\ &= \frac{1}{2}\langle (A+B)x, x\rangle \end{aligned}$$

i.e.,

$$|\langle Cx, x\rangle| \leqslant \frac{1}{2}\langle (A+B)x, x\rangle .$$

By taking the supremum over all unit vectors $x \in \mathbb{H}$, we obtain

$$\omega(C) \leqslant \frac{1}{2}\|A+B\| .$$

Since h is an increasing convex function on $[0, \infty)$, we can write

$$\begin{aligned} h(\omega(C)) &\leqslant h\left(\frac{1}{2}\|A+B\|\right) \\ &= \left\| h\left(\frac{A+B}{2}\right)\right\| \\ &\leqslant \frac{1}{2}\|h(A)+h(B)\| \quad \text{(by Lemma 1.2).} \end{aligned}$$

Consequently,

$$h(\omega(C)) \leqslant \frac{1}{2}\|h(A)+h(B)\| .$$

Thus,

$$\begin{aligned} h^2(\omega(C)) &\leqslant \frac{1}{4}\|h(A)+h(B)\|^2 \\ &= \frac{1}{4}\left\|(h(A)+h(B))^2\right\| \\ &= \frac{1}{4}\left\|h^2(A)+h^2(B)+h(A)h(B)+h(B)h(A)\right\| \\ &= \frac{1}{4}\left\|h^2(A)+h^2(B)+2\Re(h(A)h(B))\right\| \\ &= \frac{1}{4}\left\|\Re\left(h^2(A)+h^2(B)+2h(A)h(B)\right)\right\| \\ &\leqslant \frac{1}{4}\omega\left(h^2(A)+h^2(B)+2h(A)h(B)\right) \quad \text{(by Lemma 1.1),} \end{aligned}$$

as required. $\square$

THEOREM 2.4. *Let $A,B,C \in \mathbb{B}(\mathbb{H})$ be such that A,B are positive and let h be an increasing doubly convex function on $[0,\infty)$. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator in $\mathbb{B}(\mathbb{H}\oplus\mathbb{H})$, then*

$$h(\omega(C)) \leqslant \frac{\sqrt{2\alpha\beta}}{2}\omega\left(\frac{h(A)}{\alpha}+i\frac{h(B)}{\beta}\right),$$

for any $\alpha,\beta > 0$.

Proof. Let $x \in \mathbb{H}$ be a unit vector. Since $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator and h is increasing geometrically convex function on $[0,\infty)$, we have

$$\begin{aligned} h(|\langle Cx,x\rangle|) &\leqslant h\left(\sqrt{\langle Ax,x\rangle\langle Bx,x\rangle}\right) \\ &\leqslant \sqrt{h(\langle Ax,x\rangle)h(\langle Bx,x\rangle)} \\ &\leqslant \sqrt{\langle h(A)x,x\rangle\langle h(B)x,x\rangle} \\ &\leqslant \frac{\sqrt{\alpha\beta}}{2}\left(\frac{\langle h(A)x,x\rangle}{\alpha}+\frac{\langle h(B)x,x\rangle}{\beta}\right) \quad \text{(by (2.4))} \\ &= \frac{\sqrt{\alpha\beta}}{2}\left\langle\left(\frac{h(A)}{\alpha}+\frac{h(B)}{\beta}\right)x,x\right\rangle \end{aligned}$$

i.e.,

$$h(|\langle Cx,x\rangle|) \leqslant \frac{\sqrt{\alpha\beta}}{2}\left\langle\left(\frac{h(A)}{\alpha}+\frac{h(B)}{\beta}\right)x,x\right\rangle.$$

By taking the supremum over all unit vectors $x \in \mathbb{H}$, we obtain

$$h(\omega(C)) \leqslant \frac{\sqrt{\alpha\beta}}{2}\left\|\frac{h(A)}{\alpha}+\frac{h(B)}{\beta}\right\|.$$

We infer the desired result if we apply (2.5) on the right side of the inequality above. This completes the proof. □

Declarations

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