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SHARP UPPER BOUNDS FOR THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

ZHONG-XUAN MAO, JUN-PING HOU* AND JING-FENG TIAN

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Abstract. The complete elliptic integral of the first kind, denoted by $\mathscr{K}(\cdot)$, is a class of special functions widely applied in mathematics, physics, and engineering. In this paper, we establish an upper bound for this function, given by

$$\mathscr{K}(r) \leqslant \frac{\pi}{2}(1-r^2)^{\frac{1}{2}(p_1+p_2r^2+p_3r^4+p_4r^6+p_5r^8)}$$

for all $r \in (0,1)$, where the parameters satisfy $p_1 \leqslant p_{1,0} = -1/2$, $p_2 \leqslant p_{2,0} = 1/32$, $p_3 \leqslant p_{3,0} = 1/64$, $p_4 \leqslant p_{4,0} = 251/24576$ and $p_5 \leqslant p_{5,0} = 123/16384$. Meanwhile, our results show that the parameters $p_{1,0}$, $p_{2,0}$, $p_{3,0}$, $p_{4,0}$ and $p_{5,0}$ are optimal and cannot be replaced by larger values. Finally, by utilizing the relationship between the complete elliptic integral of the first kind and the Gauss arithmetic-geometric mean, we establish sharp lower bounds for the Gauss arithmetic-geometric mean.

1. Introduction

Special functions are fundamental in mathematics, physics, and engineering, owing to their profound connections with complex analysis, number theory, and optimization. Famous special functions include the Gaussian hypergeometric function [8, 10, 14, 17], confluent hypergeometric function [4, 9, 12], modified Bessel functions [6, 7, 18, 23], elliptic integrals [15, 16, 22, 25], the Gamma and Beta functions [24, 27], as well as Lommel function [3, 11].

The complete elliptic integrals of the first kind $\mathscr{K}(r)$ is defined by

$$\mathscr{K}(r) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-r^2\sin^2 t}}\mathrm{d}t.$$

It can be expressed by the Gaussian hypergeometric function

$$\mathscr{K}(r) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\Gamma^2(\frac{1}{2}+n)}{\Gamma^2(\frac{1}{2})\Gamma^2(n+1)}r^{2n}, \tag{1.1}$$

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* Corresponding author.

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where

$$F(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k$$

is the Gaussian hypergeometric function,

$$\Gamma(x)=\int_0^{\infty}s^{x-1}e^{-s}\mathrm{d}s$$

is the gamma function and

$$(a)_0=1,\quad (a)_k:=a(a+1)\cdots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)},$$

is the Pochhammer symbol.

Since elliptic integrals cannot be expressed in terms of elementary functions such as polynomials, trigonometric functions, or exponential functions, research has mainly focused on establishing elementary upper and lower bounds for them. Qiu and Vamanamurthy [13] provided the following upper bounds for $\mathscr{K}(r)$:

$$\mathscr{K}(r)<\left(1+\frac{(r')^2}{4}\right)\ln\left(\frac{4}{r'}\right):=M_1(r),\quad r\in(0,1),$$

and

$$\mathscr{K}(r)<\frac{9.096}{8+r^2}\ln\left(\frac{4}{r'}\right):=M_2(r),\quad r\in(0,1),$$

where here and after we denote that $r'=\sqrt{1-r^2}\in(0,1)$. Then, Alzer and Qiu [1] proposed two double bounds for $\mathscr{K}(r)$, namely,

$$\left(\frac{\operatorname{arth}(r)}{r}\right)^{\alpha_1}\leqslant\frac{2}{\pi}\mathscr{K}(r)\leqslant\left(\frac{\operatorname{arth}(r)}{r}\right)^{\beta_1}:=\frac{2}{\pi}M_3(\beta_1,r),\quad r\in(0,1),$$

with the best constants $\alpha_1=3/4$, $\beta_1=1$ and

$$\frac{\alpha_2\ln(r')}{r'-1}+\frac{2(1-\alpha_2)}{1+r'}\leqslant\frac{2}{\pi}\mathscr{K}(r)\leqslant\frac{\beta_2\ln(r')}{r'-1}+\frac{2(1-\beta_2)}{1+r'},\quad r\in(0,1),$$

with the best constants $\alpha_2=2/\pi$, $\beta_2=3/4$, where

$$\operatorname{arth}(r)=\frac{1}{2}\ln\frac{1+r}{1-r}$$

is the inverse hyperbolic tangent function. Furthermore, Yang et al. [20] proved that the following inequality

$$\frac{1+(6p-7)r'}{p+(5p-5)r'}\frac{\pi\operatorname{arth}r}{2r}<\mathscr{K}(r)<\frac{1+(6q-7)r'}{q+(5q-6)r'}\frac{\pi\operatorname{arth}r}{2r}:=M_4(q,r),\quad r\in(0,1),$$

if and only if $p\geqslant\pi/2$ and $q\leqslant 89/69$. Besides, some scholars have also provided various bounds for $\mathscr{K}(r)$. For example, Andás and Baricz [2] established sharp bounds for

$\mathscr{K}(r)$ using hypergeometric functions. Yang et al. [19] derived bounds involving logarithmic functions for $\mathscr{K}(r)$ by proving the monotonicity of the function $r \mapsto r'^{p} e^{\mathscr{K}(r)}$. Yang and Tian [21] established some new sharp lower and upper bounds for the generalized elliptic integrals of the first kind. Moreover, Yang et al. [26] derived lower bounds for $\mathscr{K}(r)$ using the inverse hyperbolic tangent function.

Inspired by these bounds, we aim to establish some highly accurate bounds near $r \to 0^+$. For convenience, we introduce some notations here.

$$m(r) = -\frac{1}{4} + \frac{r^2}{64} + \frac{r^4}{128} + \frac{251r^6}{49152} + \frac{123r^8}{32768}, \tag{1.2}$$

$$m_1(p_1,r) = \frac{p_1}{2} + \frac{r^2}{64} + \frac{r^4}{128} + \frac{251r^6}{49152} + \frac{123r^8}{32768}, \tag{1.3}$$

$$m_2(p_2,r) = -\frac{1}{4} + \frac{p_2 r^2}{2} + \frac{r^4}{128} + \frac{251r^6}{49152} + \frac{123r^8}{32768}, \tag{1.4}$$

$$m_3(p_3,r) = -\frac{1}{4} + \frac{r^2}{64} + \frac{p_3 r^4}{2} + \frac{251r^6}{49152} + \frac{123r^8}{32768}, \tag{1.5}$$

$$m_4(p_4,r) = -\frac{1}{4} + \frac{r^2}{64} + \frac{r^4}{128} + \frac{p_4 r^6}{2} + \frac{123r^8}{32768}, \tag{1.6}$$

$$m_5(p_5,r) = -\frac{1}{4} + \frac{r^2}{64} + \frac{r^4}{128} + \frac{251r^6}{49152} + \frac{p_5 r^8}{2}, \tag{1.7}$$

and

$$F(p_1,p_2,p_3,p_4,p_5;r) := (1-r^2)^{\frac{1}{2}(p_1+p_2r^2+p_3r^4+p_4r^6+p_5r^8)}.$$

To be specific, we will prove the following theorem.

THEOREM 1.1. *The inequality*

$$\frac{2}{\pi}\mathscr{K}(r) \leqslant (1-r^2)^{\frac{1}{2}(p_1+p_2r^2+p_3r^4+p_4r^6+p_5r^8)} = F(p_1,p_2,p_3,p_4,p_5;r)$$

holds for all $r \in (0,1)$ *if*

$$p_1 \leqslant p_{0,1} = -\frac{1}{2}, \quad p_2 \leqslant p_{0,2} = \frac{1}{32}, \quad p_3 \leqslant p_{0,3} = \frac{1}{64},$$
$$p_4 \leqslant p_{0,4} = \frac{251}{24576}, \quad p_5 \leqslant p_{0,5} = \frac{123}{16384}.$$

Moreover, we have $\mathscr{K}(r) < \frac{\pi}{2}(1-r^2)^{m(r)}$ *for all* $r \in (0,1)$.

We will also prove that the parameters $p_{1,0}$, $p_{2,0}$, $p_{3,0}$, $p_{4,0}$ and $p_{5,0}$ in above theorem are optimal and cannot be replaced by larger values. Using Theorems 1.1, we will establish some inequalities involving Gauss arithmetic-geometric mean.

2. Lemmas

In this section, we present several lemmas that will be used in the next section.

LEMMA 2.1. *Define functions R and R_1 on $\mathbb{N}$ by*

$$R(k) := \begin{cases} \frac{1}{4k} - \frac{1}{64(k-1)} - \frac{1}{128(k-2)} - \frac{251}{49152(k-3)} - \frac{123}{32768(k-4)}, & k \geqslant 5, \\ 0, & k \leqslant 4, \end{cases} \tag{2.1}$$

and

$$R_1(k) = \begin{cases} \frac{1}{4}R(k-1) + \frac{7}{64}R(k-2) + \frac{13}{192}R(k-3) + \frac{791}{16384}R(k-4), & k \geqslant 6, \\ 0, & k \leqslant 5. \end{cases} \tag{2.2}$$

Then we have

$$m(r)\ln(1-r^2) = \frac{r^2}{4} + \frac{7r^4}{64} + \frac{13r^6}{192} + \frac{791r^8}{16384} + \frac{1523r^{10}}{40960} + \frac{97349r^{12}}{2949120} + \sum_{k=7}^{\infty} R(k)r^{2k},$$

and

$$\Big(m(r)\ln(1-r^2)\Big)^2 > \Big(\frac{r^2}{4} + \frac{7r^4}{64} + \frac{13r^6}{192} + \frac{791r^8}{16384}\Big)^2 + \sum_{k=6}^{\infty} R_1(k)r^{2k}, \quad r > 0.$$

Proof. According to the expansion of function $\ln(1-r^2)$ at $r=1$, we obtain

$$\begin{aligned} & m(r)\ln(1-r^2) \\ = & -m(r)\sum_{k=1}^{\infty}\frac{r^{2k}}{k} \\ = & \sum_{k=1}^{\infty}\frac{r^{2k}}{4k} - \sum_{k=1}^{\infty}\frac{r^{2k+2}}{64k} - \sum_{k=1}^{\infty}\frac{r^{2k+4}}{128k} - \sum_{k=1}^{\infty}\frac{251r^{2k+6}}{49152k} - \sum_{k=1}^{\infty}\frac{123r^{2k+8}}{32768k} \\ = & \frac{r^2}{4} + \frac{7r^4}{64} + \frac{13r^6}{192} + \frac{791r^8}{16384} + \frac{1523r^{10}}{40960} + \frac{97349r^{12}}{2949120} \\ & + \sum_{k=7}^{\infty}\Big(\frac{1}{4k} - \frac{1}{64(k-1)} - \frac{1}{128(k-2)} - \frac{251}{49152(k-3)} - \frac{123}{32768(k-4)}\Big)r^{2k} \\ = & \frac{r^2}{4} + \frac{7r^4}{64} + \frac{13r^6}{192} + \frac{791r^8}{16384} + \frac{1523r^{10}}{40960} + \frac{97349r^{12}}{2949120} + \sum_{k=7}^{\infty} R(k)r^{2k}. \end{aligned}$$

Meantime, we have

$$\begin{aligned} & \Big(m(r)\ln(1-r^2)\Big)^2 \\ = & \Big(\frac{r^2}{4} + \frac{7r^4}{64} + \frac{13r^6}{192} + \frac{791r^8}{16384} + \sum_{k=5}^{\infty} R(k)r^{2k}\Big)^2 \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}\right)^2 \\
&\quad + \left(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}\right)\sum_{k=5}^{\infty} R(k)r^{2k} + \left(\sum_{k=5}^{\infty} R(k)r^{2k}\right)^2 \\
&> \left(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}\right)^2 + \left(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}\right)\sum_{k=5}^{\infty} R(k)r^{2k} \\
&= \left(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}\right)^2 + \sum_{k=6}^{\infty} R_1(k)r^{2k}. \quad \square
\end{aligned}
$$

REMARK 2.1. Substituting $R(k)$ into $R_1(k)$, $R_1(k)$ changes into

$$
R_1(k) = \begin{cases}
0, & k = 1,2,3,4,5, \\
\dfrac{109846369}{11833835520}, & k = 6, \\
\dfrac{2329767599}{189341368320}, & k = 7, \\
\dfrac{106313523593}{7952337469440}, & k = 8, \\
\dfrac{R_{11}(k)}{9694278057984\prod_{j=1}^{8}(k-j)}, & k \geqslant 9,
\end{cases}
$$

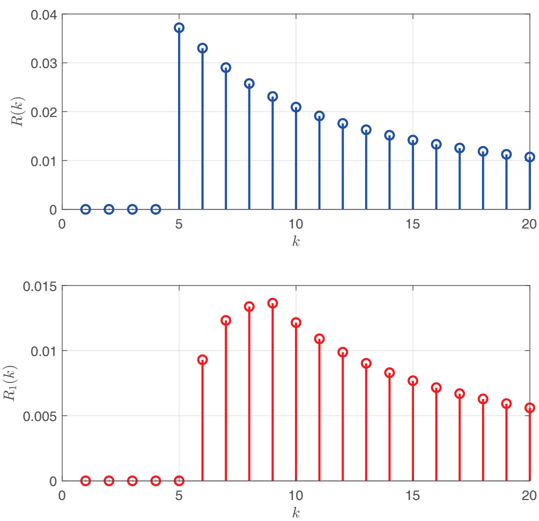

Figure 1: *Plots of functions* R *and* R_1

where

$$R_{11}(k)=\begin{pmatrix}1002746896395k^7-34559510561332k^6\\ +494700205483734k^5-3799171689851704k^4\\ +16834596177554907k^3-42823415193408580k^2\\ +57547130155136388k-31273980479514768\end{pmatrix},$$

see Figure 1.

LEMMA 2.2. *The inequalities*

$$\left(m(r)\ln(1-r^2)\right)^3>\frac{r^6}{64}+\frac{21r^8}{1024}+\frac{355r^{10}}{16384}+\frac{1407r^{12}}{65536},$$

$$\left(m(r)\ln(1-r^2)\right)^4>\frac{r^8}{256}+\frac{7r^{10}}{1024}+\frac{857r^{12}}{98304},$$

and

$$\left(m(r)\ln(1-r^2)\right)^5>\frac{r^{10}}{1024}+\frac{35r^{12}}{16384},$$

are valid for all $r\in(0,1)$.

Proof. Noting that

$$\begin{aligned}R(k)&=\frac{1}{4k}-\frac{1}{64(k-1)}-\frac{1}{128(k-2)}-\frac{251}{49152(k-3)}-\frac{123}{32768(k-4)}\\ &\geqslant\frac{1}{4k}-\frac{1}{64k}-\frac{1}{128k}-\frac{251}{49152k}-\frac{123}{32768k}=\frac{42916623}{197230592k}\geqslant 0,\end{aligned}$$

we obtain

$$m(r)\ln(1-r^2)\geqslant\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}+\frac{1523r^{10}}{40960}+\frac{97349r^{12}}{2949120}.$$

Thus, we have

$$\begin{aligned}&\left(m(r)\ln(1-r^2)\right)^3\\ &>\left(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}+\frac{1523r^{10}}{40960}+\frac{97349r^{12}}{2949120}\right)^3\\ &>\frac{r^6}{64}+\frac{21r^8}{1024}+\frac{355r^{10}}{16384}+\frac{1407r^{12}}{65536},\end{aligned}$$

$$\begin{aligned}&\left(m(r)\ln(1-r^2)\right)^4\\ &>\left(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}+\frac{1523r^{10}}{40960}+\frac{97349r^{12}}{2949120}\right)^4>\frac{r^8}{256}+\frac{7r^{10}}{1024}+\frac{857r^{12}}{98304},\end{aligned}$$

and

$$\begin{aligned}&\Big(m(r)\ln(1-r^2)\Big)^5\\ &> \Big(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}+\frac{1523r^{10}}{40960}+\frac{97349r^{12}}{2949120}\Big)^5 > \frac{r^{10}}{1024}+\frac{35r^{12}}{16384}. \qquad \square\end{aligned}$$

LEMMA 2.3. *The inequality*

$$R(k)+R_1(k)-\frac{\Gamma^2(\frac{1}{2}+k)}{\Gamma^2(\frac{1}{2})\Gamma^2(k+1)} > 0$$

is valid for all $k \geqslant 9$*, where* $R(k)$ *and* $R_1(k)$ *are defined by* (2.1) *and* (2.2)*, respectively.*

Proof. Based on the following estimate [5] of the ratio of the gamma functions

$$\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} < \frac{1}{\sqrt{k+\frac{1}{4}}} < \frac{1}{\sqrt{k}},$$

we have

$$\begin{aligned}&R(k)+R_1(k)-\frac{\Gamma^2(\frac{1}{2}+k)}{\Gamma^2(\frac{1}{2})\Gamma^2(k+1)}\\ &= \frac{1551936317k^8-55398840414k^7+830978041428k^6-6803742743478k^5+33025734909435k^4}{4831838208(k-8)(k-7)(k-6)(k-5)(k-4)(k-3)(k-2)(k-1)k}\\ &\quad+\frac{-96420321761868k^3+163218018346084k^2-143643270800784k+48704929136640}{4831838208(k-8)(k-7)(k-6)(k-5)(k-4)(k-3)(k-2)(k-1)k}\\ &\quad-\frac{\Gamma^2(\frac{1}{2}+k)}{\Gamma^2(\frac{1}{2})\Gamma^2(k+1)}\\ &> \frac{1551936317k^8-55398840414k^7+830978041428k^6-6803742743478k^5+33025734909435k^4}{4831838208(k-8)(k-7)(k-6)(k-5)(k-4)(k-3)(k-2)(k-1)k}\\ &\quad+\frac{-96420321761868k^3+163218018346084k^2-143643270800784k+48704929136640}{4831838208(k-8)(k-7)(k-6)(k-5)(k-4)(k-3)(k-2)(k-1)k}-\frac{1}{\pi k}\\ &= \frac{h(k)}{4831838208(k-8)(k-7)(k-6)(k-5)(k-4)(k-3)(k-2)(k-1)k},\end{aligned}$$

where

$$\begin{aligned}h(k) = {} & (1551936317\pi-4831838208)k^8-6(9233140069\pi-28991029248)k^7\\ &+36(23082723373\pi-73282879488)k^6\\ &+(21917218111488-6803742743478\pi)k^5\\ &+3(11008578303145\pi-36156645310464)k^4\\ &-36(2678342271163\pi-9030705610752)k^3\\ &+4(40804504586521\pi-142689014120448)k^2\\ &-48(2992568141683\pi-11031086628864)k\\ &+48704929136640(\pi-4).\end{aligned}$$

We claim that $h(k) > 0$ for all $k \geqslant 9$. In fact, expanding $h(k)$ at $k = 9$ yields

$$h(k) = \sum_{w=0}^{8} l_w (k-9)^w,$$

where

$$\begin{aligned}
l_8 &= 1551936317\pi - 4831838208 > 0,\\
l_7 &= 6(9390095735\pi - 28991029248) > 0,\\
l_6 &= 860642662302\pi - 2638183661568 > 0,\\
l_5 &= 36(199774782073\pi - 608811614208) > 0,\\
l_4 &= 35752010933325\pi - 108469935931392 > 0,\\
l_3 &= 18(5970985718545\pi - 18061411221504) > 0,\\
l_2 &= 8(23622454376171\pi - 71344507060224) > 0,\\
l_1 &= 768(228306889829\pi - 689442914304) > 0,\\
l_0 &= 13824(4661206657\pi - 14092861440) > 0. \quad \square
\end{aligned}$$

3. Sharp upper bounds of the complete elliptic integrals of the first kind

In this section, we provide some upper bounds for $\mathscr{K}(r)$. Taking partial derivations on $F(p_1, p_2, p_3, p_4, p_5; r)$ with respect to p_i yields

$$\frac{\partial}{\partial p_i} F(p_1, p_2, p_3, p_4, p_5; r) = \frac{1}{2} r^{2(i-1)} \ln\left(1-r^2\right) \left(1-r^2\right)^{\frac{1}{2}\left(p_1 + p_2 r^2 + p_3 r^4 + p_4 r^6 + p_5 r^8\right)} < 0,$$

namely, $F(p_1, p_2, p_3, p_4, p_5; r)$ is decreasing with respect to p_1, p_2, p_3, p_4 and p_5. As a result, Theorem 1.1 is equivalent to the following theorem.

THEOREM 3.1. *The inequality*

$$\frac{2}{\pi}\mathscr{K}(r) \leqslant (1-r^2)^{m(r)} := F_m(r)$$

is valid for all $r \in (0,1)$, *where function* $m(r)$ *is defined by* (1.2).

Proof. By employing Lemma 2.2, we have

$$\begin{aligned}
&\frac{1}{6}\left(m(r)\ln(1-r^2)\right)^3 + \frac{1}{24}\left(m(r)\ln(1-r^2)\right)^4 + \frac{1}{120}\left(m(r)\ln(1-r^2)\right)^5\\
&> \frac{r^6}{384} + \frac{11r^8}{3072} + \frac{1919r^{10}}{491520} + \frac{9341r^{12}}{2359296}.
\end{aligned}$$

Noting that $m(r)\ln(1-r^2) > 0$ and

$$F_m(r) = (1-r^2)^{m(r)} = \exp\left(m(r)\ln(1-r^2)\right),$$

we have

$$
\begin{aligned}
& F_m(r)-\frac{2}{\pi}\mathscr{K}(r)=\exp\Big(m(r)\ln(1-r^2)\Big)-\frac{2}{\pi}\mathscr{K}(r) \\
> & \ 1+\sum_{l=1}^{5}\frac{1}{l!}\Big(m(r)\ln(1-r^2)\Big)^l-\frac{2}{\pi}\mathscr{K}(r) \\
> & \ 1+m(r)\ln(1-r^2)+\frac{1}{2}\Big(m(r)\ln(1-r^2)\Big)^2 \\
& +\frac{r^6}{384}+\frac{11r^8}{3072}+\frac{1919r^{10}}{491520}+\frac{9341r^{12}}{2359296}-\frac{2}{\pi}\mathscr{K}(r).
\end{aligned}
$$

Further using Lemma 2.2, we obtain

$$
\begin{aligned}
& F_m(r)-\frac{2}{\pi}\mathscr{K}(r) \\
> & \ 1+m(r)\ln(1-r^2)+\frac{1}{2}\Big(m(r)\ln(1-r^2)\Big)^2+\frac{r^6}{384}+\frac{11r^8}{3072}+\frac{1919r^{10}}{491520} \\
& +\frac{9341r^{12}}{2359296}-\frac{2}{\pi}\mathscr{K}(r) \\
= & \ 1+\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}+\frac{1523r^{10}}{40960}+\frac{97349r^{12}}{2949120}+\sum_{k=7}^{\infty}R(k)r^{2k} \\
& +\frac{1}{2}\left(\left(\frac{r^2}{4}+\frac{7r^4}{64}+\frac{13r^6}{192}+\frac{791r^8}{16384}\right)^2+\sum_{k=6}^{\infty}R_1(k)r^{2k}+\sum_{k=10}^{\infty}R_2(k)r^{2k}\right) \\
& +\frac{r^6}{384}+\frac{11r^8}{3072}+\frac{1919r^{10}}{491520}+\frac{9341r^{12}}{2359296}-\frac{2}{\pi}\mathscr{K}(r) \\
= & \ \frac{34777r^{12}}{11796480}+\frac{10283r^{14}}{3145728}+\frac{625681r^{16}}{536870912}+\sum_{k=7}^{\infty}R(k)r^{2k}+\sum_{k=7}^{\infty}R_1(k)r^{2k} \\
& +\sum_{k=10}^{\infty}R_2(k)r^{2k}-\sum_{k=6}^{\infty}\frac{\Gamma^2(\frac{1}{2}+k)}{\Gamma^2(\frac{1}{2})\Gamma^2(k+1)}r^{2k} \\
> & \ \frac{34777r^{12}}{11796480}+\frac{10283r^{14}}{3145728}+\frac{625681r^{16}}{536870912}+\sum_{k=7}^{\infty}R(k)r^{2k}+\sum_{k=7}^{\infty}R_1(k)r^{2k} \\
& -\sum_{k=6}^{\infty}\frac{\Gamma^2(\frac{1}{2}+k)}{\Gamma^2(\frac{1}{2})\Gamma^2(k+1)}r^{2k} \\
> & \ \sum_{k=9}^{\infty}R(k)r^{2k}+\sum_{k=9}^{\infty}R_1(k)r^{2k}-\sum_{k=9}^{\infty}\frac{\Gamma^2(\frac{1}{2}+k)}{\Gamma^2(\frac{1}{2})\Gamma^2(k+1)}r^{2k}>0,
\end{aligned}
$$

where the final step follows from Lemma 2.3. □

Several inequalities can be easily derived from Theorem 3.1, which we present in the form of corollaries.

COROLLARY 3.1. *The following four inequalities hold for all* $r \in (0,1)$.

$$\frac{2}{\pi}\mathscr{K}(r) \leqslant (1-r^2)^{-\frac{1}{4}},$$
$$\frac{2}{\pi}\mathscr{K}(r) \leqslant (1-r^2)^{-\frac{1}{4}+\frac{r^2}{64}},$$
$$\frac{2}{\pi}\mathscr{K}(r) \leqslant (1-r^2)^{-\frac{1}{4}+\frac{r^2}{64}+\frac{r^4}{128}},$$
$$\frac{2}{\pi}\mathscr{K}(r) \leqslant (1-r^2)^{-\frac{1}{4}+\frac{r^2}{64}+\frac{r^4}{128}+\frac{251r^6}{49152}}.$$

Furthermore, we prove the following theorem.

THEOREM 3.2. *The inequality*

$$\frac{2}{\pi}\mathscr{K}(r) \leqslant (1-r^2)^{m_i(p_i;r)}, \quad r \in (0,1),$$

is valid for all $i = 1,2,3,4,5$ *if and only if* $p_i \leqslant p_{0,i}$, *where*

$$p_{0,1} = -\frac{1}{2}, \quad p_{0,2} = \frac{1}{32}, \quad p_{0,3} = \frac{1}{64}, \quad p_{0,4} = \frac{251}{24576}, \quad p_{0,5} = \frac{123}{16384}.$$

Proof. The necessity follows from Theorem 1.1. We will now demonstrate the sufficiency. Define $F_{m,i}(p_i;r) = (1-r^2)^{m_i(p_i;r)}$.

(1) Expanding $F_{m1}(p_1;r) - \frac{2}{\pi}\mathscr{K}(r)$ yields

$$\begin{aligned} F_{m,1}(p_1;r) - \frac{2}{\pi}\mathscr{K}(r) &= \left(1-r^2\right)^{\frac{p_1}{2}+\frac{123r^8}{32768}+\frac{251r^6}{49152}+\frac{r^4}{128}+\frac{r^2}{64}} - \frac{2}{\pi}\mathscr{K}(r) \\ &= \left(-\frac{p_1}{2}-\frac{1}{4}\right)r^2 + O(r^4). \end{aligned}$$

Since

$$\lim_{r\to 0^+} r^{-2}\left(\left(1-r^2\right)^{\frac{p_1}{2}+\frac{123r^8}{32768}+\frac{251r^6}{49152}+\frac{r^4}{128}+\frac{r^2}{64}} - \frac{2}{\pi}\mathscr{K}(r)\right) \geqslant 0,$$

we have $p_1 < -\frac{1}{2}$.

(2) Noting that

$$F_{m,2}(p_2;r) - \frac{2}{\pi}\mathscr{K}(r) = \frac{1}{64}(1-32p_2)r^4 + O(r^6),$$

and

$$\lim_{r\to 0^+} r^{-4}\left(F_{m,2}(p_2;r) - \frac{2}{\pi}\mathscr{K}(r)\right) = \frac{1}{64}(1-32p_2) \geqslant 0,$$

we obtain $p_2 \leqslant \frac{1}{32}$.

(3) Noting that

$$F_{m,3}(p_3;r)-\frac{2}{\pi}\mathscr{K}(r)=\frac{1}{128}(1-64p_3)r^6+O(r^8),$$

and

$$\lim_{r\to 0^+} r^{-6}\left(F_{m,3}(p_3;r)-\frac{2}{\pi}\mathscr{K}(r)\right)=\frac{1}{128}(1-64p_3)\geqslant 0,$$

we obtain $p_3\leqslant \frac{1}{64}$.

(4) Noting that

$$F_{m,4}(p_4;r)-\frac{2}{\pi}\mathscr{K}(r)=\frac{251-24576p_4}{49152}r^8+O(r^{10}),$$

and

$$\lim_{r\to 0^+} r^{-8}\left(F_{m,4}(p_4;r)-\frac{2}{\pi}\mathscr{K}(r)\right)=\frac{251-24576p_4}{49152}\geqslant 0,$$

we obtain $p_4\leqslant \frac{251}{24576}$.

(5) Noting that

$$F_{m,5}(p_5;r)-\frac{2}{\pi}\mathscr{K}(r)=\frac{123-16384p_5}{32768}r^{10}+O(r^{12}),$$

and

$$\lim_{r\to 0^+} r^{-10}\left(F_{m,5}(p_5;r)-\frac{2}{\pi}\mathscr{K}(r)\right)=\frac{123-16384p_5}{32768}\geqslant 0,$$

we obtain $p_5\leqslant \frac{123}{16384}$. □

4. Sharp lower bounds of Gauss arithmetic-geometric mean

The Gauss arithmetic-geometric mean (AGM) is defined by

$$AGM=AGM(x,y)=\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n,$$

where $x_0=x$, $y_0=y$ and

$$x_{n+1}=A(x_n,y_n)=\frac{x_n+y_n}{2},\quad y_{n+1}=G(x_n,y_n)=\sqrt{x_ny_n},\quad n\in\mathbb{N}. \tag{4.1}$$

An interesting link between the complete elliptic integrals of the first kind $\mathscr{K}(r)$ and Gauss arithmetic-geometric mean $AGM(x,y)$ is given by Gauss' formula

$$AGM\left(1,r'\right)=\frac{\pi}{2\mathscr{K}(r)}. \tag{4.2}$$

Hence, if there exists a function $u(r)$ such that

$$\frac{2}{\pi}\mathscr{K}(r)<u(r)$$

for all $r \in (0,1)$, then

$$AGM(1,r) > \frac{1}{u(\sqrt{1-r^2})}$$

for all $r \in (0,1)$.

Then Theorem 1.1, 3.1 and 3.2 give the following theorems, respectively.

THEOREM 4.1. *The inequality*

$$AGM(1,r) > \exp\Big(G(p_1,p_2,p_3,p_4,p_5;r)\ln r\Big)$$

holds for all $r \in (0,1)$ *if*

$$p_1 \leqslant -\frac{1}{2}, \quad p_2 \leqslant \frac{1}{32}, \quad p_3 \leqslant \frac{1}{64}, \quad p_4 \leqslant \frac{251}{24576}, \quad p_5 \leqslant \frac{123}{16384},$$

where

$$\begin{aligned} G(p_1,p_2,p_3,p_4,p_5;r) = &-(p_1+p_2+p_3+p_4+p_5)+(p_2+2p_3+3p_4+4p_5)r^2 \\ &-(p_3-3p_4-6p_5)r^4+(p_4+4p_5)r^6-p_5r^8. \end{aligned}$$

THEOREM 4.2. *The inequality*

$$AGM(1,r) > \exp\bigg(\Big(\frac{21401}{49152}+\frac{1009r^2}{8192}-\frac{187r^4}{2048}+\frac{989r^6}{24576}-\frac{123r^8}{16384}\Big)\ln r\bigg)$$

holds for all $r \in (0,1)$.

THEOREM 4.3. *The inequality*

$$AGM(1,r) > \exp\bigg(\Big(-p_1-\frac{3175}{49152}+\frac{1009r^2}{8192}-\frac{187r^4}{2048}+\frac{989r^6}{24576}-\frac{123r^8}{16384}\Big)\ln r\bigg)$$

holds for all $r \in (0,1)$ *if and only if* $p_1 \leqslant -1/2$. *The inequality*

$$AGM(1,r) > \exp\bigg(\Big(\frac{22937}{49152}-p_2+\Big(\frac{753}{8192}+p_2\Big)r^2-\frac{187r^4}{2048}+\frac{989r^6}{24576}-\frac{123r^8}{16384}\Big)\ln r\bigg)$$

holds for all $r \in (0,1)$ *if and only if* $p_2 \leqslant 1/32$. *The inequality*

$$\begin{aligned} AGM(1,r) > \exp\bigg(\Big(&\frac{22169}{49152}-p_3+\Big(\frac{753}{8192}+2p_3\Big)r^2-\Big(\frac{155}{2048}-p_3\Big)r^4 \\ &+\frac{989r^6}{24576}-\frac{123r^8}{16384}\Big)\ln r\bigg) \end{aligned}$$

holds for all $r \in (0,1)$ *if and only if* $p_3 \leqslant 1/64$. *The inequality*

$$\begin{aligned} AGM(1,r) > \exp\bigg(\Big(&\frac{7301}{16384}-p_4+\frac{379r^2}{4096}+3p_4r^2-\frac{497r^4}{8192}-3p_4r^4 \\ &+\frac{123r^6}{4096}+p_4r^6-\frac{123r^8}{16384}\Big)\ln r\bigg) \end{aligned}$$

holds for all $r \in (0,1)$ *if and only if* $p_4 \leqslant 251/24576$. *The inequality*

$$AGM(1,r) > \exp\left(\left(\frac{10885}{24576} - p_5 + \frac{763r^2}{8192} + 4p_5r^2 - \frac{379r^4}{8192} - 6p_5r^4 + \frac{251r^6}{24576} + 4p_5r^6 - p_5r^8\right)\ln r\right)$$

holds for all $r \in (0,1)$ *if and only if* $p_5 \leqslant 123/16384$.

5. Comparisons and conclusions

Previous studies have proposed several upper bounds for $\mathscr{K}(r)$, such as $M_1(r)$, $M_2(r)$, $M_3(1,r)$, and $M_4(89/69,r)$, as mentioned in the introduction. Here, we compare these upper bounds with our bound $\pi/2(1-r^2)^{m(r)}$ in Theorem 1.1 through graphical illustrations, as shown in Figure 2. The results indicate that our upper bound performs exceptionally well over the interval $r \in (0,1)$, demonstrating higher accuracy than the other bounds.

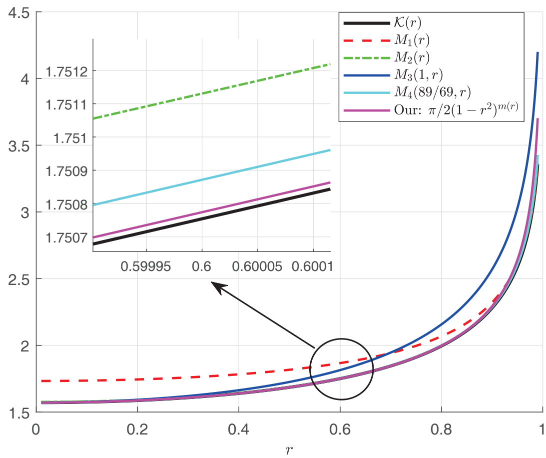

Figure 2: *Comparison of upper bounds for complete elliptic integral* $\mathscr{K}(r)$

Specifically, we have

$$M_1(r) - \mathscr{K}(r) \sim \frac{5\log(2)}{2} - \frac{\pi}{2} + o(r^2),$$

$$M_2(r) - \mathscr{K}(r) \sim 2.274\ln(2) - \frac{\pi}{2} + o(r^2),$$

$$M_3(1,r) - \mathscr{K}(r) \sim \frac{\pi}{24}r^2 + o(r^4),$$

$$M_4(89/69,r)-\mathscr{K}(r)\sim\frac{437\pi}{1075200}r^6+o(r^8),$$
$$\frac{\pi}{2}(1-r^2)^{m(r)}-\mathscr{K}(r)\sim\frac{34781\pi}{23592960}r^{12}+o(r^{14}).$$

Moreover, numerical calculations at $r=0.2$ yield

$$\begin{aligned}M_1(0.2)-\mathscr{K}(0.2)&\approx 0.157447,\\ M_2(0.2)-\mathscr{K}(0.2)&\approx 0.00459881,\\ M_3(1,0.2)-\mathscr{K}(0.2)&\approx 0.00538991,\\ M_4(89/69,0.2)-\mathscr{K}(0.2)&\approx 8.71761\times 10^{-8},\\ \frac{\pi}{2}(1-0.2^2)^{m(0.2)}-\mathscr{K}(0.2)&\approx 2.02214\times 10^{-11}.\end{aligned}$$

However, we must also acknowledge that our upper bound performs less effectively than some other bounds for $r\in(0.8,1)$, such as numerical calculations at $r=0.9$ yield

$$\begin{aligned}M_1(0.9)-\mathscr{K}(0.9)&\approx 0.0414022,\\ M_2(0.9)-\mathscr{K}(0.9)&\approx 0.0080705,\\ M_3(1,0.9)-\mathscr{K}(0.9)&\approx 0.288959,\\ M_4(89/69,0.9)-\mathscr{K}(0.9)&\approx 0.00616693,\\ \frac{\pi}{2}(1-0.9^2)^{m(0.9)}-\mathscr{K}(0.9)&\approx 0.0128702.\end{aligned}$$

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Zhong-Xuan Mao
Department of Mathematics and Physics
North China Electric Power University
Yonghua Street 619, 071003, Baoding, P. R. China
e-mail: maozhongxuan000@gmail.com

Jun-Ping Hou
Department of Mathematics and Physics
North China Electric Power University
Yonghua Street 619, 071003, Baoding, P. R. China
e-mail: jphou@ncepu.edu.cn

Jing-Feng Tian
Hebei Key Laboratory of Physics and Energy Technology
Department of Mathematics and Physics
North China Electric Power University
Yonghua Street 619, 071003, Baoding, P. R. China
e-mail: tianjf@ncepu.edu.cn