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LOGARITHMICALLY ABSOLUTE MONOTONICITY OF THE RATIO BETWEEN NORMALIZED REMAINDERS FOR A FUNCTION IN AN INTEGRAL REPRESENTATION OF THE RECIPROCAL OF THE GAMMA FUNCTION

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Abstract. Let $\Phi(t)=1-\frac{t}{\tan t}+\ln\frac{t}{\sin t}$ for $t\in(-\pi,\pi)$. In the paper, in light of a theorem on signs of coefficients in power series and with aid of the Wronski formula, the authors prove that the normalized remainder $T_{2n+1}[\Phi(t)]$ for $n\in\mathbb{N}_0$ is a logarithmically absolutely monotonic function in $t\in(0,\pi)$ and a logarithmically completely monotonic function in $t\in(-\pi,0)$, that the ratio $\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]}$ for $n\in\mathbb{N}_0$ is an absolutely monotonic function in $t\in(0,\pi)$ and a completely monotonic function in $t\in(-\pi,0)$, and that the normalized remainder $T_{2n+1}[\Phi(t)]$ and the ratio $\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]}$ for $n\in\mathbb{N}_0$ can be extended analytically into the complex z-plane and are analytic in the disc $|z|<\pi$. Moreover, the authors expand $\frac{1}{\Phi(t)}$ for $0<|t|<\pi$ into a Laurent series. These results verify a guess and generalize the corresponding ones in a paper published on Math. Inequal. Appl. **28** (2025), no. 2, 343–354.

1. Introduction

We first recall from [1, Section 5], [11, Definition 1], [12, Section 1], [15, Section 1], [18, Sections 1.9 and 1.10], [24, Remarks 2 and 4], and [31, Section 1] the definition of Qi's normalized remainders of the Maclaurin expansions of functions as follows.

DEFINITION 1. Let f be a real infinitely differentiable function on an interval $I\subseteq\mathbb{R}$ such that the origin 0 is an interior point of I. If $f^{(n+1)}(0)\ne0$ for some $n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}=\{0,1,2,\dots\}$, then we call the function

$$T_n[f(x)]=\begin{cases}\dfrac{1}{f^{(n+1)}(0)}\dfrac{(n+1)!}{x^{n+1}}\left[f(x)-\displaystyle\sum_{j=0}^{n}f^{(j)}(0)\frac{x^j}{j!}\right], & x\ne0\\ 1, & x=0\end{cases}\tag{1}$$

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for $x \in I$ the nth normalized remainder or the nth normalized tail of the Maclaurin expansion of the function f.

The normalized remainders have been investigated in [14, 27], for example.
We also recall from the monograph [21] and [25, Chapter IV] that

1. an infinitely differentiable real function $f(x)$ defined on an interval I is said to be absolutely monotonic in $x \in I$ if and only if all of its derivatives satisfy $f^{(k)}(x) \geqslant 0$ for $k \in \mathbb{N}_0$ and $x \in I$,

2. an infinitely differentiable function $f(x)$ defined on I is said to be completely monotonic in $x \in I$ if and only if all of its derivatives satisfy $(-1)^k f^{(k)}(x) \geqslant 0$ for $k \in \mathbb{N}_0$ and $x \in I$.

A function $f(x)$ is completely monotonic on (a,b) if and only if it is absolutely monotonic on $(-b,-a)$; see [25, p. 145, Definition 2c].

In [4, Definition 1] and [19, Definition 1], the notions of logarithmically absolutely (completely) monotonic functions were defined as follows:

1. A positive function $f(x)$ is said to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(x)]^{(k)} \geqslant 0$ for $x \in I$ and $k \in \mathbb{N}$.

2. A positive function $f(x)$ is said to be logarithmically completely monotonic on an interval I if it has derivatives of all orders and $(-1)^k[\ln f(x)]^{(k)} \geqslant 0$ for $x \in I$ and $k \in \mathbb{N}$.

In [4, Theorem 1], the authors proved that a logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I, but not conversely. In [2], [4, Theorem 4], and [19, Theorem 1], the authors proved that a logarithmically completely monotonic function on an interval I is also completely monotonic on I, but not conversely.

The classical Euler gamma function $\Gamma(z)$ can be defined [23, Chapter 3] by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}.$$

It is general knowledge for scientists that the gamma function $\Gamma(z)$ has had very extensive applications in mathematical sciences, including physics and engineering, in the past centuries. In [23, p. 71, Eq. (3.38)], we find the integral representation

$$\frac{1}{\Gamma(z)} = \frac{e^z z^{1-z}}{\pi} \int_0^{\pi} e^{-z\Phi(t)}\,\mathrm{d}t, \quad \Re(z) \geqslant 0,$$

where

$$\Phi(t) = 1 - \frac{t}{\tan t} + \ln\frac{t}{\sin t} = \sum_{j=1}^{\infty} \frac{2j+1}{2j} |B_{2j}| \frac{(2t)^{2j}}{(2j)!} \tag{2}$$

for $|t| < \pi$ and B_{2j} denotes the classical Bernoulli numbers generated [23, p. 3] by

$$\frac{1}{T_0[e^x]} = \frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!} = 1 - \frac{x}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{x^{2j}}{(2j)!}, \quad |x| < 2\pi. \tag{3}$$

In the paper [30], the authors considered the normalized remainder

$$T_{2n+1}[\Phi(t)] = \begin{cases} \frac{2n+2}{2n+3} \frac{1}{|B_{2n+2}|} \frac{(2n+2)!}{(2t)^{2n+2}} \left[\Phi(t) - \sum_{j=1}^{n} \frac{2j+1}{2j} |B_{2j}| \frac{(2t)^{2j}}{(2j)!} \right], & t \ne 0 \\ 1, & t = 0 \end{cases}$$

for $n \in \mathbb{N}_0$ and $t \in (-\pi, \pi)$.

It is obvious that

$$T_{2n+1}[\Phi(t)] = \frac{2n+2}{2n+3} \frac{(2n+2)!}{|B_{2n+2}|} \sum_{j=0}^{\infty} \frac{2j+2n+3}{2j+2n+2} \frac{|B_{2j+2n+2}|}{(2j+2n+2)!} (2t)^{2j} \tag{4}$$

for $n \in \mathbb{N}_0$ and $|t| < \pi$. Hence, it is easy to see that Qi's normalized remainder $T_{2n+1}[\Phi(t)]$ for $n \in \mathbb{N}_0$ is even in $t \in (-\pi, \pi)$, absolutely monotonic in $t \in (0, \pi)$, and completely monotonic in $t \in (0, \pi)$.

In the paper [30], the authors mainly obtained the following results:

1. For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(t)]$ is logarithmically convex in $t \in (-\pi, \pi)$. See [30, Theorem 1].

2. For $n \in \mathbb{N}_0$, the ratio $\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]}$ is decreasing in $t \in (-\pi, 0)$ and increasing in $t \in (0, \pi)$. See [30, Theorem 2].

In [30, Remark 5], the authors guessed that the ratio $\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]}$ for $n \in \mathbb{N}_0$ should be convex, even logarithmically convex, in $t \in (-\pi, \pi)$. For example, when $n = 0$, we have

$$\begin{aligned} \frac{T_3[\Phi(t)]}{T_1[\Phi(t)]} &= 9\left[\frac{2}{t^2} - \frac{1}{\Phi(t)}\right] \\ &= 9\left(\frac{2}{t^2} - \frac{1}{\frac{t^2}{2} + \frac{t^4}{36} + \frac{t^6}{405} + \frac{t^8}{4200} + \frac{t^{10}}{42525} + \cdots}\right) \\ &= \frac{18}{t^2}\left(1 - \frac{1}{1 + \frac{t^2}{18} + \frac{2t^4}{405} + \frac{t^6}{2100} + \frac{2t^8}{42525} + \cdots}\right) \\ &= 1 + \frac{t^2}{30} + \frac{101t^4}{56700} + \frac{109t^6}{1020600} + \frac{15979t^8}{2357586000} + \cdots \end{aligned}$$

for $t \in (-\pi, \pi)$, where we used the Maclaurin expansion in (2). This implies that the ratio $\frac{T_3[\Phi(t)]}{T_1[\Phi(t)]}$ is possibly convex in $t \in (-\pi, \pi)$.

In this paper, we will prove the following results which are stronger than the above guess and [30, Theorem 1 and 2].

THEOREM 1. *For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(t)]$ is a logarithmically absolutely monotonic function in $t \in (0,\pi)$ and a logarithmically completely monotonic function in $t \in (-\pi,0)$.*

For $n \in \mathbb{N}_0$, the function $\frac{1}{t}\frac{\mathrm{d}\ln T_{2n+1}[\Phi(t)]}{\mathrm{d}t}$ is an absolutely monotonic function in $(0,\pi)$ and a completely monotonic function in $(-\pi,0)$.

For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(t)]$ and the function $\frac{1}{t}\frac{\mathrm{d}\ln T_{2n+1}[\Phi(t)]}{\mathrm{d}t}$ can be extended analytically into the complex z-plane and are analytic in the disc $|z| < \pi$.

THEOREM 2. *For $n \in \mathbb{N}_0$, the ratio $\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]}$*

1. *is an absolutely monotonic function in $t \in (0,\pi)$,*
2. *is a completely monotonic function in $t \in (-\pi,0)$,*
3. *can be extended analytically into the complex z-plane and is analytic in the disc $|z| < \pi$.*

We will also establish a Laurent series expansion of the function $\frac{1}{\Phi(t)}$ for $0 < |t| < \pi$, whose coefficients are expressed in terms of determinants.

THEOREM 3. *The function $\frac{1}{\Phi(t)}$ for $0 < |t| < \pi$ can be expanded into*

$$\begin{aligned}\frac{1}{\Phi(t)} &= \frac{2}{t^2} + \sum_{j=0}^{\infty} b_{j+1} t^{2j} \\ &= \frac{2}{t^2} - \frac{1}{9} - \frac{1}{270}t^2 - \frac{101}{510300}t^4 - \frac{109}{9185400}t^6 - \frac{15979}{21218274000}t^8 - \cdots\end{aligned}$$

such that $b_j < 0$ for $j \in \mathbb{N}$, where b_j is a $j \times j$ order determinant defined by

$$b_j = (-1)^j 2^{j+1} \left|\sigma_{i,k}\right|_{j\times j}, \quad j \in \mathbb{N}, \tag{5}$$

$$\sigma_{i,k} = \begin{cases} a_{i-k+1}, & i-k+1 \geqslant 0; \\ 0, & i-k+1 < 0, \end{cases}$$

and

$$a_\ell = \frac{2\ell+3}{\ell+1}\frac{2^{2\ell+1}}{(2\ell+2)!}|B_{2\ell+2}|, \quad \ell \in \mathbb{N}_0.$$

2. Lemmas

For smoothly proceeding, we recall the following lemmas.

LEMMA 1. ([30, Lemma 3]) *The sequence*

$$\frac{j(2j+3)}{[(j+1)(2j+1)]^2}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

is increasing in $j\in\mathbb{N}_0$.

LEMMA 2. ([8, Theorems 1 and 2]) *Let*

$$k(t)=\sum_{\ell=0}^{\infty}k_\ell t^\ell,\quad q(t)=\sum_{\ell=0}^{\infty}q_\ell t^\ell,\quad p(t)=\sum_{\ell=0}^{\infty}p_\ell t^\ell$$

be formal series such that $k(t)=\frac{q(t)}{p(t)}$ *and* $p_\ell>0$ *for* $\ell\in\mathbb{N}_0$.

1. *If both of the sequences* $\frac{p_{\ell+1}}{p_\ell}$ *and* $\frac{q_\ell}{p_\ell}$ *are increasing in* $\ell\in\mathbb{N}_0$*, then* $k_\ell\geqslant 0$ *for* $\ell\in\mathbb{N}$.
2. *If the sequence* $\frac{p_{\ell+1}}{p_\ell}$ *is increasing in* $\ell\in\mathbb{N}_0$ *and the sequence* $\frac{q_\ell}{p_\ell}$ *is decreasing in* $\ell\in\mathbb{N}_0$*, then* $k_\ell\leqslant 0$ *for* $\ell\in\mathbb{N}$.

REMARK 1. The case $q(t)=1$ and $k_\ell\leqslant 0$ of Lemma 2 appeared in [5, p. 68, Theorem 22] and [9, 22], see also [6, p. 13, Problem 6] and [10, p. 331]. This special case was applied in [13, 16, 17]. Lemma 2 was modified in [28, Proposition 2].

3. Proofs of theorems

We are now in a position to prove our theorems.

Proof of Theorem 1. Directly computing yields

$$\begin{aligned}
\frac{\mathrm{d}\ln T_{2n+1}[\Phi(t)]}{\mathrm{d}t}&=\frac{T_{2n+1}'[\Phi(t)]}{T_{2n+1}[\Phi(t)]}\\
&=\frac{\sum_{j=1}^{\infty}\frac{2j+2n+3}{2j+2n+2}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}(2j)t^{2j-1}}{\sum_{j=0}^{\infty}\frac{2j+2n+3}{2j+2n+2}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}t^{2j}}\\
&=8t\frac{\sum_{j=0}^{\infty}\frac{2j+2n+5}{j+n+2}\frac{|B_{2j+2n+4}|}{(2j+2n+4)!}2^{2j}(j+1)t^{2j}}{\sum_{j=0}^{\infty}\frac{2j+2n+3}{j+n+1}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}t^{2j}},
\end{aligned}$$

where we used the series representation (4). Let

$$p_n(j) = \frac{2j+2n+3}{j+n+1} \frac{|B_{2j+2n+2}|}{(2j+2n+2)!} 2^{2j}$$

and

$$q_n(j) = \frac{2j+2n+5}{j+n+2} \frac{|B_{2j+2n+4}|}{(2j+2n+4)!} 2^{2j}(j+1)$$

for $j,n \in \mathbb{N}_0$. Then

$$\frac{q_n(j)}{p_n(j)} = \frac{j+1}{2(2j+2n+3)} \frac{(j+n+1)(2j+2n+5)}{(j+n+2)^2(2j+2n+3)} \left| \frac{B_{2j+2n+4}}{B_{2j+2n+2}} \right|$$

and

$$\frac{p_n(j+1)}{p_n(j)} = \frac{2(j+n+1)(2j+2n+5)}{(j+n+2)^2(2j+2n+3)^2} \left| \frac{B_{2j+2n+4}}{B_{2j+2n+2}} \right|$$

for $j,n \in \mathbb{N}_0$. Making use of Lemma 1, we see that the sequences $\frac{q_n(j)}{p_n(j)}$ and $\frac{p_n(j+1)}{p_n(j)}$ for $n \in \mathbb{N}_0$ are increasing in $j \in \mathbb{N}_0$. Employing Lemma 2, we conclude that

$$\frac{1}{8t} \frac{\mathrm{d} \ln T_{2n+1}[\Phi(t)]}{\mathrm{d}t} = \frac{(n+1)(2n+5)}{2(n+2)^2(2n+3)^2} \left| \frac{B_{2n+4}}{B_{2n+2}} \right| + \sum_{j=1}^{\infty} k_n(j) t^{2j}$$

such that $k_n(j) \geqslant 0$ for $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$. This means that the function $\frac{1}{t} \frac{\mathrm{d} \ln T_{2n+1}[\Phi(t)]}{\mathrm{d}t}$ is absolutely monotonic in $t \in (0,\pi)$ and

$$\frac{\mathrm{d} \ln T_{2n+1}[\Phi(t)]}{\mathrm{d}t} = \frac{4(n+1)(2n+5)}{(n+2)^2(2n+3)^2} \left| \frac{B_{2n+4}}{B_{2n+2}} \right| t + 8 \sum_{j=1}^{\infty} k_n(j) t^{2j+1}$$

for $n \in \mathbb{N}_0$ is absolutely monotonic in $t \in (0,\pi)$. Hence, the normalized remainder $T_{2n+1}[\Phi(t)]$ for $n \in \mathbb{N}_0$ is logarithmically absolutely monotonic in $t \in (0,\pi)$.

Since the normalized remainder $T_{2n+1}[\Phi(t)]$ and the function $\frac{1}{t} \frac{\mathrm{d} \ln T_{2n+1}[\Phi(t)]}{\mathrm{d}t}$ are even, they are logarithmically completely monotonic function and completely monotonic function in $t \in (-\pi,0)$, respectively.

By [25, p. 146, Theorem 3a], we see that the normalized remainder $T_{2n+1}[\Phi(t)]$ and the function $\frac{1}{t} \frac{\mathrm{d} \ln T_{2n+1}[\Phi(t)]}{\mathrm{d}t}$ can be extended analytically into the complex z-plane and they are analytic in the disc $|z| < \pi$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. It is easy to see that

$$T_{2n+1}[\Phi(t)] = \frac{2n+2}{2n+3} \frac{(2n+2)!}{|B_{2n+2}|} \sum_{j=0}^{\infty} \frac{2j+2n+3}{2j+2n+2} \frac{|B_{2j+2n+2}|}{(2j+2n+2)!} (2t)^{2j}$$

for $n \in \mathbb{N}_0$ and $|t| < \pi$. Then

$$\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]} = \frac{[(2n+3)(2n+4)]^2}{(2n+2)(2n+5)} \frac{|B_{2n+2}|}{|B_{2n+4}|} \frac{\sum_{j=0}^{\infty} \frac{2j+2n+5}{2j+2n+4} \frac{|B_{2j+2n+4}|}{(2j+2n+4)!} (2t)^{2j}}{\sum_{j=0}^{\infty} \frac{2j+2n+3}{2j+2n+2} \frac{|B_{2j+2n+2}|}{(2j+2n+2)!} (2t)^{2j}}.$$

Let

$$q_{j,n} = \frac{2j+2n+5}{2j+2n+4} \frac{|B_{2j+2n+4}|}{(2j+2n+4)!}$$

and

$$p_{j,n} = \frac{2j+2n+3}{2j+2n+2} \frac{|B_{2j+2n+2}|}{(2j+2n+2)!}$$

for $j,n \in \mathbb{N}_0$. The ratios

$$\frac{p_{j+1,n}}{p_{j,n}} = \frac{(2j+2n+2)(2j+2n+5)}{[(2j+2n+3)(2j+2n+4)]^2} \frac{|B_{2j+2n+4}|}{|B_{2j+2n+2}|} = \frac{q_{j,n}}{p_{j,n}}$$

for $j,n \in \mathbb{N}_0$. By virtue of Lemma 1, we see that the ratios $\frac{p_{j+1,n}}{p_{j,n}}$ and $\frac{q_{j,n}}{p_{j,n}}$ is increasing in $j \in \mathbb{N}_0$ for fixed $n \in \mathbb{N}_0$. In view of Lemma 2, we see that the series expansion

$$\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]} = \frac{2[(2n+3)(n+2)]^2}{(n+1)(2n+5)} \frac{|B_{2n+2}|}{|B_{2n+4}|} \sum_{j=0}^{\infty} k_{j,n}(2t)^j, \quad |t| < \pi$$

satisfies $k_{j,n} \geqslant 0$ for $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$. It is clear that

$$k_{0,n} = \lim_{t\to 0} \frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]} = \frac{\lim_{t\to 0} T_{2n+3}[\Phi(t)]}{\lim_{t\to 0} T_{2n+1}[\Phi(t)]} = 1.$$

Accordingly, the ratio $\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]}$ for $n \in \mathbb{N}_0$ is absolutely monotonic in $t \in (0,\pi)$. By [25, p. 145, Definition 2c], it is completely monotonic in $t \in (-\pi,0)$. By [25, p. 146, Theorem 3a], it can be extended analytically into the complex z-plane and the ratio $\frac{T_{2n+3}[\Phi(z)]}{T_{2n+1}[\Phi(z)]}$ is analytic in the disc $|z| < \pi$. The proof of Theorem 2 is complete. □

Proof of Theorem 3. In [6, p. 17, Theorem 1.3], [7, p. 347], [20, Section 2], and the old book [26] in 1881, we find the following proposition.

> If $a_0 \neq 0$ and $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is a formal series, then the coefficients of the reciprocal series $\frac{1}{P(x)} = b_0 + b_1 x + b_2 x^2 + \cdots$ are given by $b_0 = \frac{1}{a_0}$ and

$$b_j = \frac{(-1)^j}{a_0^{j+1}} \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{j-2} & a_{j-3} & a_{j-4} & a_{j-5} & \cdots & a_1 & a_0 & 0 \\ a_{j-1} & a_{j-2} & a_{j-3} & a_{j-4} & \cdots & a_2 & a_1 & a_0 \\ a_j & a_{j-1} & a_{j-2} & a_{j-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \quad j \in \mathbb{N}. \tag{6}$$

We call the determinantal formula (6) the Wronski formula. It was cited and applied in [13, Lemma 1], [16, Lemma 2.1], and [17, Lemma 2.1].

From the series expansion (2), it follows that

$$\frac{t^2}{\Phi(t)} = \frac{1}{\sum_{j=0}^{\infty} \frac{2j+3}{j+1} \frac{2^{2j+1}}{(2j+2)!} |B_{2j+2}| t^{2j}} = \sum_{j=0}^{\infty} b_j t^{2j}.$$

Taking $a_j = \frac{2j+3}{j+1} \frac{2^{2j+1}}{(2j+2)!} |B_{2j+2}|$ for $j \in \mathbb{N}_0$ in the determinantal formula (6) yields the determinantal formula (5) for $j \in \mathbb{N}$. Then we obtain

$$\begin{aligned}
\frac{1}{\Phi(t)} &= \frac{1}{t^2} \sum_{j=0}^{\infty} b_j t^{2j} \\
&= \frac{b_0}{t^2} + \sum_{j=1}^{\infty} b_j t^{2j-2} \\
&= \frac{2}{t^2} - \frac{1}{9} - \frac{1}{270} t^2 - \frac{101}{510300} t^4 - \frac{109}{9185400} t^6 - \frac{15979}{21218274000} t^8 - \cdots
\end{aligned}$$

for $0 < |t| < \pi$.

From Theorem 2, we immediately deduce that the function $\frac{2}{t^2} - \frac{1}{\Phi(t)}$ is absolutely monotonic on $(0, \pi)$ and completely monotonic on $(-\pi, 0)$. Consequently, the sequence b_j for $j \in \mathbb{N}$ is negative. The proof of Theorem 3 is complete. □

4. Guesses and more remarks

4.1. Guesses on $T_{2n-1}[\tan^2 x]$

In the paper [3, p. 798], we find the Maclaurin power series expansion

$$\begin{aligned}
\tan^2 x &= \sum_{j=1}^{\infty} \frac{2^{2j+2}(2^{2j+2}-1)(2j+1)}{(2j+2)!} |B_{2j+2}| x^{2j} \\
&= x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \frac{1382x^{10}}{14175} + \frac{21844x^{12}}{467775} + \cdots
\end{aligned} \tag{7}$$

for $|x| < \frac{\pi}{2}$, where the Bernoulli numbers B_j are generated by (3). Making use of the series expansion (7) and taking $f(x) = \tan^2 x$ in (1) lead to the normalized remainder

$$T_{2n-1}[\tan^2 x] = \begin{cases} \dfrac{(2n+2)! \left[\tan^2 x - \sum_{j=1}^{n-1} \frac{2^{2j+2}(2^{2j+2}-1)(2j+1)}{(2j+2)!} |B_{2j+2}| x^{2j}\right]}{2^{2n+2}(2^{2n+2}-1)(2n+1)|B_{2n+2}| x^{2n}}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{8}$$

for $n \in \mathbb{N}$ and $|x| < \frac{\pi}{2}$. For more information on normalized remainders $T_n[\tan^2 x]$ and

$$T_{2n-1}[\tan^2 x] = T_{2n-2}\left[\frac{\tan^2 x}{x}\right] = T_{2n-3}\left[\left(\frac{\tan x}{x}\right)^2\right] = T_{2n-1}[\sec^2 x], \quad n \geqslant 2,$$

please refer to [14, 27, 31] and closely related references.

Theorem 1 in [31] reads that the normalized remainder $T_{2n-1}\left[\tan^2 x\right]$ for $n\in\mathbb{N}$ is a logarithmically convex function in $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. In [31, Theorem 2], the logarithm $\ln T_{2n-1}\left[\tan^2 x\right]$ for $n\in\mathbb{N}$ was expanded into a Maclaurin power series. In [12, Theorem 1.1], the ratio $\frac{T_{2n+1}[\tan^2 x]}{T_{2n-1}[\tan^2 x]}$ for $n\in\mathbb{N}$ was proved to be increasing in $x\in\left(-\frac{\pi}{2},0\right)$ and decreasing in $x\in\left(0,\frac{\pi}{2}\right)$.

In [12, Remark 4.1], the authors proposed the following guess.

GUESS 1. ([12, Remark 4.1]) *The ratio* $\frac{T_{2n+1}[\tan^2 x]}{T_{2n-1}[\tan^2 x]}$ *for* $n\in\mathbb{N}$ *is concave in* $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

We now propose one more stronger guess as follows.

GUESS 2. *For* $n\in\mathbb{N}$, *the ratio* $\frac{T_{2n-1}[\tan^2 x]}{T_{2n+1}[\tan^2 x]}$ *is completely monotonic in* $x\in\left(-\frac{\pi}{2},0\right)$ *and is absolutely monotonic in* $x\in\left(0,\frac{\pi}{2}\right)$.

REMARK 2. The normalized remainder $T_{2n-1}\left[\tan^2 x\right]$ defined by (8) for $n\in\mathbb{N}$ has a series representation

$$T_{2n-1}\left[\tan^2 x\right]=\frac{(2n+2)!}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\times\sum_{j=0}^{\infty}\frac{(2n+2j+1)\left(2^{2n+2j+2}-1\right)}{(2n+2j+2)!}|B_{2n+2j+2}|(2x)^{2j} \tag{9}$$

for $|x|<\frac{\pi}{2}$. From the series representation (9), it follows that

$$\frac{T_{2n-1}[\tan^2 x]}{T_{2n+1}[\tan^2 x]}=\frac{1}{2(2n+1)(n+2)}\frac{2^{2n+4}-1}{2^{2n+2}-1}\left|\frac{B_{2n+4}}{B_{2n+2}}\right|\times\frac{\sum_{j=0}^{\infty}\frac{(2n+2j+1)(2^{2n+2j+2}-1)}{(2n+2j+2)!}|B_{2n+2j+2}|(2x)^{2j}}{\sum_{j=0}^{\infty}\frac{(2n+2j+3)(2^{2n+2j+4}-1)}{(2n+2j+4)!}|B_{2n+2j+4}|(2x)^{2j}}$$

for $n\in\mathbb{N}$ and $|x|<\frac{\pi}{2}$.

Let

$$q_n(j)=\frac{(2n+2j+1)\left(2^{2n+2j+2}-1\right)}{(2n+2j+2)!}|B_{2n+2j+2}|$$

and $p_n(j)=q_n(j+1)$ for $n\in\mathbb{N}$ and $j\in\mathbb{N}_0$. Then

$$\frac{p_n(j+1)}{p_n(j)}=\frac{1}{2(2n+2j+3)(n+j+3)}\frac{2^{2n+2j+6}-1}{2^{2n+2j+4}-1}\left|\frac{B_{2n+2j+6}}{B_{2n+2j+4}}\right|$$

and

$$\frac{q_n(j)}{p_n(j)}=2(2n+2j+1)(n+j+2)\frac{2^{2n+2j+2}-1}{2^{2n+2j+4}-1}\left|\frac{B_{2n+2j+2}}{B_{2n+2j+4}}\right|$$

for $n\in\mathbb{N}$ and $j\in\mathbb{N}_0$. Since the sequence

$$\frac{1}{(2j-1)(j+1)}\frac{2^{2j+2}-1}{2^{2j}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

is decreasing in $j\in\mathbb{N}$, see [12, Lemma 2.1], we conclude that the sequence $\frac{p_n(j+1)}{p_n(j)}$ for fixed $n\in\mathbb{N}$ is decreasing in $j\in\mathbb{N}_0$ and that the sequence $\frac{q_n(j)}{p_n(j)}$ for fixed $n\in\mathbb{N}$ is increasing in $j\in\mathbb{N}_0$. This means that we cannot use Lemma 2 to confirm Guess 2.

4.2. Guesses on $T_n[\mathrm{e}^x]$

In the papers [1, 11, 15, 29] and [18, Section 1.7], the normalized remainder $T_n[\mathrm{e}^x]$ was investigated systematically. The normalized remainder $T_n[\mathrm{e}^x]$ for $n\in\mathbb{N}_0$ was proved to be logarithmically convex and absolutely monotonic in $x\in\mathbb{R}$ and the ratio $\frac{T_{n+1}[\mathrm{e}^x]}{T_n[\mathrm{e}^x]}$ for $n\in\mathbb{N}_0$ was verified to be decreasing in $x\in\mathbb{R}$. We propose the following guess.

GUESS 3. *For $n\in\mathbb{N}_0$, the normalized remainder $T_n[\mathrm{e}^x]$ is a logarithmically absolutely monotonic function in $x\in\mathbb{R}$. For $n\in\mathbb{N}_0$, the ratio $\frac{T_n[\mathrm{e}^x]}{T_{n+1}[\mathrm{e}^x]}$ is an absolutely monotonic function in $x\in\mathbb{R}$.*

REMARK 3. It is clear that, for $n\in\mathbb{N}_0$,

$$T_n[\mathrm{e}^x]=\sum_{j=0}^{\infty}\frac{1}{\binom{j+n+1}{n+1}}\frac{x^j}{j!},\quad x\in\mathbb{R}.$$

Standard computation gives

$$\begin{aligned}\frac{\mathrm{d}\ln T_n[\mathrm{e}^x]}{\mathrm{d}x}&=\frac{T_n'[\mathrm{e}^x]}{T_n[\mathrm{e}^x]}\\&=\frac{\sum_{j=1}^{\infty}\frac{1}{\binom{j+n+1}{n+1}}\frac{x^{j-1}}{(j-1)!}}{\sum_{j=0}^{\infty}\frac{1}{\binom{j+n+1}{n+1}}\frac{x^j}{j!}}\\&=\frac{\sum_{j=0}^{\infty}\frac{1}{\binom{j+n+2}{n+1}}\frac{x^j}{j!}}{\sum_{j=0}^{\infty}\frac{1}{\binom{j+n+1}{n+1}}\frac{x^j}{j!}}\end{aligned}$$

for $n\in\mathbb{N}_0$ and $x\in\mathbb{R}$.

Let

$$p_n(j)=\frac{1}{\binom{j+n+1}{n+1}}\frac{1}{j!}\quad\text{and}\quad q_n(j)=\frac{1}{\binom{j+n+2}{n+1}}\frac{1}{j!}$$

for $j,n \in \mathbb{N}_0$. Then

$$\begin{aligned}\frac{p_n(j+1)}{p_n(j)} &= \frac{1}{\binom{j+n+2}{n+1}} \frac{1}{(j+1)!} \binom{j+n+1}{n+1} j! \\ &= \frac{1}{j+n+2}\end{aligned}$$

is decreasing in $j \in \mathbb{N}_0$ and

$$\begin{aligned}\frac{q_n(j)}{p_n(j)} &= \frac{1}{\binom{j+n+2}{n+1}} \frac{1}{j!} \binom{j+n+1}{n+1} j! \\ &= \frac{j+1}{j+n+2}\end{aligned}$$

is increasing in $j \in \mathbb{N}_0$ for fixed $j \in \mathbb{N}_0$. This means that we cannot employ Lemma 2 to confirm Guess 3.

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