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BOUNDEDNESS OF AVERAGING OPERATORS ON NON–DOUBLING MANIFOLDS WITH ENDS

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Abstract. In this paper, we explicitly calculate the best constant for weak-type of the operator S_δ which averages $f \in L^p(\mathbb{R}^n)$ over $B(x,\delta|x|)$, introduced by Christ and Grafakos in Proc. Amer. Math. Soc. 123 (1995) 1687–1693. Let M be a non-doubling manifold with two ends $\mathbb{R}^m \sharp \mathscr{R}^n$ with $m > n \geqslant 2$. We also show the weak type of the operator S_δ on $L^p(M)$ and $L^p(M)$ boundedness of the operators S_1 and S_2.

1. Introduction

Let f be a locally integrable function on $\mathbb{R}^n$. In [2], Christ and Grafakos considered the following two averaging operators:

$$(T_\delta f)(x) = \frac{1}{|B(0,\delta|x|)|}\int_{B(0,\delta|x|)} f(y)dy$$

and

$$(S_\delta f)(x) = \frac{1}{|B(x,\delta|x|)|}\int_{B(x,\delta|x|)} f(y)dy$$

for any $\delta > 0$. For $p > 1$, they proved that the operator norm of T_1 on $L^p(\mathbb{R}^n)$ is equal to $\frac{p}{p-1}$, which means that it is the same as in the usual one dimensional case. Since $(T_\delta f)(x) = (T_1 f)(\delta x)$, it is immediate that the operator norm of T_δ on $L^p(\mathbb{R}^n)$ is $\frac{p}{p-1}\delta^{-\frac{n}{p}}$. In some sense, the operator S_δ lies between the identity operator and the Hardy-Littlewood maximal function M, and that Mf is not much larger than f.

Let $1 < p < \infty$ and $c_{p,n} = p'\frac{\omega_{n-2}}{\omega_{n-1}}2^{\frac{n}{p'}-1}B(\frac{1}{2}(\frac{n}{p'}-1),\frac{n-3}{2})$, Christ and Grafakos in [2] obtained:

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|B(x,|x|)|}\int_{B(x,|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}} \leqslant c_{p,n}\left(\int_{\mathbb{R}^n}|f(y)|^p dy\right)^{\frac{1}{p}}$$

for all f in $L^p(\mathbb{R}^n)$ and the constant $c_{p,n}$ is the best possible. More generally, they also obtained

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THEOREM A. *For $\delta > 1$, the operator norm of S_δ on $L^p(\mathbb{R}^n)$ is*

$$p' \frac{\omega_{n-2}}{\omega_{n-1}} \frac{1}{\delta^n} \int_{-1}^{1} (1-s^2)^{\frac{n-3}{2}} (s+\sqrt{s^2+\delta^2-1})^{\frac{n}{p'}} \, ds.$$

For $\delta < 1$, the operator norm of S_δ on $L^p(\mathbb{R}^n)$ is

$$p' \frac{\omega_{n-2}}{\omega_{n-1}} \frac{1}{\delta^n} \int_{\sqrt{1-\delta^2}}^{1} \Big[(s+\sqrt{s^2+\delta^2-1})^{\frac{n}{p'}} - (s-\sqrt{s^2+\delta^2-1})^{\frac{n}{p'}} \Big] \, ds.$$

The authors in [3] and [4] introduced the generalized averaging operators T_δ and S_δ and derived the related mixed means inequalities. In [11], the authors obtained that the sharp bound for the weak-type (p,p) inequality for T_1 is 1, where $1 \leqslant p \leqslant \infty$. In fact, we could easily obtained the sharp bound for the weak-type (p,p) inequality for T_δ is $\delta^{-\frac{n}{p}}$. Some improved results and further extended to other function spaces can be found in [5, 8, 9, 10] and the references therein.

Let us recall manifolds with ends as in [7]. Let M be a complete non-compact Riemannian manifold. Let $K \subset M$ be a compact set with non-empty interior and smooth boundary such that $M \setminus K$ has k connected components $E_1, \ldots, E_k$ and each E_i is non-compact. We say in such a case that M has k ends with respect to K and refer to K as the central part of M. In many cases, each E_i is isometric to the exterior of a compact set in another manifold M_i. In such case, we write $M = M_1 \sharp M_2 \sharp \cdots \sharp M_k$ and refer to M as a connected sum of the manifolds M_i, $i = 1, 2, \cdots, k$.

Following [7] we consider the following model case. Fix a large integer N (which will be the topological dimension of M) and, for any integer $m \in [2, N]$, define the manifold $\mathscr{R}^m$ by

$$\mathscr{R}^m = \mathbb{R}^m \times \mathbb{S}^{N-m},$$

where $\mathbb{S}^{N-m}$ is the unit sphere in $\mathbb{R}^{m-n}$. The manifold $\mathscr{R}^m$ has topological dimension N but its "dimension at infinity" is m in the sense that $V(x,r) \approx r^m$ for $r \geqslant 1$, see [7]. Thus, for different values of m, the manifold $\mathscr{R}^m$ have different dimension at infinity but the same topological dimension N, this enables us to consider finite connected sums of the $\mathscr{R}^m$'s.

Fix N and k integers $N_1, N_2, \cdots, N_k \in [2, N]$ such that

$$N = \max\{N_1, N_2, \cdots, N_k\}.$$

Next consider the manifold

$$M = \mathscr{R}^{N_1} \sharp \mathscr{R}^{N_2} \sharp \cdots \sharp \mathscr{R}^{N_k}.$$

In [7] Grigor'yan and Saloff-Coste establish both the global upper bound and lower bound for the heat kernel acting on this model class. Now we recall the first part of their results with the hypothesi that

$$n := \min_{1 \leqslant i \leqslant k} N_i > 2.$$

Let K be the central part of M and $E_1, E_2, \cdots, E_K$ be the ebds of M so that E_i is isometric to the complement of a compact set in $\mathscr{R}^{N_i}$. With $E_i = \mathscr{R}^{N_i} \setminus K$. Thus, $x \in \mathscr{R}^{N_i} \setminus K$ means that the point $x \in M$ belongs to the end associated with $\mathscr{R}^{N_i}$. For any $x \in M$, define

$$|x| := \sup_{z \in K} d(x,z),$$

where $d = d(x,y)$ is the geodesic distance in M. One can see that $|x|$ is separated from zero on M and $|x| \approx 1 + d(x,K)$ where $d(x,K) = \inf\{d(x,y) : y \in K\}$.

For $x \in M$, let

$$B(x,r) := \{y \in M : d(x,y) < r\}$$

be the geodesic ball with center $x \in M$ and radius $r > 0$ and let $V(x,r) = \mu(B(x,r))$ where μ is a Riemannian measure on M.

Throughout the paper, we take the simple case $k = 2$ for the model of metric spaces with non-doubling measure, i.e., we set $M = \mathbb{R}^m \sharp \mathscr{R}^n$ with $2 \leqslant n < m$. From the construction of the manifold M, we see that M does not satisfy the doubling condition since

1. $V(x,r) \approx r^m$ for all $x \in M$, when $r \leqslant 1$;
2. $V(x,r) \approx r^n$ for $B(x,r) \subset \mathscr{R}^n$, when $r > 1$;
3. $V(x,r) \approx r^m$ for $x \in \mathscr{R}^n \setminus K$, $r > 2|x|$, or $x \in \mathbb{R}^m$, $r > 1$.

In [6], Duong, Li and Sikora studied the boundedness of certain maximal functions on non-doubling manifolds with ends. The authors in [1] showed certain singular integrals with non-smooth kernels acting on non-doubling spaces. More generally, they obtained the holomorphic functional calculus of Laplace transform type for operators with suitable heat kernel upper bounds such as the Schrödinger operator on a non-doubling manifold with two ends.

The paper is organized as follows: in section 2 we obtain the weak-type estimates for S_δ on $L^p(\mathbb{R}^n)$ and calculate the weak-type norm. We also get the weak-type boundedness of S_δ on M. In section 3, we consider the boundedness of S_1 on $L^p(M)$. In section 4, we consider the boundedness of S_2 on $L^p(M)$.

Let us introduce some notation. ω_{n-1} will denote the area of the unit sphere S^{n-1} and v_n the volume of the unit ball in $\mathbb{R}^n$. Let $B(s,t)$ denote the usual beta-function $\int_0^1 x^t(1-x)^s dx$. We denote by $|A|$ the Lebesgue measure of the set A and by χ_A its characteristic function.

2. Weak type bounds for Hardy operator

PROPOSITION 2.1. *For $1 \leqslant p \leqslant \infty$ and $\delta > 0$, the following inequality*

$$\|S_\delta f\|_{L^{p,\infty}(\mathbb{R}^n)} \leqslant \delta^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}$$

holds. Moreover,

$$\|S_\delta\|_{L^p(\mathbb{R}^n) \to L^{p,\infty}(\mathbb{R}^n)} = \delta^{-\frac{n}{p}}.$$

Proof. We only give the proof for the case $1<p<\infty$, with the usual modifications made when $p=1$ or $p=\infty$. For $0<\lambda<\infty$, we have

$$\begin{aligned}
&|\{x\in\mathbb{R}^n : |S_\delta f(x)|>\lambda\}| \\
&\leqslant \left|\left\{x\in\mathbb{R}^n : \frac{1}{v_n|\delta x|^n}\left(\int_{B(x,\delta|x|)}|f(y)|^p dy\right)^{\frac{1}{p}}\left(\int_{B(x,\delta|x|)}dy\right)^{\frac{1}{p'}}>\lambda\right\}\right| \\
&\leqslant \left|\left\{x\in\mathbb{R}^n : \lambda(v_n|\delta x|^n)^{\frac{1}{p}}<\|f\|_{L^p(\mathbb{R}^n)}\right\}\right| \\
&=\delta^{-n}\frac{\|f\|^p_{L^p(\mathbb{R}^n)}}{\lambda^p}.
\end{aligned}$$

On the other hand, we will show that the constant $\delta^{-\frac{n}{p}}$ is the best possible. For any $\varepsilon>0$, taking $f_\varepsilon(x)=\chi_{[0,\varepsilon]}(|x|)$, we obtain $\|f_\varepsilon\|^p_{L^p(\mathbb{R}^n)}=v_n\varepsilon^n$ and $S_\delta f_\varepsilon(x)\leqslant 1$.

For $0<\lambda<1$, we divide ε into two cases:

(i) If $\varepsilon\geqslant|x|$, then $S_\delta f_\varepsilon(x)=1$ and

$$|\{x\in\mathbb{R}^n : |S_\delta f_\varepsilon(x)|>\lambda\}|=|\{x\in\mathbb{R}^n : 0<|x|\leqslant\varepsilon\}|=v_n\varepsilon^n.$$

(ii) If $0<\varepsilon<|x|$, then $S_\delta f_\varepsilon(x)=\frac{\varepsilon^n}{|\delta x|^n}$ and

$$|\{x\in\mathbb{R}^n : |S_\delta f_\varepsilon(x)|>\lambda\}|=\left|\left\{x\in\mathbb{R}^n : \frac{\varepsilon^n}{|\delta x|^n}>\lambda, 0<\varepsilon<|x|\right\}\right|=v_n\varepsilon^n\left(\frac{1}{\lambda\delta^n}-1\right).$$

From the above results, we have

$$|\{x\in\mathbb{R}^n : |S_\delta f_\varepsilon(x)|>\lambda\}|=\frac{v_n\varepsilon^n}{\lambda\delta^n}=\frac{1}{\lambda\delta^n}\|f_\varepsilon\|^p_{L^p(\mathbb{R}^n)}.$$

It implies that for $1<p<\infty$,

$$\sup_{0<\lambda<1}\lambda|\{x\in\mathbb{R}^n : |S_\delta f_\varepsilon(x)|>\lambda\}|^{\frac{1}{p}}=\sup_{0<\lambda<1}\lambda^{1-\frac{1}{p}}\delta^{-\frac{n}{p}}\|f_\varepsilon\|_{L^p(\mathbb{R}^n)}=\delta^{-\frac{n}{p}}\|f_\varepsilon\|_{L^p(\mathbb{R}^n)}.$$

Hence, we finish the proof. □

PROPOSITION 2.2. *Let* $1\leqslant p\leqslant\infty$ *and* $x\in M$. *Then*

(i) If $0<\delta\leqslant\frac{1}{|x|}$, *we have*

$$\|S_\delta f\|_{L^{p,\infty}(M)}\lesssim\delta^{-\frac{m}{p}}\|f\|_{L^p(M)}.$$

(ii) If $\delta\geqslant\frac{1}{|x|}$, *we have*

$$\|S_\delta f\|_{L^{p,\infty}(M)}\lesssim\max\{\delta^{-\frac{m}{p}},\delta^{-\frac{n}{p}}\}\|f\|_{L^p(M)}$$

Proof. (i) If $0 < \delta \leqslant \frac{1}{|x|}$, then $|B(x, \delta|x|)| = c_m \delta^m |x|^m$. Similarly as the proof of Proposition 2.1, we can finish the proof of (i).

(ii) If $\delta > \frac{1}{|x|}$, we split M into three components K, $\mathbb{R}^m \backslash K$ and $\mathscr{R}^n \backslash K$. Then

$$\begin{aligned}
&|\{x \in M : |S_\delta f(x)| > \lambda\}| \\
&\leqslant |\{x \in K : |S_\delta(f)(x)| > \lambda\}| + |\{x \in \mathbb{R}^m \backslash K : |S_\delta(f)(x)| > \lambda\}| \\
&\qquad\qquad + |\{x \in \mathscr{R}^n \backslash K : |S_\delta(f)(x)| > \lambda\}| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

To estimate I_1, we note that the measure of K is finite. Therefore $I_1 \leqslant |K|$. By Hölder's inequality, we have

$$|S_\delta(f)(x)| \leqslant |B(x, \delta|x|)|^{-\frac{1}{p}} \|f\|_{L^p(M)}.$$

For $x \in K$, we have $|B(x, \delta|x|)| = c_m \delta^m$. So

$$I_1 \leqslant |\{x \in K : \lambda < c_m^{-\frac{1}{p}} \delta^{-\frac{m}{p}} \|f\|_{L^p(M)}\}.$$

If $\lambda > c_m^{-\frac{1}{p}} \delta^{-\frac{m}{p}} \|f\|_{L^p(M)}$, then $I_1 = 0$. If $\lambda < c_m^{-\frac{1}{p}} \delta^{-\frac{m}{p}} \|f\|_{L^p(M)}$, then

$$\sup_{\lambda > 0} \lambda I_1^{\frac{1}{p}} \leqslant c_m^{-\frac{1}{p}} \delta^{-\frac{m}{p}} \|f\|_{L^p(M)}.$$

To estimate I_2, we have $|B(x, \delta|x|)| = c_m \delta^m |x|^m$ for $x \in \mathbb{R}^m \backslash K$. Therefore

$$|S_\delta(f)(x)| \leqslant c_m^{-\frac{1}{p}} \delta^{-\frac{m}{p}} |x|^{-\frac{m}{p}} \|f\|_{L^p(M)}.$$

Hence,

$$\begin{aligned}
I_2 &\leqslant |\{x \in \mathbb{R}^m \backslash K : c_m^{-\frac{1}{p}} \delta^{-\frac{m}{p}} |x|^{-\frac{m}{p}} \|f\|_{L^p(M)} > \lambda\}| \\
&\lesssim c_m \delta^{-m} \frac{\|f\|_{L^p(M)}^p}{\lambda^p}.
\end{aligned}$$

To estimate I_3, we consider three cases.

Case 1: $\delta > 1$ and $c_m(\delta - 1)^m |x|^m \leqslant |B(x, |x|)| = c_n |x|^n$. That is

$$1 < \delta < c_{m,n}^{\frac{1}{m}} |x|^{\frac{n}{m} - 1} + 1.$$

Hence, $|B(x, \delta|x|)| = c_n \delta^n |x|^n$ for $x \in \mathscr{R}^n \backslash K$. Therefore

$$\begin{aligned}
I_3 &\leqslant |\{x \in \mathscr{R}^n \backslash K : c_n^{-\frac{1}{p}} \delta^{-\frac{n}{p}} |x|^{-\frac{n}{p}} \|f\|_{L^p(M)} > \lambda\}| \\
&\lesssim c_n \delta^{-n} \frac{\|f\|_{L^p(M)}^p}{\lambda^p}.
\end{aligned}$$

Case 2: $\delta > c_{m,n}^{\frac{1}{m}}|x|^{\frac{n}{m}-1}+1$. For $x \in \mathscr{R}^n\backslash K$, we have $|B(x,\delta|x|)| = c_m\delta^m|x|^m$. Therefore

$$\begin{aligned} I_3 &\leqslant |\{x \in \mathscr{R}^n\backslash K : c_m^{-\frac{1}{p}}\delta^{-\frac{m}{p}}|x|^{-\frac{m}{p}}\|f\|_{L^p(M)} > \lambda\}| \\ &\lesssim c_m\delta^{-m}\frac{\|f\|_{L^p(M)}^p}{\lambda^p}. \end{aligned}$$

Case 3: $\frac{1}{|x|} < \delta < 1$. For $x \in \mathscr{R}^n\backslash K$, we have $|B(x,\delta|x|)| = c_n\delta^n|x|^n$. Therefore

$$\begin{aligned} I_3 &\leqslant |\{x \in \mathscr{R}^n\backslash K : c_n^{-\frac{1}{p}}\delta^{-\frac{n}{p}}|x|^{-\frac{n}{p}}\|f\|_{L^p(M)} > \lambda\}| \\ &\lesssim c_n\delta^{-n}\frac{\|f\|_{L^p(M)}^p}{\lambda^p}. \end{aligned}$$

Combining the estimates of I_1, I_2 and I_3, we have

$$\|S_\delta f\|_{L^{p,\infty}(M)} \lesssim \max\{\delta^{-\frac{m}{p}}, \delta^{-\frac{n}{p}}\}\|f\|_{L^p(M)}. \quad \square$$

3. The boundedness of the Hardy operator S_1

THEOREM 3.1. *Let* $1 < p < \infty$ *and* $f \in L^p(M)$. *Then the following inequality holds:*

$$\|S_1 f\|_{L^p(M)} \leqslant C\|f\|_{L^p(M)},$$

where

$$\begin{aligned} C = \max\Bigg\{ &\frac{|K|}{c_m}, \frac{|K|^{\frac{1}{p'}}}{c_n}\left(\frac{\omega_{m-n}\omega_{n-1}}{n(p-1)}\right)^{\frac{1}{p}}, p'\frac{2^{\frac{n}{p'}-1}\omega_{n-2}}{nc_n}B\left(\frac{1}{2}\left(\frac{n}{p'}-1\right), \frac{n-3}{2}\right), \\ &\frac{1}{c_n}(\omega_{m-n}\omega_{n-1}\ln 2)^{\frac{1}{p}}, \frac{|K|^{\frac{1}{p'}}}{c_m}\left(\frac{\omega_{m-1}}{m(p-1)}\right)^{\frac{1}{p}} p'\frac{2^{\frac{m}{p'}-1}\omega_{m-2}}{mc_m}B\left(\frac{1}{2}\left(\frac{m}{p'}-1\right), \frac{m-3}{2}\right)\Bigg\}. \end{aligned}$$

Proof. First we split M into three components K, $\mathbb{R}^m\backslash K$ and $\mathscr{R}^n\backslash K$, and denote their characteristic functions by χ_1, χ_2 and χ_3, respectively.

$$\begin{aligned} &\left(\int_M \left(\frac{1}{|B(x,|x|)|}\int_{B(x,|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_{\mathscr{R}^n\backslash K} \left(\frac{1}{|B(x,|x|)|}\int_{B(x,|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}} \\ &\quad + \left(\int_{\mathbb{R}^m\backslash K} \left(\frac{1}{|B(x,|x|)|}\int_{B(x,|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}} \\ &\quad + \left(\int_K \left(\frac{1}{|B(x,|x|)|}\int_{B(x,|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

To estimate I_1, we note that for all $x \in \mathscr{R}^n \backslash K$,

$$\begin{aligned} I_1 \leqslant & \left(\int_{\mathscr{R}^n \backslash K} \left(\frac{1}{|B(x,|x|)|} \int_{B(x,|x|)} |\chi_1 f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\ & + \left(\int_{\mathscr{R}^n \backslash K} \left(\frac{1}{|B(x,|x|)|} \int_{B(x,|x|)} |(\chi_2 f)(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\ & + \left(\int_{\mathscr{R}^n \backslash K} \left(\frac{1}{|B(x,|x|)|} \int_{B(x,|x|)} |(\chi_3 f)(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\ =: & \, I_{11} + I_{12} + I_{13}. \end{aligned}$$

For all $x \in \mathscr{R}^n \backslash K,\ |B(x,|x|)| = c_n |x|^n$. So

$$\begin{aligned} I_{11} & \leqslant \left(\int_{\mathscr{R}^n \backslash K} \frac{1}{c_n^p |x|^{np}} \int_K |f(y)|^p dy \left(\int_K dy \right)^{\frac{p}{p'}} dx \right)^{\frac{1}{p}} \\ & \leqslant \frac{|K|^{\frac{1}{p'}}}{c_n} \left(\int_{S^{m-n}} \int_{S^{n-1}} \int_1^\infty \frac{1}{r^{np}} r^{n-1} dr d\sigma_1 d\sigma_2 \right)^{\frac{1}{p}} \|f\|_{L^p(M)} \\ & = \frac{|K|^{\frac{1}{p'}}}{c_n} \left(\frac{\omega_{m-n}\omega_{n-1}}{n(p-1)} \right)^{\frac{1}{p}} \|f\|_{L^p(M)}. \end{aligned}$$

For all $x \in \mathscr{R}^n \backslash K,\ B(x,|x|) \cap (\mathbb{R}^m \backslash K) = \emptyset$. Therefore, $I_{12} = 0$.

To estimate I_{13}, we consider two cases. Fix f and g positive and continuous with $\|g\|_{L^{p'}(\mathscr{R}^n \backslash K)} \leqslant 1$. We express both g and Sf in polar coordinates by writing $x = (r\phi, \sigma_1)$ and $y = (t\theta, \sigma_2)$, where $\sigma_i \in S^{m-n}$ $(i = 1,2)$. From $|x-y| \leqslant |x|$, we can get

$$\theta \cdot \phi \geqslant \frac{t}{2r} + \frac{1}{2rt}(1 - 2\sigma_1 \cdot \sigma_2) \geqslant \frac{1}{2r}\left(t - \frac{1}{t} \right).$$

Case 1: We suppose $r \geqslant 2$ and $t \geqslant 2$, so $\theta \cdot \phi \geqslant \frac{t}{4r}$. Let $\mathscr{S}_1$ be the set of all these selected points in $\mathscr{R}^n \backslash K$.

$$\begin{aligned} & \int_{\mathscr{S}_1} g(x) S(\chi_{\mathscr{S}_1} f)(x) dx \\ & = \int_{\mathscr{S}_1} \int_{\mathscr{S}_1} \frac{1}{c_n |x|^n} g(x) f(y) \chi_{B(x,|x|)}(y) dx dy \\ & \leqslant \frac{1}{c_n} \iint_{(S^{m-n})^2} \iint_{(S^{n-1})^2} \int_1^\infty \int_0^{4r} f(t\theta, \sigma_1) g(r\phi, \sigma_2) \chi_{\theta\cdot\phi \geqslant t/4r} t^n \frac{dt}{t} \frac{dr}{r} d\phi d\theta d\sigma_1 d\sigma_2 \\ & \leqslant \frac{4^{\frac{n}{p'}}}{c_n} \iint_{(S^{m-n})^2} \iint_{(S^{n-1})^2} \int_1^\infty g(r\phi, \sigma_2) r^{\frac{n}{p'}} \left(\int_0^1 f(4rt\theta, \sigma_1)(4rt)^{\frac{n}{p}} \chi_{\theta\cdot\phi \geqslant t} t^{\frac{n}{p'}} \frac{dt}{t} \right) \frac{dr}{r} \\ & \quad \times d\phi d\theta d\sigma_1 d\sigma_2 \end{aligned}$$

$$\leqslant \frac{4^{\frac{n}{p'}}}{c_n} \iint_{(S^{m-n})^2} \iint_{(S^{n-1})^2} G(\phi,\sigma_2) \left[\int_1^\infty \left(\int_0^1 f(4rt\theta,\sigma_1)(4rt)^{\frac{n}{p}} \chi_{\theta\cdot\phi\geqslant t} t^{\frac{n}{p'}} \frac{dt}{t} \right)^p \frac{dr}{r} \right]^{\frac{1}{p}}$$
$$\times d\phi d\theta d\sigma_1 d\sigma_2,$$

where $G(\phi,\sigma_2) = (\int_1^\infty g(r\phi,\sigma_2)^{p'} r^n \frac{dr}{r})^{1/p'}$. Notice that the above bracketed expression is the L^p norm of the group $(\mathbb{R}^+, \frac{dt}{t})$ convolution of the function $t \to f(rt\theta,\sigma_1)(t)^{\frac{n}{p}}$ with the kernel $\chi_{[0,\theta\cdot\phi]}(t) t^{\frac{n}{p'}}$ at $4r$. Therefore we have the following estimate

$$\int_{\mathscr{S}_1} g(x) S(\chi_{\mathscr{S}_1} f)(x) dx$$
$$\leqslant \frac{4^{\frac{n}{p'}}}{c_n} \iint_{(S^{m-n})^2} \iint_{(S^{n-1})^2} G(\phi,\sigma_2) F(\theta,\sigma_1) \left(\int_0^{\theta\cdot\phi} t^{\frac{n}{p'}} \frac{dt}{t} \right) d\phi d\theta d\sigma_1 d\sigma_2,$$

where $F(\theta,\sigma_1) = (\int_1^\infty f(r\theta,\sigma_1)^p r^n \frac{dr}{r})^{1/p}$. It follows from [2], we have

$$\int_{\mathscr{S}_1} g(x) S(\chi_{\mathscr{S}_1} f)(x) dx$$
$$\leqslant \frac{p' 4^{\frac{n}{p'}} \omega_{n-2}}{2nc_n} B\left(\frac{n-p'}{2p'}, \frac{n-3}{2} \right) \iint_{(S^{m-n})^2} \|F(\theta,\sigma_1)\|_{L^p(S^{n-1})} \|G(\phi,\sigma_2)\|_{L^{p'}(S^{n-1})} d\sigma_1 d\sigma_2$$
$$= p' \frac{4^{\frac{n}{p'}-\frac{1}{2}} \omega_{n-2}}{nc_n} B\left(\frac{1}{2}\left(\frac{n}{p'} - 1 \right), \frac{n-3}{2} \right) \|f\|_{L^p(\mathscr{R}^n \setminus K)} \|g\|_{L^{p'}(\mathscr{R}^n \setminus K)}.$$

Therefore by duality we obtain

$$\left(\int_{\mathscr{S}_1} \left(\frac{1}{|B(x,|x|)|} \int_{B(x,|x|)\cap\mathscr{S}_1} |f(y)| dy \right)^p dx \right)^{\frac{1}{p}}$$
$$\leqslant p' \frac{4^{\frac{n}{p'}-\frac{1}{2}} \omega_{n-2}}{nc_n} B\left(\frac{1}{2}\left(\frac{n}{p'} - 1 \right), \frac{n-3}{2} \right) \|f\|_{L^p(M)}$$

Case 2: We suppose $1 < r < 2$ and $1 < t < 2$. Obviously, the measure of $\mathscr{S}_2 := (\mathscr{R}^n \setminus K) \setminus \mathscr{S}_1$ is finite. Therefore

$$\left(\int_{\mathscr{S}_2} \left(\frac{1}{|B(x,|x|)|} \int_{B(x,|x|)\cap\mathscr{S}_2} |f(y)| dy \right)^p dx \right)^{\frac{1}{p}}$$
$$\leqslant \left(\int_{\mathscr{S}_2} \frac{1}{c_n^p |x|^{np}} \left(\int_{B(x,|x|)} |f(y)|^p dy \right) \left(\int_{B(x,|x|)} dy \right)^{\frac{p}{p'}} dx \right)^{\frac{1}{p}}$$
$$\leqslant \frac{1}{c_n} \left(\int_{S^{m-n}} \int_{S^{n-1}} \int_1^2 \frac{1}{r^n} r^{n-1} dr d\sigma_1 d\sigma_2 \right)^{\frac{1}{p}} \|f\|_{L^p(M)}$$
$$= \frac{1}{c_n} (\omega_{m-n}\omega_{n-1} \ln 2)^{\frac{1}{p}} \|f\|_{L^p(M)}.$$

We have

$$I_{13} \leqslant \left(p' \frac{4^{\frac{n}{p'}-\frac{1}{2}} \omega_{n-2}}{nc_n} B\left(\frac{1}{2}\left(\frac{n}{p'}-1\right), \frac{n-3}{2}\right) + \frac{1}{c_n} (\omega_{m-n}\omega_{n-1}\ln 2)^{\frac{1}{p}} \right) \|f\|_{L^p(M)}.$$

Combining the estimates of I_{11}, I_{12} and I_{13}, we have

$$I_1 \leqslant \left(\frac{|K|^{\frac{1}{p'}}}{c_n} \left(\frac{\omega_{m-n}\omega_{n-1}}{n(p-1)}\right)^{\frac{1}{p}} + p' \frac{4^{\frac{n}{p'}-\frac{1}{2}} \omega_{n-2}}{nc_n} B\left(\frac{1}{2}\left(\frac{n}{p'}-1\right), \frac{n-3}{2}\right) \right.$$
$$\left. + \frac{1}{c_n} \left(\omega_{m-n}\omega_{n-1}\ln 2\right)^{\frac{1}{p}} \right) \|f\|_{L^p(M)}.$$

To estimate I_2, we note that for all $x \in \mathbb{R}^m \backslash K$,

$$\begin{aligned}
I_2 \leqslant & \left(\int_{\mathbb{R}^m \backslash K} \left(\frac{1}{|B(x,|x|)|} \int_{B(x,|x|)} |\chi_1 f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\
& + \left(\int_{\mathbb{R}^m \backslash K} \left(\frac{1}{|B(x,|x|)|} \int_{B(x,|x|)} |(\chi_2 f)(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\
& + \left(\int_{\mathbb{R}^m \backslash K} \left(\frac{1}{|B(x,|x|)|} \int_{B(x,|x|)} |(\chi_3 f)(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\
=: & I_{21} + I_{22} + I_{23}.
\end{aligned}$$

For all $x \in \mathbb{R}^m \backslash K$, $|B(x,|x|)| = c_m |x|^m$. So

$$\begin{aligned}
I_{21} & \leqslant \left(\int_{\mathbb{R}^m \backslash K} \frac{1}{c_m^p |x|^{mp}} \int_K |f(y)|^p dy \left(\int_K dy \right)^{\frac{p}{p'}} dx \right)^{\frac{1}{p}} \\
& \leqslant \frac{|K|^{\frac{1}{p'}}}{c_m} \left(\int_{S^{m-1}} \int_1^\infty \frac{1}{r^{mp}} r^{m-1} dr d\sigma_1 d\sigma_2 \right)^{\frac{1}{p}} \|f\|_{L^p(M)} \\
& = \frac{|K|^{\frac{1}{p'}}}{c_m} \left(\frac{\omega_{m-1}}{m(p-1)} \right)^{\frac{1}{p}} \|f\|_{L^p(M)}.
\end{aligned}$$

To estimate I_{22}, we adopt the same method in [2]. Note that $|B(x,|x|)| = c_m |x|^m$, so we have the following estimate:

$$I_{22} \leqslant p' \frac{2^{\frac{m}{p'}-1} \omega_{m-2}}{mc_m} B\left(\frac{1}{2}\left(\frac{m}{p'}-1\right), \frac{m-3}{2}\right) \|f\|_{L^p(M)}.$$

For $x \in \mathbb{R}^m \backslash K$, we have $B(x,|x|) \cap (\mathscr{R}^n \backslash K) = \emptyset$. Therefore $I_{23} = 0$. Combining the estimates of I_{21}, I_{22} and I_{23}, we have

$$I_2 \leqslant \left(\frac{|K|^{\frac{1}{p'}}}{c_m} \left(\frac{\omega_{m-1}}{m(p-1)} \right)^{\frac{1}{p}} + p' \frac{2^{\frac{m}{p'}-1} \omega_{m-2}}{mc_m} B\left(\frac{1}{2}\left(\frac{m}{p'}-1\right), \frac{m-3}{2}\right) \right) \|f\|_{L^p(M)}.$$

To estimate I_3, we note that for all $x \in K$, $|x| = 1$ and $|B(x,|x|)| = c_m$. Therefore

$$I_3 \leqslant \frac{1}{c_m}\left(\int_K \int_{B(x,1)\cap K} |f(y)|^p dy \left(\int_{B(x,1)\cap K} dy\right)^{\frac{p}{p'}} dx\right)^{\frac{1}{p}}$$
$$\leqslant \frac{|K|}{c_m}\|f\|_{L^p(M)}.$$

Combining the estimates of I_1, I_2 and I_3, we have

$$\left(\int_M \left(\frac{1}{|B(x,|x|)|}\int_{B(x,|x|)} |f(y)|dy\right)^p dx\right)^{\frac{1}{p}} \leqslant C\|f\|_{L^p(M)},$$

where

$$C = \max\left\{\frac{|K|}{c_m}, \frac{|K|^{\frac{1}{p'}}}{c_n}\left(\frac{\omega_{m-n}\omega_{n-1}}{n(p-1)}\right)^{\frac{1}{p}}, p'\frac{2^{\frac{n}{p'}-1}\omega_{n-2}}{nc_n}B\left(\frac{1}{2}\left(\frac{n}{p'}-1\right),\frac{n-3}{2}\right),\right.$$
$$\frac{1}{c_n}\left(\omega_{m-n}\omega_{n-1}\ln 2\right)^{\frac{1}{p}},$$
$$\left.\frac{|K|^{\frac{1}{p'}}}{c_m}\left(\frac{\omega_{m-1}}{m(p-1)}\right)^{\frac{1}{p}} p'\frac{2^{\frac{m}{p'}-1}\omega_{m-2}}{mc_m}B\left(\frac{1}{2}\left(\frac{m}{p'}-1\right),\frac{m-3}{2}\right)\right\}. \quad \square$$

4. The boundedness of the Hardy operator S_2

THEOREM 4.1. *Let $1 < p < \infty$ and $f \in L^p(M)$. Then the following inequality holds:*

$$\|S_2 f\|_{L^p(M)} \leqslant C\|f\|_{L^p(M)},$$

where

$$C = \max\left\{\frac{|K|}{c_m}, \frac{|K|^{\frac{1}{p'}}}{2^m c_m}\left(\frac{\omega_{m-1}}{m(p-1)}\right)^{\frac{1}{p}}, p'\frac{\nu_n}{c_m}\left(\frac{\omega_{m-1}}{\omega_{n-1}}\right)^{\frac{1}{p}}\omega_{m-n}^{\frac{1}{p'}}, \frac{|K|^{\frac{1}{p'}}}{c_m}\left(\frac{\omega_{m-n}\omega_{n-1}}{mp-n}\right)^{\frac{1}{p}},\right.$$
$$(3\sqrt{2})^{\frac{n}{p'}}\frac{p'\nu_n\omega_{m-n}}{c_m 2^m}, p'\frac{\nu_m}{c_m}\left(\frac{\omega_{m-n}\omega_{n-1}}{\omega_{m-1}}\right)^{\frac{1}{p}},$$
$$\left.p'\frac{\omega_{m-2}}{2^m mc_m}\int_{-1}^{1}(1-s^2)^{\frac{m-3}{2}}(s+\sqrt{s^2+3})^{\frac{m}{p'}}ds\right\}.$$

Proof. Similarly as the proof of Theorem 3.1, we have

$$
\begin{aligned}
&\left(\int_M \left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\
&\leqslant \left(\int_{\mathscr{R}^n\setminus K} \left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\
&\quad + \left(\int_{\mathbb{R}^m\setminus K} \left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\
&\quad + \left(\int_{K} \left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|f(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\
&=: J_1+J_2+J_3.
\end{aligned}
$$

To estimate J_1, we note that for all $x\in\mathscr{R}^n\setminus K$,

$$
\begin{aligned}
J_1 &\leqslant \left(\int_{\mathscr{R}^n\setminus K} \left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|f_1(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\
&\quad + \left(\int_{\mathscr{R}^n\setminus K} \left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|f_2(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\
&\quad + \left(\int_{\mathscr{R}^n\setminus K} \left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|f_3(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\
&=: J_{11}+J_{12}+J_{13},
\end{aligned}
$$

where $f_1(x)=\chi_K(x)$, $f_2(x)=\chi_{\mathbb{R}^m\setminus K}(x)$ and $f_3(x)=\chi_{\mathscr{R}^n\setminus K}(x)$. For all $x\in\mathscr{R}^n\setminus K$, $|B(x,2|x|)|=c_n|2x|^n+c_m|x|^m$. We obtain

$$
\begin{aligned}
J_{11} &\leqslant \left(\int_{\mathscr{R}^n\setminus K}\frac{1}{c_m^p|x|^{mp}}\int_K|f(y)|^p dy\left(\int_K dy\right)^{\frac{p}{p'}}dx\right)^{\frac{1}{p}}\\
&\leqslant \frac{|K|^{\frac{1}{p'}}}{c_m}\left(\int_{S^{m-n}}\int_{S^{n-1}}\int_1^\infty \frac{1}{r^{mp}}r^{n-1}dr\,d\sigma_1 d\sigma_2\right)^{\frac{1}{p}}\|f\|_{L^p(M)}\\
&= \frac{|K|^{\frac{1}{p'}}}{c_m}\left(\frac{\omega_{m-n}\omega_{n-1}}{mp-n}\right)^{\frac{1}{p}}\|f\|_{L^p(M)}.
\end{aligned}
$$

For all $x\in\mathscr{R}^n\setminus K$, $B(x,2|x|)\cap(\mathbb{R}^m\setminus K)=\{y\in\mathbb{R}^m\setminus K : |y|\leqslant|x|\}$. Therefore by polar coordinates, we obtain

$$
\begin{aligned}
J_{12}^p &\leqslant \int_1^\infty\int_{S^{n-1}}\int_{S^{m-n}}\left(\frac{1}{c_m r^m}\int_0^{\sqrt{r^2+1}}\int_{S^{m-1}}|f(t\theta)|t^{m-1}d\theta\,dt\right)^p r^{n-1}d\sigma\,d\phi\,dr\\
&\leqslant \frac{\omega_{m-n}\omega_{n-1}}{c_m^p}\int_1^\infty\left(\int_{S^{m-1}}\int_0^1|f(t\theta\sqrt{r^2+1})|(t\sqrt{r^2+1})^{\frac{m}{p}}t^{\frac{m}{p'}}\frac{dt}{t}d\theta\right)^p\frac{dr}{r}.
\end{aligned}
$$

We apply Hölder's inequality with exponents $\frac{1}{p}+\frac{1}{p'}=1$ to the functions 1 and $\theta \to \int_0^1 |f(t\theta\sqrt{r^2+1})|(t\sqrt{r^2+1})^{\frac{m}{p}} t^{\frac{m}{p'}} \frac{dt}{t}$ and then to Fubini's theorem to interchange the integrals in θ and r. We obtain

$$J_{12}^p \leqslant \frac{\omega_{m-n}\omega_{n-1}}{c_m^p} \omega_{m-1}^{\frac{p}{p'}} \int_{S^{m-1}} \int_0^\infty \left(\int_0^1 |f(t\theta\sqrt{r^2+1})|(t\sqrt{r^2+1})^{\frac{m}{p}} t^{\frac{m}{p'}} \frac{dt}{t} \right)^p \frac{dr}{r} d\theta.$$

Let $\mathbb{R}^+$ denote the multiplicative group of positive real numbers with Haar measure $\frac{dt}{t}$. The function $t^{\frac{m}{p'}} \chi_{[0,1]}(t)$ is in $L^1(\mathbb{R}^+, \frac{dt}{t})$ with norm $\frac{p'}{m}$. By the group inequality $\|g * K\|_{L^p} \leqslant \|g\|_{L^p} \|K\|_{L^1}$, we have

$$\int_0^\infty \left(\int_0^1 |f(t\theta\sqrt{r^2+1})|(t\sqrt{r^2+1})^{\frac{m}{p}} t^{\frac{m}{p'}} \frac{dt}{t} \right)^p \frac{dr}{r} \leqslant \left(\frac{p'}{m} \right)^p \int_0^\infty |f(r\theta)|^p r^m \frac{dr}{r}.$$

Therefore

$$J_{12} \leqslant p' \frac{v_m}{c_m} \left(\frac{\omega_{m-n}\omega_{n-1}}{\omega_{m-1}} \right)^{\frac{1}{p}} \|f\|_{L^p(M)}.$$

Therefore From $|x-y| \leqslant |2x|$, we obtain $|y| \leqslant 3|x|$. Without loss of generality, assume that f is nonegative. Therefore

$$\begin{aligned}
J_{13}^p &\leqslant \int_{\mathscr{R}^n \setminus K} \left(\frac{1}{|B(x,2|x|)|} \int_{B(0,3|x|)} f(y)dy \right)^p dx \\
&\leqslant \int_1^\infty \int_{S^{n-1}} \int_{S^{m-n}} \left(\frac{2^{-m}}{c_m r^m} \int_0^{3\sqrt{r^2+1}} \int_{S^{n-1}} \int_{S^{m-n}} f(t\theta,\sigma_2) t^n d\sigma_2 d\theta \frac{dt}{t} \right)^p r^n d\sigma_1 d\phi \frac{dr}{r} \\
&\leqslant \frac{\omega_{m-n}\omega_{n-1}}{c_m^p 2^{mp}} \int_1^\infty \left(r^{-n} \int_0^{3\sqrt{r^2+1}} \int_{S^{n-1}} \int_{S^{m-n}} f(t\theta,\sigma_2) t^n d\sigma_2 d\theta \frac{dt}{t} \right)^p r^n \frac{dr}{r} \\
&\leqslant \frac{\omega_{m-n}\omega_{n-1}}{c_m^p 2^{mp}} \int_1^\infty \left(\int_{S^{n-1}} \int_0^1 \int_{S^{m-n}} f(3\sqrt{r^2+1}t\theta,\sigma_2)(3\sqrt{r^2+1}t)^{\frac{n}{p}} t^{\frac{n}{p'}} d\sigma_2 \frac{dt}{t} d\theta \right)^p \\
&\qquad \left(\frac{3\sqrt{r^2+1}}{r} \right)^{\frac{np}{p'}} \frac{dr}{r}.
\end{aligned}$$

For $r \in (1,\infty)$, we have $\frac{3\sqrt{r^2+1}}{r} \leqslant 3\sqrt{2}$. Hence,

$$J_{13}^p \leqslant \frac{\omega_{m-n}\omega_{n-1}(3\sqrt{2})^{\frac{np}{p'}}}{c_m^p 2^{mp}} I,$$

where

$$I = \int_1^\infty \left(\int_{S^{n-1}} \int_0^1 \int_{S^{m-n}} f(3\sqrt{r^2+1}t\theta,\sigma_2)(3\sqrt{r^2+1}t)^{\frac{n}{p}} t^{\frac{n}{p'}} d\sigma_2 \frac{dt}{t} d\theta \right)^p \frac{dr}{r}.$$

We apply Hölder's inequality with exponents $\frac{1}{p}+\frac{1}{p'}=1$ to the functions 1 and $\theta \to \int_0^1 \int_{S^{m-n}} f(3\sqrt{r^2+1}t\theta,\sigma_2)(3\sqrt{r^2+1}t)^{\frac{n}{p}} t^{\frac{n}{p'}} d\sigma_2 \frac{dt}{t}$ and then to Fubini's theorem to interchange the integrals in θ and r. We obtain

$$I \leqslant \omega_{n-1}^{\frac{p}{p'}} \int_{S^{n-1}} \int_1^\infty \left(\int_0^1 \int_{S^{m-n}} f(3\sqrt{r^2+1}t\theta,\sigma_2)(3\sqrt{r^2+1}t)^{\frac{n}{p}} t^{\frac{n}{p'}} d\sigma_2 \frac{dt}{t} \right)^p \frac{dr}{r} d\theta.$$

Let $\mathbb{R}^+$ denote the multiplicative group of positive real numbers with Haar measure $\frac{dt}{t}$. The function $t^{\frac{n}{p'}}\chi_{[0,1]}(t)$ is in $L^1(\mathbb{R}^+,\frac{dt}{t})$ with norm $\frac{p'}{n}$. By the group inequality $\|g*K\|_{L^p}\leqslant\|g\|_{L^p}\|K\|_{L^1}$, we have

$$\begin{aligned}&\left(\int_1^\infty\left(\int_0^1\int_{S^{m-n}} f(3\sqrt{r^2+1}t\theta,\sigma_2)(3\sqrt{r^2+1}t)^{\frac{n}{p}}t^{\frac{n}{p'}}d\sigma_2\frac{dt}{t}\right)^p\frac{dr}{r}\right)^{\frac{1}{p}}\\ &\leqslant\frac{p'}{n}\left(\int_1^\infty\left(\int_{S^{m-n}} f(r\theta,\sigma_2)r^{\frac{n}{p}}d\sigma_2\right)^p\frac{dr}{r}\right)^{\frac{1}{p}}.\end{aligned}$$

By Minkowski's integral inequality and Hölder's inequality, we obtain

$$\begin{aligned}&\left(\int_1^\infty\left(\int_{S^{m-n}} f(r\theta,\sigma_2)r^{\frac{n}{p}}d\sigma_2\right)^p\frac{dr}{r}\right)^{\frac{1}{p}}\\ &\leqslant\int_{S^{m-n}}\left(\int_1^\infty f(r\theta,\sigma_2)^p r^{n-1}dr\right)^{\frac{1}{p}}d\sigma_2\\ &\leqslant\omega_{m-n}^{\frac{1}{p'}}\left(\int_{S^{m-n}}\int_1^\infty f(r\theta,\sigma_2)^p r^{n-1}drd\sigma_2\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore

$$J_{13}\leqslant(3\sqrt{2})^{\frac{n}{p'}}\frac{p'\nu_n\omega_{m-n}}{c_m2^m}\|f\|_{L^p(M)}.$$

Combining the estimates of J_{11}, J_{12} and J_{13}, we have

$$J_1\leqslant\left(\frac{|K|^{\frac{1}{p'}}}{c_m}\left(\frac{\omega_{m-n}\omega_{n-1}}{mp-n}\right)^{\frac{1}{p}}+p'\frac{\nu_m}{c_m}\left(\frac{\omega_{m-n}\omega_{n-1}}{\omega_{m-1}}\right)^{\frac{1}{p}}+(3\sqrt{2})^{\frac{n}{p'}}\frac{p'\nu_n\omega_{m-n}}{c_m2^m}\right)\|f\|_{L^p(M)}.$$

To estimate J_2, we note that for all $x\in\mathbb{R}^m\backslash K$,

$$\begin{aligned}J_2\leqslant&\left(\int_{\mathbb{R}^m\backslash K}\left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|\chi_1f(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\ &+\left(\int_{\mathbb{R}^m\backslash K}\left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|(\chi_2f)(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\ &+\left(\int_{\mathbb{R}^m\backslash K}\left(\frac{1}{|B(x,2|x|)|}\int_{B(x,2|x|)}|(\chi_3f)(y)|dy\right)^p dx\right)^{\frac{1}{p}}\\ =:&\,J_{21}+J_{22}+J_{23}.\end{aligned}$$

For all $x \in \mathbb{R}^m \backslash K,\ |B(x,2|x|)| = c_m|2x|^m + c_n|x|^n$. So

$$J_{21} \leqslant \left(\int_{\mathbb{R}^m \backslash K} \frac{1}{c_m^p |2x|^{mp}} \int_K |f(y)|^p dy \left(\int_K dy \right)^{\frac{p}{p'}} dx \right)^{\frac{1}{p}}$$

$$\leqslant \frac{|K|^{\frac{1}{p'}}}{2^m c_m} \left(\int_{S^{m-1}} \int_1^\infty \frac{1}{r^{mp}} r^{m-1} dr d\sigma_1 d\sigma_2 \right)^{\frac{1}{p}} \|f\|_{L^p(M)}$$

$$= \frac{|K|^{\frac{1}{p'}}}{2^m c_m} \left(\frac{\omega_{m-1}}{m(p-1)} \right)^{\frac{1}{p}} \|f\|_{L^p(M)}.$$

To estimate J_{22}, we adopt the same method in [2]. We have the following estimate:

$$J_{22} \leqslant p' \frac{\omega_{m-2}}{2^m m c_m} \int_{-1}^1 (1-s^2)^{\frac{m-3}{2}} (s+\sqrt{s^2+3})^{\frac{m}{p'}} ds \|f\|_{L^p(M)}.$$

For $x \in \mathbb{R}^m \backslash K,$ we have $B(x,2|x|) \cap (\mathscr{R}^n \backslash K) = \{y \in \mathscr{R}^n \backslash K : |y| \leqslant |x|\}$. Therefore by polar coordinates, we obtain

$$J_{23}^p \leqslant \int_1^\infty \int_{S^{m-1}} \left(\frac{1}{c_m r^m} \int_0^r \int_{S^{n-1}} \int_{S^{m-n}} |f(t\theta,\sigma)| t^{n-1} d\sigma d\theta dt \right)^p r^{m-1} d\phi dr$$

$$\leqslant \frac{\omega_{m-1}}{c_m^p} \int_1^\infty \left(\int_{S^{n-1}} \int_0^1 \int_{S^{m-n}} |f(tr\theta,\sigma)| (tr)^{\frac{n}{p}} t^{\frac{n}{p'}} d\sigma \frac{dt}{t} d\theta \right)^p \frac{dr}{r}.$$

We apply Hölder's inequality with exponents $\frac{1}{p} + \frac{1}{p'} = 1$ to the functions 1 and $\theta \to \int_0^1 \int_{S^{m-n}} |f(tr\theta,\sigma)|(tr)^{\frac{m}{p}} t^{\frac{n}{p'}} d\sigma \frac{dt}{t}$ and then to Fubini's theorem to interchange the integrals in θ and r. We obtain

$$J_{23}^p \leqslant \frac{\omega_{m-1}}{c_m^p} \omega_{n-1}^{\frac{p}{p'}} \int_{S^{n-1}} \int_1^\infty \left(\int_0^1 \int_{S^{m-n}} |f(tr\theta,\sigma)| (tr)^{\frac{n}{p}} t^{\frac{n}{p'}} d\sigma \frac{dt}{t} \right)^p \frac{dr}{r} d\theta.$$

Let $\mathbb{R}^+$ denote the multiplicative group of positive real numbers with Haar measure $\frac{dt}{t}$. The function $t^{\frac{n}{p'}} \chi_{[0,1]}(t)$ is in $L^1(\mathbb{R}^+, \frac{dt}{t})$ with norm $\frac{p'}{n}$. By the group inequality $\|g * K\|_{L^p} \leqslant \|g\|_{L^p} \|K\|_{L^1}$, we have

$$\int_1^\infty \left(\int_0^1 \int_{S^{m-n}} |f(tr\theta,\sigma)| (tr)^{\frac{n}{p}} t^{\frac{n}{p'}} d\sigma \frac{dt}{t} \right)^p \frac{dr}{r}$$

$$\leqslant \left(\frac{p'}{n} \right)^p \int_1^\infty \left(\int_{S^{m-n}} |f(r\theta,\sigma)| r^{\frac{n}{p}} d\sigma \right)^p \frac{dr}{r}.$$

By Minkowski's integral inequality and Hölder's inequality, we obtain

$$\left(\int_1^\infty \left(\int_{S^{m-n}} |f(r\theta,\sigma)| r^{\frac{n}{p}} d\sigma \right)^p \frac{dr}{r} \right)^{\frac{1}{p}}$$

$$\leqslant \int_{S^{m-n}} \left(\int_1^\infty |f(r\theta,\sigma)|^p r^{n-1} dr \right)^{\frac{1}{p}} d\sigma$$

$$\leqslant \omega_{m-n}^{\frac{1}{p'}} \left(\int_{S^{m-n}} \int_1^\infty |f(r\theta,\sigma)|^p r^{n-1} dr d\sigma \right)^{\frac{1}{p}}.$$

Therefore

$$J_{23} \leqslant p' \frac{\nu_n}{c_m} \left(\frac{\omega_{m-1}}{\omega_{n-1}} \right)^{\frac{1}{p}} \omega_{m-n}^{\frac{1}{p'}} \|f\|_{L^p(M)}.$$

Combining the estimates of J_{21}, J_{22} and J_{23}, we have

$$\begin{aligned} J_2 \leqslant & \Bigg(\frac{|K|^{\frac{1}{p'}}}{2^m c_m} \left(\frac{\omega_{m-1}}{m(p-1)} \right)^{\frac{1}{p}} + p' \frac{\nu_n}{c_m} \left(\frac{\omega_{m-1}}{\omega_{n-1}} \right)^{\frac{1}{p}} \omega_{m-n}^{\frac{1}{p'}} \\ & + p' \frac{\omega_{m-2}}{2^m m c_m} \int_{-1}^{1} (1-s^2)^{\frac{m-3}{2}} (s+\sqrt{s^2+3})^{\frac{m}{p'}} ds \Bigg) \|f\|_{L^p(M)}. \end{aligned}$$

To estimate J_3, we note that for all $x \in K$, $|x| = 1$ and $|B(x,2|x|)| = 2^m c_m$. Therefore

$$\begin{aligned} J_3 &\leqslant \frac{1}{c_m} \left(\int_K \int_{B(x,2)\cap K} |f(y)|^p dy \left(\int_{B(x,2)\cap K} dy \right)^{\frac{p}{p'}} dx \right)^{\frac{1}{p}} \\ &\leqslant \frac{|K|}{c_m} \|f\|_{L^p(M)}. \end{aligned}$$

Combining the estimates of J_1, J_2 and J_3, we have

$$\left(\int_M \left(\frac{1}{|B(x,2|x|)|} \int_{B(x,2|x|)} |f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \leqslant C \|f\|_{L^p(M)},$$

where

$$\begin{aligned} C = \max \Bigg\{ & \frac{|K|}{c_m}, \frac{|K|^{\frac{1}{p'}}}{2^m c_m} \left(\frac{\omega_{m-1}}{m(p-1)} \right)^{\frac{1}{p}}, p' \frac{\nu_n}{c_m} \left(\frac{\omega_{m-1}}{\omega_{n-1}} \right)^{\frac{1}{p}} \omega_{m-n}^{\frac{1}{p'}}, \frac{|K|^{\frac{1}{p'}}}{c_m} \left(\frac{\omega_{m-n}\omega_{n-1}}{mp-n} \right)^{\frac{1}{p}}, \\ & p' \frac{\nu_m}{c_m} \left(\frac{\omega_{m-n}\omega_{n-1}}{\omega_{m-1}} \right)^{\frac{1}{p}}, p' \frac{\omega_{m-2}}{2^m m c_m} \int_{-1}^{1} (1-s^2)^{\frac{m-3}{2}} (s+\sqrt{s^2+3})^{\frac{m}{p'}} ds \Bigg\}. \quad \square \end{aligned}$$

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