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SINGULAR VALUE INEQUALITIES RELATED TO THE GEOMETRIC MEAN

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Abstract. In this short note, we obtain some singular value inequalities associated with the geometric mean of two positive definite matrices. Our results are refinements of some known inequalities.

1. Introduction

Let $\mathbf{M}_n$ be the set of $n\times n$ complex matrices. For $A\in\mathbf{M}_n$, $A\geqslant 0$ (resp. $A>0$) will be used to mean that A is positive semidefinite (resp. positive definite). The conjugate transpose of A is denoted by A^*. If A is any Hermitian matrix, we enumerate eigenvalues of A in nonincreasing order $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\ldots\geqslant\lambda_n(A)$. The singular values of A, denoted by $s_1(A),s_2(A),\ldots,s_n(A)$, are the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{1/2}$, arranged in nonincreasing order and repeated according to multiplicity as $s_1(A)\geqslant s_2(A)\geqslant\ldots\geqslant s_n(A)$. We denote $\lambda(A)=(\lambda_1(A),\ldots,\lambda_n(A))$ and $s(A)=(s_1(A),\ldots,s_n(A))$.

Let $A,B>0$. Then for $t\in[0,1]$, the t-geometric mean of A,B is

$$A\sharp_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}.$$

When $t=1/2$, $A\sharp_{1/2}B$ is called the geometric mean of A and B, which is often denoted by $A\sharp B$. See [4, 6, 12].

Recall that a norm $\|\cdot\|$ on $\mathbf{M}_n$ is unitarily invariant if $\|UAV\|=\|A\|$ for any $A\in\mathbf{M}_n$ and all unitary $U,V\in\mathbf{M}_n$. The unitarily invariant norms of matrices are determined by nonzero singular values of the matrices via symmetric gauge functions (see, e.g. [13, Theorems 10.37 and 10.38]). The trace norms, a special class of unitarily invariant norms, are defined as $||\cdot||_1=\sum_{j=1}^n s_j(A)$.

Let $x=(x_1,x_2,\ldots,x_n)$, $y=(y_1,y_2,\ldots,y_n)\in\mathbf{R}^n$. We rearrange the components of x in nonincreasing order as $x_{[1]}\geqslant x_{[2]}\geqslant\ldots\geqslant x_{[n]}$. If

$$\sum_{j=1}^k x_{[j]}\leqslant\sum_{j=1}^k y_{[j]},\qquad k=1,2,\ldots,n,$$

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we say that x is *weakly majorized* by y, denoted by $x \prec_w y$. If, in addition, the last inequality is an equality, i.e., $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$, we say that x is *majorized* by y, written as $x \prec y$. If $x_j, y_j > 0$ for $j = 1, \ldots, n$,

$$\prod_{j=1}^{k} x_{[j]} \leqslant \prod_{j=1}^{k} y_{[j]}, \qquad k = 1, 2, \ldots, n,$$

we say that x is *weakly log majorized* by y, denoted by $x \prec_{wlog} y$. If $x \prec_{wlog} y$ and $\prod_{j=1}^{n} x_j = \prod_{j=1}^{n} y_j$, then we say that x is *log majorized* by y, denoted by $x \prec_{log} y$ (see, e.g. [13, p. 326]). As we know, (weak) log-majorization implies weak majorization. It is well known (see, e.g. [13, p. 368]) that for $A, B \in \mathbf{M}_n$, $\|A\| \leqslant \|B\|$ for all unitarily invariant norms $\|\cdot\|$ on $\mathbf{M}_n$ if and only if $s(A) \prec_w s(B)$.

It has been shown in [1] and [2] that if $A, B \geqslant 0$, then for any $t \in [0,1]$ and $s > 0$, the following log-majorization relationship holds:

$$\lambda(A\natural_t B) \prec_{log} \lambda(e^{(1-t)logA+tlogB}) \prec_{log} \lambda(B^{ts/2}A^{(1-t)s}B^{ts/2})^{1/s} \prec_{log} s((A^{(1-t)s}B^{ts})^{1/s}).$$

This implies that

$$\|A\natural_t B\| \leqslant \|(B^{ts/2}A^{(1-t)s}B^{ts/2})^{1/s}\| \leqslant \|(A^{(1-t)s}B^{ts})^{1/s}\|. \tag{1}$$

The following theorem is well known and it has various generalizations; see for example [5, 10, 11].

THEOREM 1.1. *Let A, B be positive definite matrices. Then*

$$\prod_{j=1}^{k} s_j(A\sharp B) \leqslant \prod_{j=1}^{k} s_j(A^{1/2}B^{1/2}), \;\; k = 1, \ldots, n. \tag{2}$$

Applying Wely theorem (see, e.g. [13, p. 353]) to inequality (2), it follows that

$$\prod_{j=1}^{k} s_j((A\sharp B)^2) \leqslant \prod_{j=1}^{k} \lambda_j(AB) \leqslant \prod_{j=1}^{k} s_j(AB), \;\; k = 1, \ldots, n. \tag{3}$$

Dinh et al. [7, Theorem 3.1] obtained a geometric mean inequality and proposed a conjecture [7, p. 787] as follows.

THEOREM 1.2. *Let $A_j, B_j \in \mathbf{M}_n$, $j = 1, \ldots, k$, be positive definite. Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbf{M}_n$, we have*

$$\begin{aligned} \left\| \sum_{j=1}^{k} (A_j \sharp B_j)^2 \right\| &\leqslant \left\| \left(\sum_{j=1}^{k} A_j\right)^{\frac{1}{2}} \left(\sum_{j=1}^{k} B_j\right) \left(\sum_{j=1}^{k} A_j\right)^{\frac{1}{2}} \right\| \\ &\leqslant \left\| \left(\sum_{j=1}^{k} A_j\right) \left(\sum_{j=1}^{k} B_j\right) \right\|. \end{aligned} \tag{4}$$

CONJECTURE 1.1. *Let $A_j, B_j \in \mathbf{M}_n$, $j = 1, \ldots, k$, be positive definite. Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbf{M}_n$, we have*

$$\begin{aligned}\left\|\sum_{j=1}^{k}(A_j^2 \sharp B_j^2)\right\| &\leqslant \left\|\left(\sum_{j=1}^{k} A_j\right)^{\frac{1}{2}}\left(\sum_{j=1}^{k} B_j\right)\left(\sum_{j=1}^{k} A_j\right)^{\frac{1}{2}}\right\| \\ &\leqslant \left\|\left(\sum_{j=1}^{k} A_j\right)\left(\sum_{j=1}^{k} B_j\right)\right\|.\end{aligned} \tag{5}$$

Recently, the conjecture 1.1 has been proven in [8].

In this paper, we first present a stronger result than (3) under positivity assumption which is clearly also a refinement of inequality (1) with $s = 1, t = \frac{1}{2}$. And then a sharper inequality than (4) will be shown. Finally, we refine Conjecture 1.1 in the case of the trace norm.

2. Main results

Before presenting the main theorems, we list some preliminary results for later convenience. The first lemma is an interesting fact that this property characterizes the geometric mean.

LEMMA 2.1. [4, Chapter 4] *Let $A, B \in \mathbf{M}_n$ be positive semidefinite. Then*

$$A \sharp B = A^{\frac{1}{2}} U B^{\frac{1}{2}}$$

for some unitary matrix U.

The following several lemmas are quite standard in matrix analysis.

LEMMA 2.2. [3, p. 253] *Let $A, B \in \mathbf{M}_n$ such that AB is normal. Then*

$$\|AB\| \leqslant \|BA\|$$

for all unitarily invariant norms.

REMARK 1. Actually, by the proof of Lemma 2.2 [3, p. 253] or antisymmetric tensor power [3, p. 18], it is easy to obtain that

$$\prod_{j=1}^{k} s_j(AB) \leqslant \prod_{j=1}^{k} s_j(BA), \quad k = 1, 2, \ldots, n,$$

whenever AB is normal.

LEMMA 2.3. [9] or [13, p. 364] *Let $A, B \in \mathbf{M}_n$. Then*

$$s(AB) \prec_{log} \{s_j(A) s_j(B)\}_{j=1}^{n}.$$

LEMMA 2.4. [13, p. 351] *Let* $A \in \mathbf{M}_n$. *Then*

$$|\lambda(A)| \prec_w s(A).$$

The last result is the famous Ando-Hiai theorem.

LEMMA 2.5. [1, Theorem 2.1] *Let* $A, B \in \mathbf{M}_n$ *be positive definite matrices,* $0 \leqslant t \leqslant 1$ *and* $r \geqslant 1$. *Then*

$$\lambda(A^r \sharp_t B^r) \prec_{log} \lambda((A \sharp_t B)^r).$$

Now we present the singular value inequalities related to geometric mean which refine inequality (3).

THEOREM 2.6. *Let* $A, B \in \mathbf{M}_n$ *be positive definite matrices. Then*

$$\prod_{j=1}^{k} s_j((A \sharp B)^2) \leqslant \prod_{j=1}^{k} s_j\left(A^{1/2}(A \sharp B)B^{1/2}\right) \leqslant \prod_{j=1}^{k} \lambda_j(AB) \leqslant \prod_{j=1}^{k} s_j(AB), \ \ k = 1, \ldots, n.$$

Proof. For $1 \leqslant k \leqslant n$, we have

$$\begin{aligned}
\prod_{j=1}^{k} s_j((A \sharp B)^2) &= \prod_{j=1}^{k} s_j\left((A \sharp B)(B^{1/2} V A^{1/2})\right) \text{ (by Lemma 2.1)} \\
&\leqslant \prod_{j=1}^{k} s_j\left(V A^{1/2}(A \sharp B)B^{1/2}\right) \text{ (by Remark 1)} \\
&= \prod_{j=1}^{k} s_j\left(A^{1/2}(A \sharp B)B^{1/2}\right) \\
&\leqslant \prod_{j=1}^{k} s_j(A^{1/2}B^{1/2}) s_j(A^{1/2}B^{1/2}) \text{ (by Lemma 2.3)} \\
&= \prod_{j=1}^{k} \lambda_j((A^{1/2}B^{1/2})^*(A^{1/2}B^{1/2})) \\
&= \prod_{j=1}^{k} \lambda_j(B^{1/2}AB^{1/2}) \\
&= \prod_{j=1}^{k} s_j(B^{1/2}AB^{1/2}) \\
&= \prod_{j=1}^{k} s_j(A^{1/2}BA^{1/2}) \\
&= \prod_{j=1}^{k} \lambda_j(AB) \\
&\leqslant \prod_{j=1}^{k} s_j(AB), \text{ (by Lemma 2.4)}
\end{aligned}$$

where V is a unitary matrix. □

The following corollary is an immediate consequence of Theorem 2.6.

COROLLARY 2.7. *Let $A,B \in \mathbf{M}_n$ be positive definite matrices. Then*

$$\|(A\sharp B)^2\| \leqslant \|A^{1/2}(A\sharp B)B^{1/2}\| \leqslant \|A^{1/2}BA^{1/2}\| \leqslant \|AB\|. \tag{6}$$

for any unitarily invariant norm $\|\cdot\|$.

REMARK 2. Obviously, our result (6) is a refinement of inequality (1) in the case of $s=1$, $t=\frac{1}{2}$.

Next we will present unitarily invariant norm inequalities which refine (4) and (5).

Observe that for any m-tuple of positive semidefinite matrices $A_j \in \mathbf{M}_n$ and for any non-negative convex function f on $[0,\infty)$ with $f(0)=0$ the inequality

$$\|f(A_1)+f(A_2)+\cdots+f(A_m)\| \leqslant \|f(A_1+A_2+\cdots+A_m)\|$$

holds for any unitarily invariant norm $\|\cdot\|$ on $\mathbf{M}_n$.

THEOREM 2.8. *Let $A_j,B_j \in \mathbf{M}_n$, $j=1,\ldots,k$, be positive definite matrices. Then*

$$\begin{aligned}
&\left\|\sum_{j=1}^{k}(A_j\sharp B_j)^2\right\| \\
&\leqslant \left\|\left(\sum_{j=1}^{k}A_j\right)^{1/2}\left(\sum_{j=1}^{k}A_j\sharp\sum_{j=1}^{k}B_j\right)\left(\sum_{j=1}^{k}B_j\right)^{1/2}\right\| \\
&\leqslant \left\|\left(\sum_{j=1}^{k}A_j\right)^{1/2}\left(\sum_{j=1}^{k}B_j\right)\left(\sum_{j=1}^{k}A_j\right)^{1/2}\right\| \\
&\leqslant \left\|\left(\sum_{j=1}^{k}A_j\right)\left(\sum_{j=1}^{k}B_j\right)\right\|
\end{aligned}$$

for any unitarily invariant norm $\|\cdot\|$.

Proof. Using the convexity of t^2, we have

$$\left\|\sum_{j=1}^{k}(A_j\sharp B_j)^2\right\| \leqslant \left\|\left(\sum_{j=1}^{k}A_j\sharp B_j\right)^2\right\|. \tag{7}$$

By the concavity of the geometric mean, we obtain

$$\left\|\sum_{j=1}^{k}A_j\sharp B_j\right\| \leqslant \left\|\left(\sum_{j=1}^{k}A_j\right)\sharp\left(\sum_{j=1}^{k}B_j\right)\right\|.$$

So

$$\left\| \left(\sum_{j=1}^{k} A_j \sharp B_j \right)^2 \right\| \leqslant \left\| \left(\left(\sum_{j=1}^{k} A_j \right) \sharp \left(\sum_{j=1}^{k} B_j \right) \right)^2 \right\|. \tag{8}$$

Applying Corollary 2.7 yields

$$\begin{aligned} &\left\| \left(\left(\sum_{j=1}^{k} A_j \right) \sharp \left(\sum_{j=1}^{k} B_j \right) \right)^2 \right\| \\ &\leqslant \left\| \left(\sum_{j=1}^{k} A_j \right)^{1/2} \left(\left(\sum_{j=1}^{k} A_j \right) \sharp \left(\sum_{j=1}^{k} B_j \right) \right) \left(\sum_{j=1}^{k} B_j \right)^{1/2} \right\| \\ &\leqslant \left\| \left(\sum_{j=1}^{k} A_j \right)^{1/2} \left(\sum_{j=1}^{k} B_j \right) \left(\sum_{j=1}^{k} A_j \right)^{1/2} \right\| \\ &\leqslant \left\| \left(\sum_{j=1}^{k} A_j \right) \left(\sum_{j=1}^{k} B_j \right) \right\|. \end{aligned} \tag{9}$$

Then by (7), (8) and (9) we get the desired result. □

REMARK 3. Obviously, the result in Theorem 2.8 is sharper than inequality (4).

Finally, we prove a result which refines inequality (5) for the trace norm $\|\cdot\|_1$.

THEOREM 2.9. *Let $A_j, B_j \in \mathbf{M}_n$, $j = 1, \ldots, k$, be positive definite matrices. Then*

$$\begin{aligned} &\left\| \sum_{j=1}^{k} (A_j^2 \sharp B_j^2) \right\|_1 \\ &\leqslant \left\| \left(\sum_{j=1}^{k} A_j \right)^{1/2} \left(\sum_{j=1}^{k} A_j \sharp \sum_{j=1}^{k} B_j \right) \left(\sum_{j=1}^{k} B_j \right)^{1/2} \right\|_1 \\ &\leqslant \left\| \left(\sum_{j=1}^{k} A_j \right)^{1/2} \left(\sum_{j=1}^{k} B_j \right) \left(\sum_{j=1}^{k} A_j \right)^{1/2} \right\|_1 \\ &\leqslant \left\| \left(\sum_{j=1}^{k} A_j \right) \left(\sum_{j=1}^{k} B_j \right) \right\|_1. \end{aligned}$$

Proof. By Lemma 2.5, we have

$$\left\| \sum_{j=1}^{k} (A_j^2 \sharp B_j^2) \right\|_1 \leqslant \left\| \sum_{j=1}^{k} (A_j \sharp B_j)^2 \right\|_1 .$$

The conclusion follows from Theorem 2.8. □

Declarations

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