Journal of
Mathematical
Inequalities
Volume 19, Number 4 (2025), 1253–1261 doi:10.7153/jmi-2025-19-81

ON THE NORMS OF GEOMETRIC CIRCULANT MATRICES WITH THE QUADRAPELL NUMBERS

ALEKSANDRA ERIĆ AND IVAN LAZAREVIĆ*

(*Communicated by T. Burić*)

Abstract. In this paper we give the some bounds of the spectral norm for r-geometric circulant matrices with the Quadrapell numbers. In addition, there are some remarks on Quadrapell numbers and a proof of the conjecture about $\lim_{n\to\infty}\frac{D_{n+1}}{D_n}$ stated by D. Tasci.

1. Introduction

Geometric circulant matrices and r-circulant matrices appear in various areas of applied mathematics and engineering. Due to their structured form, they are particularly useful in numerical linear algebra, signal and image processing, and the design of fast algorithms. In addition, these matrices find applications in coding theory, cryptography, and the modelling of cyclic systems in physics and biology.

In this paper we calculated the upper and lower bound for spectral norm for geometric circulant matrices with the Quadrapell numbers.

Matrix norms are fundamental tools in numerical linear algebra, providing a means to quantify the size of a matrix. They are crucial for analyzing the stability and convergence of numerical algorithms, as well as for evaluating the sensitivity of solutions to perturbations and estimating error bounds. Matrix norms are widely applied in areas such as control theory, signal processing, optimization, and machine learning.

Many authors have studied matrix norms, with particular attention given to the norms of circulant matrices, (see, [6], [8], [9]). For further discussion, it is especially interesting to consider the computation of norms of circulant matrices whose entries are taken from generalizations of Fibonacci numbers (see, [1], [3], [5]).

DEFINITION 1.1. A circulant matrix C is a matrix where each row is a right cyclic shift of the previous one. If there exists vector $c=(c_0,c_1,c_2,\ldots,c_{n-1})$ than the following matrix is a circulant:

$$C=\begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1}\\ c_{n-1} & c_0 & \cdots & c_{n-2}\\ \vdots & \vdots & \ddots & \vdots\\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}.$$

Mathematics subject classification (2020): 15A60, 15B99, 11B39.
Keywords and phrases: Quadrapell numbers, geometric circulant matrix, spectral norms.
* Corresponding author.

DEFINITION 1.2. A matrix with dimension $n \times n$ is called a geometric circulant matrix if exists $r \neq 0$ and is given by:

$$C_{r^*} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & \cdots & c_{n-2} \\ r^2 \cdot c_{n-2} & r \cdot c_{n-1} & \cdots & c_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ r^{n-1} \cdot c_{n-1} & r^{n-2} \cdot c_2 & \cdots & c_0 \end{bmatrix}.$$

These matrices are a generalization of r-circulant matrices, which are given as follows:

$$C_r = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & \cdots & c_{n-2} \\ r \cdot c_{n-2} & r \cdot c_{n-1} & \cdots & c_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ r \cdot c_{n-1} & r \cdot c_2 & \cdots & c_0 \end{bmatrix}.$$

For $r = 1$ we get a circulant matrix.

The singular values $\sigma_1 \geqslant \sigma_2 \geqslant \ldots. \geqslant \sigma_n$ of matrix A are the square roots of eigenvalues of $A \cdot A^T$. The Hadamar product of matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ is matrix $C = A \circ B = [a_{ij} \cdot b_{ij}]_{m\times n}$.

DEFINITION 1.3. Let $A = [a_{ij}]_{n\times n}$ be any matrix, we define norm with label $\| A \|_1$ on the following way:

$$\| A \|_1 = \sum_{i,j=1}^{n} |a_{ij}|.$$

DEFINITION 1.4. Euclidean norm of matrix $A = [a_{ij}]_{n\times n}$ is:

$$\| A \|_E = \sqrt{\sum_{i,j=1}^{n} (a_{ij})^2}.$$

In the literature, this norm is often denoted as the Frobenius norm or the Hilbert-Schmidt norm.

DEFINITION 1.5. For matrix $A = [a_{ij}]_{n\times n}$, we define the spectral norm (operator norm) $\| A \|_2$ in the following way:

$$\| A \|_2 = \sup_{|x|=1} |Ax|.$$

Note that the spectral norm is equal to the largest singular value of the matrix:

$$\| A \|_2 = \sigma_1.$$

It is sometimes important to understand the relationship between different norms of the same matrix. The following result provides an inequality between the spectral norm and the Euclidean norm.

If $A = [a_{ij}]_{n\times n}$ matrix , then following inequalities holds, for spectral and Euclidean norm of this matrix:

$$\frac{1}{\sqrt{n}} \parallel A \parallel_E \leqslant \parallel A \parallel_2 \leqslant \parallel A \parallel_E$$

$$\parallel A \parallel_2 \leqslant \parallel A \parallel_E \leqslant \sqrt{n} \parallel A \parallel_2 .$$

In paper [5], authors give the upper and lower bounds for the spectral norms of the geometric circulant matrices and symmetric geometric circulant matrices with Tribonacci numbers. In this paper we give the upper and lower bounds for the spectral norms of the geometric circulant matrices with Quadrapell numbers.

The following lemma from paper [2] is also used by the authors of paper [5], as well as in our main result.

LEMMA 1.1. *Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices. Thus:*

$$\parallel A \circ B \parallel_2 \leqslant r(A)\cdot c(B)$$

where:

$$r(A) = \max_{1\leqslant i\leqslant m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}; \qquad c(B) = \max_{1\leqslant j\leqslant n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}.$$

2. Preliminaries

Recurrence relations have many interesting properties and applications in many fields of science. The Fibonacci sequence is one of the most prominent examples of a recurrence relation, with notable applications in the natural sciences, including physics, chemistry, and biology. The Fibonacci sequence first appears in The Book of Calculation, 1202. years by Fibonacci (Leonardo Bonacci), where it is used to calculate the growth of rabbit populations. The Fibonacci sequence, denote F_n is defined by recurrence relation:

$$F_n = F_{n-1} + F_{n-2}.$$

The explicit formula for the n-th terms of the Fibonacci sequence, without using recursion is:

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

The formula is called Binet's formula.

The next two identities provide the sum and the sum of the squares of the first n Fibonacci numbers.

1.
$$\sum_{k=0}^{n-1} F_k = F_{n+2} - 1$$

2.
$$\sum_{k=0}^{n-1} F_k^2 = F_n \cdot F_{n-1}$$

The previous two identities will be used in the chapter Main Results. Among the most notable generalizations of the Fibonacci sequence (see, [4], [7], [11]) are the Tribonacci sequence, the Lucas numbers, and the Quadrapell sequence. The Tribonacci sequence, denote T_n is defined by recurrence relation:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3.}$$

DEFINITION 2.1. [10] The Quadrapell numbers D_n are defined by the recurrence relation

$$D_n = D_{n-2} + 2D_{n-3} + D_{n-4}, \quad n \geqslant 4$$

where $D_0 = D_1 = D_2 = 1$, $D_3 = 2$.

The first few Quadrapell numbers D_n are: 1, 1, 1, 2, 4, 5, 9, 15, 23, 38, 62, 99, 161, 261, 421,...

The characteristic equation of the Quadrapell recurrence relation is given by:

$$x^4 - x^2 - 2x - 1 = 0.$$

The real roots of this equation are given by: $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ and the imaginary roots are given by: $\gamma = \dfrac{-1+\sqrt{3}i}{2}$ and $\delta = \dfrac{-1-\sqrt{3}i}{2}$.

We can observe that α is equal to the Golden ratio. Rations of consecutive Fibonacci numbers and Lucas numbers converge to the Golden ratio. In paper [10] author posed the conjecture that the rations of consecutive Quadrapell number converges to the limit α, where α is Golden ratio, i.e the following equality is valid:

$$\lim_{n\to\infty} \frac{D_n}{D_{n-1}} = \frac{1+\sqrt{5}}{2}$$

The following lemma give a affirmative answer for this conjecture.

LEMMA 2.1. *Let D_n be a recurrence sequences of Quadrapell numbers then:*

$$\lim_{n\to\infty} \frac{D_n}{D_{n-1}} = \frac{1+\sqrt{5}}{2}.$$

Proof. Let α,β,γ and δ are roots of characteristic equation of the Quadrapell recurrence relation. The following equality is the result from paper [10]:

$$D_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - \frac{1}{2}\cdot\frac{\alpha^n-\beta^n}{\alpha-\beta} + \frac{1}{2}\cdot\frac{\gamma^n-\delta^n}{\gamma-\delta}.$$

In trigonometry format of complex numbers: $\gamma = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and $\delta = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$. According to Moivres formula: $\gamma^n = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$ and $\delta^n = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$.

From the previous equalities follows:

$$D_n = \frac{\alpha^{n+1}-\beta^{n+1}-\frac{1}{2}(\alpha^n-\beta^n)}{\sqrt{5}} + \frac{1}{2}\cdot\frac{\sin\frac{2n\pi}{3}}{\sin\frac{2\pi}{3}}.$$

Now, we calculate the limit value:

$$\lim_{n\to\infty}\frac{D_n}{D_{n-1}} = \lim_{n\to\infty}\frac{\dfrac{\alpha^{n+1}-\beta^{n+1}-\frac{1}{2}(\alpha^n-\beta^n)}{\sqrt{5}} + \dfrac{\sin\frac{2n\pi}{3}}{\sqrt{3}}}{\dfrac{\alpha^n-\beta^n-\frac{1}{2}(\alpha^{n-1}-\beta^{n-1})}{\sqrt{5}} + \dfrac{\sin\frac{2(n-1)\pi}{3}}{\sqrt{3}}}$$

$$= \lim_{n\to\infty}\frac{\dfrac{\alpha-\beta\left(\frac{\beta}{\alpha}\right)^n - \frac{1}{2} + \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^n}{\sqrt{5}} + \dfrac{\sin\frac{2n\pi}{3}}{\sqrt{3}\alpha^n}}{\dfrac{1-\left(\frac{\beta}{\alpha}\right)^n - \frac{1}{2\alpha} + \frac{1}{2\alpha}\left(\frac{\beta}{\alpha}\right)^{n-1}}{\sqrt{5}} + \dfrac{\sin\frac{2(n-1)\pi}{3}}{\sqrt{3}\alpha^n}}$$

$$= \frac{\frac{1}{\sqrt{5}}\left(\alpha-\frac{1}{2}\right)}{\frac{1}{\sqrt{5}}\left(1-\frac{1}{2\alpha}\right)} = \frac{\frac{\sqrt{5}}{2}}{\frac{\sqrt{5}}{1+\sqrt{5}}} = \frac{1+\sqrt{5}}{2}. \quad \square$$

Consequently, the final result coincides with the value of the golden ratio, thereby confirming the validity of the conjecture. The following lemma is the result from [10].

LEMMA 2.2. *Let D_n be a sequence of quadruples of numbers. Then, the following equality hold:*

$$\sum_{i=1}^{n} D_i = \frac{1}{3}(D_n + D_{n+2} + D_{n+3} - 4).$$

The following lemma gives a very interesting properties of Quadrapell number.

LEMMA 2.3. *Let D_n be a sequence of quadruples of numbers. Then, the following equalities hold:*

1. $D_{3k} = D_{3k-1} + D_{3k-2}$

2. $D_{3k+1} = D_{3k} + D_{3k-1} + 1$

3. $D_{3k+2} = D_{3k} + D_{3k+1} - 1$.

Proof. Let us prove the first equality using mathematical induction.
Base case: For $k = 1$, the statement clearly holds: $D_3 = D_1 + D_2 = 1 + 1 = 2$.
Induction hypothesis: Assume that the statement holds for some arbitrary $k = n$.
Inductive step: We now prove that the statement holds for $k = n+1$ (i.e $D_{3k+3} = D_{3k+2} + D_{3k+1}$), assuming it holds for $k = n$.

We begin with the definition of Quadrapell numbers, followed by the use of the induction hypothesis.

$$\begin{aligned} D_{3k+3} &= D_{3k+1} + 2D_{3k} + D_{3k-1} = D_{3k+1} + D_{3k} + D_{3k} + D_{3k-1} \\ &= D_{3k+1} + D_{3k} + D_{3k-1} + D_{3k-2} + D_{3k-1} \\ &= D_{3k+1} + D_{3k} + 2D_{3k-1} + D_{3k-2} = D_{3k+1} + D_{3k+2}. \end{aligned}$$

This completes the proof of the first equality. The second and third equalities can be proved analogous. □

COROLLARY 1. *The difference between the sequences D_n and F_n $(n \geqslant 3)$ results in a new sequence of numbers that can be interpreted as a variant of the Quadrapell numbers* $(0,\ 1,\ 0,\ 1,\ 2,\ 2,\ 4,\ 7,\ 10,\ 17\ldots)$ *with different initial conditions.*

The proof is straightforward and can be obtained using mathematical induction.

COROLLARY 2. *Based on the previous lemma, it can be observed that the following inequality holds for $n \geqslant 1$:*

$$F_n \leqslant D_n \leqslant F_{n+1}.$$

3. Main results

In this chapter, we derive upper and lower bounds for the spectral norm of geometric r-circulant matrices with Quadruple numbers.

Let us consider the sequence of numbers D_n with real roots of characteristic equation $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

THEOREM 3.1. *Let us consider the given geometric r-circulant matrix D_{r^*} with Quadrapell numbers, and spectral norm $\| D_{r^*} \|_2$, then holds:*

1. If $|r|<1$ then:

$$\frac{|r|}{\sqrt{5}}\sqrt{\left(\frac{\alpha^{2n}-r^{2n}}{\alpha^2-r^2}+\frac{\beta^{2n}-r^{2n}}{\beta^2-r^2}+2\frac{(-1)^n-r^{2n}}{1+r^2}\right)}$$
$$\leqslant \| D_{r^*} \|_2 \leqslant \frac{1}{3}(D_{n-1}+D_{n+1}+D_{n+2}-4).$$

2. If $|r|>1$ then: $\sqrt{F_n\cdot F_{n-1}} \leqslant \| D_{r^*} \|_2 \leqslant \sqrt{\frac{r^{2n}-1}{r^2-1}}\sqrt{F_n\cdot F_{n+1}}.$

Proof.

1. $|r|<1$:

Define the matrices A and B as follows

$$A=\begin{bmatrix} \sqrt{D_0} & \sqrt{D_1} & \cdots & \sqrt{D_{n-1}} \\ \sqrt{D_{n-1}} & \sqrt{D_0} & \cdots & \sqrt{D_{n-2}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{D_1} & \sqrt{D_2} & \cdots & \sqrt{D_0} \end{bmatrix}$$

$$B=\begin{bmatrix} \sqrt{D_0} & \sqrt{D_1} & \cdots & \sqrt{D_{n-1}} \\ r\cdot\sqrt{D_{n-1}} & \sqrt{D_0} & \cdots & \sqrt{D_{n-2}} \\ \vdots & \vdots & \ddots & \vdots \\ r^{n-1}\sqrt{D_1} & r^{n-2}\cdot\sqrt{D_2} & \cdots & \sqrt{D_0} \end{bmatrix}$$

$$\| D_{r^*} \|_2 = \| A\circ B \|_2 \leqslant r(A)\cdot c(B) = \sqrt{\sum_{k=0}^{n-1} D_k}\cdot\sqrt{\sum_{k=0}^{n-1} D_k}$$
$$=\sum_{k=0}^{n-1} D_k = \frac{1}{3}(D_{n-1}+D_{n+1}+D_{n+2}-4).$$

Lower bound:

$$\| D_{r^*} \|_2^2 \geqslant \frac{1}{n}\| D_{r^*} \|_E^2 = \frac{1}{n}\sum_{k=0}^{n-1}(n-k)D_k^2+\frac{1}{n}\sum_{k=0}^{n-1}k|r^{n-k}|^2D_k^2$$
$$\geqslant n\cdot\frac{1}{n}\sum_{k=0}^{n-1}|r^{n-k}|^2D_k^2 = r^{2n}\sum_{k=0}^{n-1}\frac{D_k^2}{r^{2k}}$$
$$\geqslant r^{2n}\sum_{k=0}^{n-1}\frac{F_k^2}{r^{2k}} = \frac{r^2}{5}\left(\frac{\alpha^{2n}-r^{2n}}{\alpha^2-r^2}+\frac{\beta^{2n}-r^{2n}}{\beta^2-r^2}+2\frac{(-1)^n-r^{2n}}{1+r^2}\right)$$
$$\| D_{r^*} \|_2 \geqslant \frac{|r|}{\sqrt{5}}\sqrt{\left(\frac{\alpha^{2n}-r^{2n}}{\alpha^2-r^2}+\frac{\beta^{2n}-r^{2n}}{\beta^2-r^2}+2\frac{(-1)^n-r^{2n}}{1+r^2}\right)}.$$

2. $|r| > 1$:

   Upper bound: Define the matrices A and B as follows:

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r & 1 & \cdots & 1 \\ r^2 & r & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & \cdots & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} D_0 & D_1 & \cdots & D_{n-1} \\ D_{n-1} & D_0 & \cdots & D_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ D_1 & D_2 & \cdots & D_0 \end{bmatrix}$$

$$\begin{aligned} \| D_{r^*} \|_2 = \| A \circ B \|_2 \leqslant r(A) \cdot c(B) &= \sqrt{\frac{r^{2n}-1}{r^2-1}} \cdot \sqrt{\sum_{k=0}^{n-1} D_k^2} \\ &\leqslant \sqrt{\frac{r^{2n}-1}{r^2-1}} \cdot \sqrt{\sum_{k=0}^{n-1} (F_{k+1})^2} \\ \| D_{r^*} \|_2 &\leqslant \sqrt{\frac{r^{2n}-1}{r^2-1}} \sqrt{F_n \cdot F_{n+1}}. \end{aligned}$$

   Lower bound:

$$\begin{aligned} \| D_{r^*} \|_2^2 \geqslant \frac{1}{n} \| D_{r^*} \|_E^2 &= \frac{1}{n} \left(\sum_{k=0}^{n-1} (n-k) D_k^2 + \sum_{k=0}^{n-1} k |r^{n-k}|^2 D_k^2 \right) \\ &\geqslant \frac{1}{n} \left(\sum_{k=0}^{n-1} (n-k) D_k^2 + \sum_{k=0}^{n-1} k D_k^2 \right) = n \cdot \frac{1}{n} \sum_{k=0}^{n-1} D_k^2 \\ &\geqslant \sum_{k=0}^{n-1} F_k^2 = F_n \cdot F_{n-1} \\ \| D_{r^*} \|_2 &\geqslant \sqrt{F_n \cdot F_{n-1}}. \quad \square \end{aligned}$$

The following numerical example illustrates the result of the preceding theorem.

EXAMPLE 1. Let D_{r^*} geometric r-circulant matrix with Quadrapell numbers:

1. $r = 1,05 \qquad (r > 1)$
   $n = 5$ it is obtained that: $3,87 \leqslant 9,39 \leqslant 15$
   $n = 10$ it is obtained that: $49 \leqslant 107,34 \leqslant 234$

2. $r = 0,9 \qquad (r < 1)$
   $n = 5$ it is obtained that: $3,3 \leqslant 7,49 \leqslant 8$
   $n = 10$ it is obtained that: $33,05 \leqslant 86,25 \leqslant 98$

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(Received June 30, 2025)

Aleksandra Erić
University of Belgrade
Faculty of Civil Engineering
Bulevar Kralja Aleksandra 73, Belgrade, Serbia
e-mail: eric@grf.bg.ac.rs

Ivan Lazarević
University of Belgrade
Faculty of Civil Engineering
Bulevar Kralja Aleksandra 73, Belgrade, Serbia
e-mail: ilazarevic@grf.bg.ac.rs