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A NEW p–NUMERICAL RADIUS

MOHAMMAD EL-HADDAD, OMAR HIRZALLAH AND FUAD KITTANEH

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Abstract. The p-numerical radius of a Hilbert space operator A is defined as

$$w_p(A) = \sup_{\theta \in \mathbb{R}} \left\| \operatorname{Re}\left(e^{i\theta}A\right) \right\|_p$$

for $0 < p \leqslant \infty$, where $\operatorname{Re}(\cdot)$ is the real part and $\|\cdot\|_p$ is the Schatten p-norm (quasi-norm). In this paper, a more natural p-numerical radius is defined as

$$\mathfrak{n}_p(A) = \sup_{\{\phi_j\} \text{ o.n.s.}} \left(\sum_j \left| \langle A\phi_j, \phi_j \rangle \right|^p \right)^{1/p}$$

for $0 < p \leqslant \infty$. Properties and inequalities related to this new p-numerical radius are given.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ be the space of all bounded linear operators acting on a separable complex Hilbert space $\mathbb{H}$ occupied with an inner product $\langle \cdot, \cdot \rangle$. The absolute value of an operator $A \in \mathbb{B}(\mathbb{H})$, denoted by $|A|$, is the positive square root of the positive operator A^*A, that is, $|A| = (A^*A)^{1/2}$. For an operator $A \in \mathbb{B}(\mathbb{H})$, let $\operatorname{Re}(A) = \frac{A+A^*}{2}$ and $\operatorname{Im}(A) = \frac{A-A^*}{2i}$ denote the real part and the imaginary part of A, respectively.

The usual operator norm, denoted by $\|\cdot\|$, is the norm on $\mathbb{B}(\mathbb{H})$ defined as

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{H}, \ \|x\| = 1\}.$$

Equivalently,

$$\|A\| = \sup\{|\langle Ax, y\rangle| : x, y \in \mathbb{H}, \|x\| = \|y\| = 1\}. \tag{1.1}$$

Moreover, if A is normal, then the usual operator norm of A becomes

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \ \|x\| = 1\}. \tag{1.2}$$

The numerical radius of an operator $A \in \mathbb{B}(\mathbb{H})$ is denoted by $w(A)$ and is defined as

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \ \|x\| = 1\}. \tag{1.3}$$

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It is known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, if $A \in \mathbb{B}(\mathbb{H})$, then

$$\frac{\|A\|}{2} \leqslant w(A) \leqslant \|A\|. \tag{1.4}$$

Moreover, if $A^2 = 0$, then $w(A) = \frac{\|A\|}{2}$. Also, if A is normal, then $w(A) = \|A\|$.

One of the distinguishable characterizations of the numerical radius is the one introduced by Yamazaki [16], which asserts that if $A \in \mathbb{B}(\mathbb{H})$, then

$$w(A) = \sup_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\| = \sup_{\theta\in\mathbb{R}} \left\|\operatorname{Im}\left(e^{i\theta}A\right)\right\|. \tag{1.5}$$

Also, the numerical radius satisfies the weak unitary invariance property, that is, if $A \in \mathbb{B}(\mathbb{H})$, then

$$w(U^*AU) = w(A) \tag{1.6}$$

for every unitary operator U in $\mathbb{B}(\mathbb{H})$. Another useful property of the numerical radius is the self-adjointness property, that is,

$$w(A^*) = w(A)$$

for every $A \in \mathbb{B}(\mathbb{H})$. This property can be easily derived from the relations (1.5).

For a compact operator $A \in \mathbb{B}(\mathbb{H})$, the Schatten p-norm, denoted by $\|\cdot\|_p$, is defined on $\mathbb{B}(\mathbb{H})$ by

$$\|A\|_p = \sup_{\{\phi_j\},\{\psi_j\}\text{ o.n.s.}} \left(\sum_j \left|\left\langle A\phi_j, \psi_j\right\rangle\right|^p\right)^{1/p} \tag{1.7}$$

for $1 \leqslant p < \infty$, where the the supremum is taken over all orthonormal sets (o.n.s.) $\{\phi_j\}, \{\psi_j\}$ in $\mathbb{H}$. In particular, when $A \in \mathbb{B}(\mathbb{H})$ is normal, we have

$$\|A\|_p = \sup_{\{\phi_j\}\text{ o.n.s.}} \left(\sum_j \left|\left\langle A\phi_j, \phi_j\right\rangle\right|^p\right)^{1/p}. \tag{1.8}$$

Note that for $0 < p < 1$, $\|\cdot\|_p$ is a quasi-norm (it does not satisfy the triangle inequality). Moreover, an important fact for the Schatten p-norms, when $2 \leqslant p < \infty$, is that

$$\|A\|_p = \sup_{\{\phi_j\}\text{ o.n.b.}} \left(\sum_j \left\|A\phi_j\right\|^p\right)^{1/p}, \tag{1.9}$$

where the the supremum is taken over all orthonormal bases (o.n.b.) $\{\phi_j\}$ for $\mathbb{H}$. Also, one of the useful properties of the Schatten p-norms (quasi-norms) is that

$$\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_p = \left\|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right\|_p = \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} \tag{1.10}$$

for $A,B\in\mathbb{B}(\mathbb{H})$ and for $0<p<\infty$. We say that an operator $A\in\mathbb{B}(\mathbb{H})$ belongs to the Schatten p-class, which is denoted by $\mathbb{B}_p(\mathbb{H})$, if $\|A\|_p<\infty$. Indeed, $\lim_{p\to\infty}\|A\|_p=\|A\|$. For comprehensive accounts of the Schatten p-norms, we refer the reader to [8], [13], and [17].

The relations (1.5), inspired Abu-Omar and Kittaneh [1] to define generalized numerical radii of operators. Among these generalized numerical radii is the p-numerical radius (see, e.g., [2], [3], [4], [7], [10], [11], and [12]), which is defined as follows: For $0<p<\infty$, define the p-numerical radius $w_p(\cdot)$ on $\mathbb{B}_p(\mathbb{H})$ as

$$w_p(A)=\sup_{\theta\in\mathbb{R}}\left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\|_p \text{ for } A\in\mathbb{B}_p(\mathbb{H}). \tag{1.11}$$

Equivalently, we have

$$w_p(A)=\sup_{\theta\in\mathbb{R}}\left\|\operatorname{Im}\left(e^{i\theta}A\right)\right\|_p \text{ for } A\in\mathbb{B}_p(\mathbb{H}). \tag{1.12}$$

Some of the nice properties of the p-numerical radius are the following (see, e.g., [5] and [9]): For $A,B\in\mathbb{B}_p(\mathbb{H})$ and $0<p<\infty$, we have

$$c_p\|A\|_p\leqslant w_p(A)\leqslant\|A\|_p, \tag{1.13}$$

where

$$c_p=\begin{cases}2^{-1/p}, & 0<p\leqslant 2\\ 2^{-1+1/p}, & 2\leqslant p\leqslant\infty.\end{cases} \tag{1.14}$$

In particular, we have

$$\frac{\|A\|_p}{2}\leqslant w_p(A)\leqslant\|A\|_p. \tag{1.15}$$

Moreover, we have

$$w_p\left(\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}\right)\leqslant\left(w_p^p(A)+w_p^p(B)\right)^{1/p} \tag{1.16}$$

and

$$w_p\left(\begin{bmatrix}0 & A\\ A & 0\end{bmatrix}\right)=2^{1/p}w_p(A). \tag{1.17}$$

It can be seen (see, e.g., [14]) that

$$w_p\left(\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right)=2^{-1+1/p}\sup_{\theta\in\mathbb{R}}\left\|A+e^{i\theta}B^*\right\|_p \tag{1.18}$$

for $0<p\leqslant\infty$.

In this paper, we focus our attention on a new generalized numerical radius. In Section 2, inspired by the relations (1.1), (1.3), and (1.7), we introduce a more natural p-numerical radius and we study some properties of this new notion. In Section 3, we focus our attention on presenting some properties of our new p-numerical radii of the diagonal parts and the off-diagonal parts of 2×2 operator matrices. In Section 4, we introduce lower and upper bounds for the new p-numerical radii of the off-diagonal parts of 2×2 operator matrices.

2. The new p-numerical radius

In accordance with the relations (1.1), (1.3), and based on the relation (1.7), one might introduce the definition of the p-numerical radius of an operator as follows.

DEFINITION 2.1. The p-numerical radius of an operator, denoted by $\mathfrak{n}_p(\cdot)$, is defined on $\mathbb{B}_p(\mathbb{H})$ by

$$\mathfrak{n}_p(A) = \sup_{\{\phi_j\} \text{ o.n.s.}} \left(\sum_j \left| \langle A\phi_j, \phi_j \rangle \right|^p \right)^{1/p}$$

for $0 < p < \infty$.

REMARK 2.2. Each of the following statements can be easily verified.

(a) $\lim_{p \to \infty} \mathfrak{n}_p(A) = w(A)$ for every $A \in \mathbb{B}_p(\mathbb{H})$.

(b) For $0 < p \leqslant \infty$, the relations (1.3), (1.7), and Definition 2.1 imply that

$$w(A) \leqslant \mathfrak{n}_p(A) \leqslant \|A\|_p \tag{2.1}$$

for every $A \in \mathbb{B}_p(\mathbb{H})$.

(c) For $0 < p \leqslant \infty$, the new p-numerical radius $\mathfrak{n}_p(\cdot)$ is weakly unitarily invariant, that is,

$$\mathfrak{n}_p(U^*AU) = \mathfrak{n}_p(A) \tag{2.2}$$

for every $A \in \mathbb{B}_p(\mathbb{H})$ and for every unitary $U \in \mathbb{B}(\mathbb{H})$.

(d) For $1 \leqslant p \leqslant \infty$, the new p-numerical radius $\mathfrak{n}_p(\cdot)$ is a norm on $\mathbb{B}_p(\mathbb{H})$.

Based on Definition 2.1, a result that relates $\mathfrak{n}_p(\cdot)$ and $w_p(\cdot)$ can be seen in the following theorem.

THEOREM 2.3. *Let $A \in \mathbb{B}_p(\mathbb{H})$. Then*

$$w_p(A) \leqslant \mathfrak{n}_p(A) \leqslant \frac{w_p(A)}{c_p} \tag{2.3}$$

for $0 < p \leqslant \infty$, where c_p is defined in the relation (1.14).

Proof. For the first inequality of (2.3), let $\theta \in \mathbb{R}$. Since $\operatorname{Re}\left(e^{i\theta}A\right)$ is self-adjoint, we have

$$\begin{aligned}
\left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\|_p &= \sup_{\{\phi_j\} \text{ o.n.s.}} \left(\sum_j \left|\left\langle \operatorname{Re}\left(e^{i\theta}A\right)\phi_j, \phi_j\right\rangle\right|^p\right)^{1/p} \quad \text{(by the relation (1.8))} \\
&= \sup_{\{\phi_j\} \text{ o.n.s.}} \left(\sum_j \left|\operatorname{Re}\left\langle e^{i\theta}A\phi_j, \phi_j\right\rangle\right|^p\right)^{1/p} \\
&\leqslant \sup_{\{\phi_j\} \text{ o.n.s.}} \left(\sum_j \left|\left\langle e^{i\theta}A\phi_j, \phi_j\right\rangle\right|^p\right)^{1/p} \\
&= \sup_{\{\phi_j\} \text{ o.n.s.}} \left(\sum_j \left|\left\langle A\phi_j, \phi_j\right\rangle\right|^p\right)^{1/p} \\
&= \mathfrak{n}_p(A) \text{ (by Definition 2.1).}
\end{aligned} \tag{2.4}$$

By taking the supremum in both sides of the inequality (2.4) over all real numbers θ in $\mathbb{R}$, we have

$$\begin{aligned}
w_p(A) &= \sup_{\theta \in \mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\|_p \\
&\leqslant \mathfrak{n}_p(A) \text{ (by the inequality (2.4)).}
\end{aligned} \tag{2.5}$$

For the second inequality of (2.3), we have

$$\begin{aligned}
\mathfrak{n}_p(A) &\leqslant \|A\|_p \text{ (by the second inequality of (2.1))} \\
&\leqslant \frac{w_p(A)}{c_p} \text{ (by the first inequality of (1.13)),}
\end{aligned}$$

as required. □

In the following example, we compute the two p-numerical radii $w_p(\cdot)$ and $\mathfrak{n}_p(\cdot)$ for certain matrices. This example shows that the inequalities (2.3) are sharp.

EXAMPLE 2.4. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, and let $0 < p \leqslant \infty$. Then

$$w_p(A) = 2^{-1+1/p} \text{ (by the relation (1.18)).}$$

Also, by direct computations, we have

$$\begin{aligned}
\mathfrak{n}_p(A) &= \sup\left\{\left(\sum_j \left|\left\langle A\phi_j, \phi_j\right\rangle\right|^p\right)^{1/p} : \{\phi_j\} \text{ is orthonormal set in } \mathbb{C}^2\right\} \\
&= 2^{-1+1/p}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned} w_p(B) &= \sup_{\theta\in\mathbb{R}} \left\| \mathrm{Re}\left(e^{i\theta}A\right)\right\|_p \\ &= \sup_{\theta\in\mathbb{R}} \left(|\cos(\theta)|^p + |\sin(\theta)|^p\right)^{1/p}. \end{aligned}$$

In particular, letting $p=2$, we have

$$w_2(B) = 1,$$

while the normality of B implies that

$$\mathfrak{n}_2(B) = \|B\|_2 = \sqrt{2}.$$

Consequently, $\mathfrak{n}_p(A) = w_p(A)$ and $\mathfrak{n}_2(B) = \frac{w_2(B)}{c_2}$. In this example, we see that the inequalities (2.3) are sharp. Also, in this example, we see that our new p-numerical radius $\mathfrak{n}_p(\cdot)$ and the old p-numerical radius $w_p(\cdot)$ are different concepts.

We need the following lemma for scalars. It follows from the convexity of the function $f(t) = t^p$ on $[0,\infty)$ for $1 \leqslant p < \infty$.

LEMMA 2.5. *Let a and b be positive real numbers. Then*

$$(a+b)^p \leqslant 2^{p-1}(a^p + b^p)$$

for $1 \leqslant p < \infty$, and

$$(a+b)^p \geqslant 2^{p-1}(a^p + b^p)$$

for $0 < p \leqslant 1$.

In the following theorem, we give some upper and lower bounds for our new p-numerical radii of operators.

THEOREM 2.6. *Let $A \in \mathbb{B}_p(\mathbb{H})$. Then*

$$\mathfrak{n}_p(A) \leqslant 2^{1/2-1/p}\left(\|\mathrm{Re}(A)\|_p^p + \|\mathrm{Im}(A)\|_p^p\right)^{1/p} \tag{2.6}$$

for $2 \leqslant p \leqslant \infty$, and

$$\mathfrak{n}_p(A) \geqslant 2^{1/2-1/p}\|\mathrm{Re}(A) + \mathrm{Im}(A)\|_p \tag{2.7}$$

for $0 < p \leqslant 1$.

Proof. Let $\{\phi_j\}$ be an orthonormal set in $\mathbb{H}$, and let $2 \leqslant p \leqslant \infty$. Then

$$\begin{aligned}
\mathfrak{n}_p^p(A) &= \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left(\left|\left\langle A\phi_j, \phi_j \right\rangle\right|^p\right) \\
&= \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left(\left|\left\langle \left(\operatorname{Re}(A) + i\operatorname{Im}(A)\right)\phi_j, \phi_j \right\rangle\right|^p\right) \\
&= \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left(\left|\operatorname{Re}\left(\left\langle A\phi_j, \phi_j \right\rangle\right) + i\operatorname{Im}\left(\left\langle A\phi_j, \phi_j \right\rangle\right)\right|^p\right) \\
&= \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left(\left(\left(\operatorname{Re}\left\langle A\phi_j, \phi_j \right\rangle\right)^2 + \left(\operatorname{Im}\left\langle A\phi_j, \phi_j \right\rangle\right)^2\right)^{p/2}\right) \qquad (2.8) \\
&\leqslant 2^{p/2-1} \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left(\left|\operatorname{Re}\left\langle A\phi_j, \phi_j \right\rangle\right|^p + \left|\operatorname{Im}\left\langle A\phi_j, \phi_j \right\rangle\right|^p\right) \\
&\qquad\qquad \text{(by Lemma 2.5)} \\
&= 2^{p/2-1} \sup_{\{\phi_j\} \text{ o.n.s.}} \left(\sum_j \left|\left\langle \operatorname{Re}(A)\phi_j, \phi_j \right\rangle\right|^p + \sum_j \left|\left\langle \operatorname{Im}(A)\phi_j, \phi_j \right\rangle\right|^p\right) \\
&\leqslant 2^{p/2-1} \left(\|\operatorname{Re}(A)\|_p^p + \|\operatorname{Im}(A)\|_p^p\right).
\end{aligned}$$

On the other hand, let $0 < p \leqslant 1$. Then

$$\begin{aligned}
\mathfrak{n}_p^p(A) &= \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left(\left(\left(\operatorname{Re}\left\langle A\phi_j, \phi_j \right\rangle\right)^2 + \left(\operatorname{Im}\left\langle A\phi_j, \phi_j \right\rangle\right)^2\right)^{p/2}\right) \\
&\qquad\qquad \text{(by the relation (2.8))} \\
&\geqslant 2^{p/2-1} \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left(\left|\operatorname{Re}\left\langle A\phi_j, \phi_j \right\rangle\right|^p + \left|\operatorname{Im}\left\langle A\phi_j, \phi_j \right\rangle\right|^p\right) \\
&\qquad\qquad \text{(by Lemma 2.5)} \\
&\geqslant 2^{p/2-1} \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left(\left|\left\langle \operatorname{Re}(A)\phi_j, \phi_j \right\rangle\right| + \left|\left\langle \operatorname{Im}(A)\phi_j, \phi_j \right\rangle\right|\right)^p \\
&\geqslant 2^{p/2-1} \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left|\left\langle \operatorname{Re}(A)\phi_j, \phi_j \right\rangle + \left\langle \operatorname{Im}(A)\phi_j, \phi_j \right\rangle\right|^p \\
&= 2^{p/2-1} \sup_{\{\phi_j\} \text{ o.n.s.}} \sum_j \left|\left\langle \left(\operatorname{Re}(A) + \operatorname{Im}(A)\right)\phi_j, \phi_j \right\rangle\right|^p \\
&= 2^{p/2-1} \left(\|\operatorname{Re}(A) + \operatorname{Im}(A)\|_p^p\right),
\end{aligned}$$

as required. □

3. Properties of the new p-numerical radii of the diagonal parts and the off-diagonal parts of 2×2 operator matrices

In this section, we are interested in studying some properties of our new p-numerical radii of the diagonal parts and the off-diagonal parts of 2×2 operator matrices. First, we start with the following theorem that involves some of the these properties.

THEOREM 3.1. *Let* $A,B \in \mathbb{B}_p(\mathbb{H})$, *and let* $0 < p \leqslant \infty$. *Then*

(a) $\mathfrak{n}_p\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \mathfrak{n}_p\left(\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}\right)$.

(b) $\mathfrak{n}_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \mathfrak{n}_p\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right)$.

(c) $\mathfrak{n}_p\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \mathfrak{n}_p\left(\begin{bmatrix} e^{i\alpha}A & 0 \\ 0 & e^{i\beta}B \end{bmatrix}\right)$ *for every* $\alpha,\beta \in \mathbb{R}$.

(d) $\mathfrak{n}_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \mathfrak{n}_p\left(\begin{bmatrix} 0 & e^{i\alpha}A \\ e^{i\beta}B & 0 \end{bmatrix}\right)$ *for every* $\alpha,\beta \in \mathbb{R}$.

Proof.
Part (a)*:* We have

$$\begin{aligned}\mathfrak{n}_p\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \mathfrak{n}_p\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\right) \\ &\qquad \text{(by the identity (2.2))} \\ &= \mathfrak{n}_p\left(\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}\right),\end{aligned}$$

where I is the identity operator in $\mathbb{B}(\mathbb{H})$.
Part (b)*:* We have

$$\begin{aligned}\mathfrak{n}_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \mathfrak{n}_p\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\right) \text{ (by the identity (2.2))} \\ &= \mathfrak{n}_p\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right).\end{aligned}$$

Part (c)*:* We have

$$\begin{aligned}\mathfrak{n}_p\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \mathfrak{n}_p\left(\begin{bmatrix} 0 & e^{i\alpha/2}I \\ e^{i\beta/2}I & 0 \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & e^{i\alpha/2}I \\ e^{i\beta/2}I & 0 \end{bmatrix}\right) \\ &\qquad \text{(by the identity (2.2))} \\ &= \mathfrak{n}_p\left(\begin{bmatrix} e^{i(\beta-\alpha)/2}B & 0 \\ 0 & e^{i(\alpha-\beta)/2}A \end{bmatrix}\right) \\ &= \mathfrak{n}_p\left(e^{i(\alpha+\beta)/2}\begin{bmatrix} e^{i(\beta-\alpha)/2}B & 0 \\ 0 & e^{i(\alpha-\beta)/2}A \end{bmatrix}\right) \\ &= \mathfrak{n}_p\left(\begin{bmatrix} e^{i\beta}B & 0 \\ 0 & e^{i\alpha}A \end{bmatrix}\right) \\ &= \mathfrak{n}_p\left(\begin{bmatrix} e^{i\alpha}A & 0 \\ 0 & e^{i\beta}B \end{bmatrix}\right) \text{ (by Part } (a)\text{)}.\end{aligned}$$

Part (d)*:* We have

$$\begin{aligned}
\mathfrak{n}_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \mathfrak{n}_p\left(\begin{bmatrix} e^{-i\alpha/2}I & 0 \\ 0 & e^{-i\beta/2}I \end{bmatrix}^* \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} e^{-i\alpha/2}I & 0 \\ 0 & e^{-i\beta/2}I \end{bmatrix}\right) \\
&\qquad \text{(by the identity (2.2))} \\
&= \mathfrak{n}_p\left(\begin{bmatrix} 0 & e^{i(\alpha-\beta)/2}A \\ e^{i(\beta-\alpha)/2}B & 0 \end{bmatrix}\right) \\
&= \mathfrak{n}_p\left(e^{i(\alpha+\beta)/2}\begin{bmatrix} 0 & e^{i(\alpha-\beta)/2}A \\ e^{i(\beta-\alpha)/2}B & 0 \end{bmatrix}\right) \\
&= \mathfrak{n}_p\left(\begin{bmatrix} 0 & e^{i\alpha}A \\ e^{i\beta}B & 0 \end{bmatrix}\right). \quad \square
\end{aligned}$$

Other properties of the new p-numerical radii of the diagonal parts of 2×2 operator matrices can be seen in the following theorem.

THEOREM 3.2. *Let* $A,B \in \mathbb{B}_p(\mathbb{H})$*, and let* $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$*. Then*

$$\mathfrak{n}_p(T) \geqslant \max\left(\mathfrak{n}_p(A), \mathfrak{n}_p(B), \frac{1}{2}\mathfrak{n}_p(A+B)\right) \tag{3.1}$$

and

$$\mathfrak{n}_p(T) \leqslant \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} \tag{3.2}$$

for $0 < p \leqslant \infty$.

Proof. Let $\{\phi_j\}$ be an orthonormal set in $\mathbb{H}$. Then the set $\{\eta_j\}$, where $\eta_j = \begin{bmatrix} \phi_j \\ 0 \end{bmatrix}$, is orthonormal in $\mathbb{H}\oplus\mathbb{H}$. It follows that

$$\begin{aligned}
\left(\sum_j |\langle A\phi_j, \phi_j\rangle|^p\right)^{1/p} &= \left(\sum_j |\langle T\eta_j, \eta_j\rangle|^p\right)^{1/p} \\
&\leqslant \sup_{\{\xi_j\}\text{ o.n.s.}} \left(\sum_j |\langle T\xi_j, \xi_j\rangle|^p\right)^{1/p} \\
&= \mathfrak{n}_p(T).
\end{aligned} \tag{3.3}$$

By taking the supremum in both sides of the inequality (3.3) over all orthonormal sets $\{\phi_j\}$ in $\mathbb{H}$, we have

$$\mathfrak{n}_p(A) \leqslant \mathfrak{n}_p(T). \tag{3.4}$$

Similarly, we have

$$\mathfrak{n}_p(B) \leqslant \mathfrak{n}_p(T). \tag{3.5}$$

Moreover, let $\varsigma_j = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_j\\ \phi_j\end{bmatrix}$. Then $\{\varsigma_j\}$ is an orthonormal set in $\mathbb{H}\oplus\mathbb{H}$. It follows that

$$\begin{aligned}\frac{1}{2}\left(\sum_j\left|\left\langle (A+B)\phi_j,\phi_j\right\rangle\right|^p\right)^{1/p} &= \left(\sum_j\left|\left\langle T\varsigma_j,\varsigma_j\right\rangle\right|^p\right)^{1/p}\\ &\leqslant \sup_{\{\xi_j\}\text{ o.n.s.}}\left(\sum_j\left|\left\langle T\xi_j,\xi_j\right\rangle\right|^p\right)^{1/p}\\ &= \mathfrak{n}_p(T). \end{aligned} \tag{3.6}$$

By taking the supremum in both sides of the inequality (3.6) over all orthonormal sets $\{\phi_j\}$ in $\mathbb{H}$, we have

$$\frac{1}{2}\mathfrak{n}_p(A+B) \leqslant \mathfrak{n}_p(T). \tag{3.7}$$

Now, the inequality (3.1) follows from the inequalities (3.4), (3.5), and (3.7).

The inequality (3.2) follows from the second inequality of (2.1) and the fact that $\|T\|_p = \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p}$. □

A direct application of Theorem 3.2 can be seen in the following corollary.

COROLLARY 3.3. *Let $A\in\mathbb{B}_p(\mathbb{H})$, and let $S=\begin{bmatrix}A & 0\\ 0 & A\end{bmatrix}$ and $T=\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}$. Then*

$$\mathfrak{n}_p(A)\leqslant\mathfrak{n}_p(S)\leqslant 2^{1/p}\|A\|_p$$

and

$$\mathfrak{n}_p(A)\leqslant\mathfrak{n}_p(T)\leqslant\|A\|_p$$

for $0<p\leqslant\infty$.

In the following theorem, we give properties of the new p-numerical radii of the off-diagonal parts of 2×2 operator matrices.

THEOREM 3.4. *Let $A\in\mathbb{B}_p(\mathbb{H})$, and let $T=\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}$. Then*

$$\frac{1}{2}\mathfrak{n}_p(A+B)\leqslant\mathfrak{n}_p(T)\leqslant\left(\|A\|_p^p+\|B\|_p^p\right)^{1/p} \tag{3.8}$$

for $0<p\leqslant\infty$.

Proof. Let $\{\phi_j\}$ be an orthonormal set in $\mathbb{H}$, and let $\varsigma_j = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_j\\ \phi_j\end{bmatrix}$. Then $\{\varsigma_j\}$ is an orthonormal set in $\mathbb{H}\oplus\mathbb{H}$. It follows that

$$\begin{aligned}\frac{1}{2}\left(\sum_j \left|\left\langle (A+B)\,\phi_j,\phi_j\right\rangle\right|^p\right)^{1/p} &= \left(\sum_j \left|\left\langle T\varsigma_j,\varsigma_j\right\rangle\right|^p\right)^{1/p}\\ &\leqslant \sup_{\{\xi_j\}\text{ o.n.s.}} \left(\sum_j \left|\left\langle T\xi_j,\xi_j\right\rangle\right|^p\right)^{1/p}\\ &= \mathfrak{n}_p(T). \qquad (3.9)\end{aligned}$$

By taking the supremum in both sides of inequalities (3.9) over all orthonormal sets $\{\phi_j\}$ in $\mathbb{H}$, we have

$$\frac{1}{2}\mathfrak{n}_p(A+B)\leqslant \mathfrak{n}_p(T),$$

which proves the first inequality of (3.8). The second inequality of (3.8) follows from the second inequality of (2.1) and the fact that $\|T\|_p = \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p}$. $\square$

A direct application of Theorem 3.4 can be seen in the following corollary.

COROLLARY 3.5. *Let $A\in\mathbb{B}_p(\mathbb{H})$, and let $S=\begin{bmatrix}0 & A\\ A & 0\end{bmatrix}$ and $T=\begin{bmatrix}0 & A\\ 0 & 0\end{bmatrix}$. Then*

$$\mathfrak{n}_p(A)\leqslant \mathfrak{n}_p(S)\leqslant 2^{1/p}\|A\|_p$$

and

$$\frac{1}{2}\mathfrak{n}_p(A)\leqslant \mathfrak{n}_p(T)\leqslant \|A\|_p$$

for $0<p\leqslant\infty$.

4. Inequalities for the new p-numerical radii of the off-diagonal parts of 2×2 operator matrices

In this section, we introduce further properties concerning our new p-numerical radii of the off-diagonal parts of 2×2 operator matrices. We start with the following Clarkson type lemma.

LEMMA 4.1. *Let a and b be complex numbers, and let $2\leqslant p<\infty$. Then*

$$2\left(|a|^p+|b|^p\right)\leqslant |a+b|^p+|a-b|^p\leqslant 2^{p-1}\left(|a|^p+|b|^p\right).$$

These inequalities can be reversed for $0<p\leqslant 2$.

Now, we have the following result.

THEOREM 4.2. *Let* $A,B \in \mathbb{B}_p(\mathbb{H})$, *and let* $R = \begin{bmatrix} 0 & A+B^* \\ 0 & 0 \end{bmatrix}$, $S = \begin{bmatrix} 0 & A-B^* \\ 0 & 0 \end{bmatrix}$, *and* $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. *Then*

(a) $\mathfrak{n}_p(T) \geqslant \max \begin{pmatrix} c_p \left(\|A\|_p^p + \|B\|_p^p \right)^{1/p}, \\ 2^{-1+1/p} \|A+B^*\|_p, 2^{-1+1/p} \|A-B^*\|_p \end{pmatrix}$ for $0 < p \leqslant \infty$, where c_p is defined in the relation (1.14).

(b) $\mathfrak{n}_p(T) \leqslant 2^{1-2/p} \left(\mathfrak{n}_p^p(R) + \mathfrak{n}_p^p(S) \right)^{1/p}$ for $2 \leqslant p \leqslant \infty$.

Proof.
Part (a)*:* We have

$$\begin{aligned} \mathfrak{n}_p(T) &\geqslant c_p \|T\|_p \text{ (by the first inequalities of (1.13) and (2.3))} \\ &= c_p \left(\|A\|_p^p + \|B\|_p^p \right)^{1/p}. \end{aligned} \tag{4.1}$$

On the other hand, we have

$$\begin{aligned} \mathfrak{n}_p(T) &\geqslant w_p(T) \text{ (by the first inequality of (2.3))} \\ &= \sup_{\theta \in \mathbb{R}} \left\| \operatorname{Re}\left(e^{i\theta} T \right) \right\|_p \\ &= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| \begin{bmatrix} 0 & e^{i\theta} A + e^{-i\theta} B^* \\ e^{-i\theta} A^* + e^{i\theta} B & 0 \end{bmatrix} \right\|_p \\ &= 2^{-1+1/p} \sup_{\theta \in \mathbb{R}} \left\| e^{i\theta} A + e^{-i\theta} B^* \right\|_p \\ &\geqslant 2^{-1+1/p} \max \left(\|A+B^*\|_p, \|A-B^*\|_p \right). \end{aligned} \tag{4.2}$$

Now, Part (a) follows from inequalities (4.1) and (4.2).

Part (b)*:* Let $T_1 = \begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix}$, and let $\{\xi_j\} = \left\{ \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} \right\}$ be an orthonormal set in $\mathbb{H} \oplus \mathbb{H}$. Then

$$\begin{aligned} &\left| \langle T\xi_j, \xi_j \rangle \right|^p + \left| \langle T_1 \xi_j, \xi_j \rangle \right|^p \\ &= \left| \langle A\psi_j, \phi_j \rangle + \langle B\phi_j, \psi_j \rangle \right|^p + \left| \langle A\psi_j, \phi_j \rangle - \langle B\phi_j, \psi_j \rangle \right|^p \\ &\leqslant 2^{p-1} \left(\left| \langle A\psi_j, \phi_j \rangle \right|^p + \left| \langle B\phi_j, \psi_j \rangle \right|^p \right) \\ &\quad \text{(by the second inequality of Lemma 4.1)} \\ &= 2^{p-1} \left(\left| \langle A\psi_j, \phi_j \rangle \right|^p + \left| \langle B^*\psi_j, \phi_j \rangle \right|^p \right) \\ &\leqslant 2^{p-2} \left(\left| \langle (A+B^*)\psi_j, \phi_j \rangle \right|^p + \left| \langle (A-B^*)\psi_j, \phi_j \rangle \right|^p \right) \qquad (4.3) \\ &\quad \text{(by the first inequality of Lemma 4.1)} \\ &= 2^{p-2} \left(\left| \langle R\xi_j, \xi_j \rangle \right|^p + \left| \langle S\xi_j, \xi_j \rangle \right|^p \right). \end{aligned}$$

Consequently,

$$\sum_j \left|\langle T\xi_j, \xi_j\rangle\right|^p \leqslant 2^{p-2}\left(\sum_j \left|\langle R\xi_j, \xi_j\rangle\right|^p + \sum_j \left|\langle S\xi_j, \xi_j\rangle\right|^p\right). \tag{4.4}$$

By taking the supremum in both sides of the inequality (4.4) over all orthonormal sets $\{\xi_j\}$ in $\mathbb{H}\oplus\mathbb{H}$, we have

$$\mathfrak{n}_p^p(T) \leqslant 2^{p-2}\left(\mathfrak{n}_p^p(R) + \mathfrak{n}_p^p(S)\right),$$

as required. □

A direct application of Part (b) of Theorem 4.2 can be stated as follows.

COROLLARY 4.3. *Let $A, B \in \mathbb{B}_p(\mathbb{H})$, and let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Then*

$$\mathfrak{n}_p(T) \leqslant 2^{1-2/p}\left(\|A+B^*\|_p^p + \|A-B^*\|_p^p\right)^{1/p} \tag{4.5}$$

for $2 \leqslant p \leqslant \infty$.

Proof. We have

$$\begin{aligned} \mathfrak{n}_p(T) &\leqslant 2^{1-2/p}\left(\mathfrak{n}_p^p(R) + \mathfrak{n}_p^p(S)\right)^{1/p} \text{ (by Theorem 4.2 } (b)) \\ &\leqslant 2^{1-2/p}\left(\|R\|_p^p + \|S\|_p^p\right)^{1/p} \text{ (by the second inequality of (2.1))} \\ &= 2^{1-2/p}\left(\|A+B^*\|_p^p + \|A-B^*\|_p^p\right)^{1/p}, \end{aligned}$$

as required. □

REMARK 4.4. Another formulation of the inequality (4.5) can be stated as follows: Let $A, B \in \mathbb{B}_p(\mathbb{H})$, and let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Then

$$\mathfrak{n}_p(T) \leqslant 2^{2-3/p}\left(\|\mathrm{Re}(T)\|_p^p + \|\mathrm{Im}(T)\|_p^p\right)^{1/p} \tag{4.6}$$

for $2 \leqslant p \leqslant \infty$. In particular, taking $B = A$, we have

$$\mathfrak{n}_p(A) \leqslant 2^{2-2/p}\left(\|\mathrm{Re}(A)\|_p^p + \|\mathrm{Im}(A)\|_p^p\right)^{1/p}. \tag{4.7}$$

In fact, the inequality (4.6) follows from the inequality (4.5), by observing that

$$\|A+B^*\|_p^p = 2^{p-1}\|\mathrm{Re}(T)\|_p^p \text{ and } \|A-B^*\|_p^p = 2^{p-1}\|\mathrm{Im}(T)\|_p^p,$$

while the inequality (4.7) follows from the inequality (4.6) by letting $B = A$ and using the first inequality of (3.8).

An application of Theorem 4.2 and Corollary 4.3 can be seen in the following corollary.

COROLLARY 4.5. *Let* $A \in \mathbb{B}_p(\mathbb{H})$, *and let* $S = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ *and* $T = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$. *Then*

$$2^{-1+2/p} \|A\|_p \leqslant \mathfrak{n}_p(S) \leqslant 2^{2-2/p} \|A\|_p \tag{4.8}$$

$$2^{-1+1/p} \|A\|_p \leqslant \mathfrak{n}_p(T) \leqslant 2^{1-1/p} \|A\|_p \tag{4.9}$$

for $2 \leqslant p \leqslant \infty$.

Proof. The first inequality of (4.8) follows by applying Part (a) of Theorem 4.2 for $2 \leqslant p \leqslant \infty$ and taking $B = A$, while the second inequality of (4.8) follows from Corollary 4.3 by taking $B = A$ and the facts that $\|\mathrm{Re}(A)\|_p \leqslant \|A\|_p$ and $\|\mathrm{Im}(A)\|_p \leqslant \|A\|_p$.

The inequalities of (4.9) follow by applying Part (a) of Theorem 4.2 for $2 \leqslant p \leqslant \infty$ and Corollary 4.3 by taking $B = 0$. □

Another result can be stated as follows.

THEOREM 4.6. *Let* $A, B \in \mathbb{B}_p(\mathbb{H})$, *and let* $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. *Then*

$$\mathfrak{n}_p^2(T) \leqslant 2 \left\| \, |A|^2 + |B^*|^2 \right\|_{p/2} \tag{4.10}$$

for $0 < p \leqslant \infty$.

Proof. Let $\{\xi_j\} = \left\{ \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} \right\}$ be an orthonormal set of $\mathbb{H} \oplus \mathbb{H}$. Then

$$\begin{aligned}
&\left|\left\langle T\xi_j, \xi_j \right\rangle\right|^2 + \left|\left\langle T_1\xi_j, \xi_j \right\rangle\right|^2 \\
&\quad \leqslant \left|\left\langle (A+B^*)\,\psi_j, \phi_j \right\rangle\right|^2 + \left|\left\langle (A-B^*)\,\psi_j, \phi_j \right\rangle\right|^2 \text{ (by the inequality (4.3))} \\
&\quad \leqslant \left(\left\|(A+B^*)\,\psi_j\right\|^2 + \left\|(A-B^*)\,\psi_j\right\|^2 \right) \left\|\phi_j\right\|^2 \\
&\quad \leqslant \left\|(A+B^*)\,\psi_j\right\|^2 + \left\|(A-B^*)\,\psi_j\right\|^2 \\
&\quad = 2\left\langle \left(|A|^2 + |B^*|^2\right) \psi_j, \psi_j \right\rangle \\
&\quad = 2\left\langle \begin{bmatrix} 0 & 0 \\ 0 & |A|^2 + |B^*|^2 \end{bmatrix} \xi_j, \xi_j \right\rangle,
\end{aligned}$$

and so

$$\sum_j \left|\left\langle T\xi_j, \xi_j \right\rangle\right|^p \leqslant 2^{p/2} \sum_j \left| \left\langle \begin{bmatrix} 0 & 0 \\ 0 & |A|^2 + |B^*|^2 \end{bmatrix} \xi_j, \xi_j \right\rangle \right|^{p/2}. \tag{4.11}$$

By taking the supremum in both sides of the inequality (4.11) over all orthonormal sets $\{\xi_j\}$ in $\mathbb{H}\oplus\mathbb{H}$, we have

$$\mathfrak{n}_p^2(T)\leqslant 2\left\|\,|A|^2+|B^*|^2\right\|_{p/2},$$

as required. □

Now, we have the following three corollaries.

COROLLARY 4.7. *Let $A\in\mathbb{B}_p(\mathbb{H})$. Then*

$$2^{-1+1/p}\|A\|_p\leqslant\mathfrak{n}_p\left(\begin{bmatrix}0 & \mathrm{Re}(A)\\ \mathrm{Im}(A) & 0\end{bmatrix}\right)\leqslant 2^{1/p}w_p(A) \tag{4.12}$$

for $0<p\leqslant\infty$.

Proof. We have

$$\begin{aligned}2^{-1+1/p}\|A\|_p&=2^{-1+1/p}\|\mathrm{Re}(A)+i\mathrm{Im}(A)\|_p\\&=2^{-1+1/p}\left\|\mathrm{Re}(A)+(-i\mathrm{Im}(A))^*\right\|_p\\&\leqslant\mathfrak{n}_p\left(\begin{bmatrix}0 & \mathrm{Re}(A)\\ -i\mathrm{Im}(A) & 0\end{bmatrix}\right)\ \text{(by Theorem 4.2 }(a))\\&=\mathfrak{n}_p\left(\begin{bmatrix}0 & \mathrm{Re}(A)\\ \mathrm{Im}(A) & 0\end{bmatrix}\right)\ \text{(by Theorem 3.1 }(d)),\end{aligned}$$

which proves the first inequality of (4.12). For the second inequality of (4.12), we have

$$\begin{aligned}\mathfrak{n}_p\left(\begin{bmatrix}0 & \mathrm{Re}(A)\\ \mathrm{Im}(A) & 0\end{bmatrix}\right)&\leqslant\left\|\begin{bmatrix}0 & \mathrm{Re}(A)\\ \mathrm{Im}(A) & 0\end{bmatrix}\right\|_p\\&\qquad\text{(by the second inequality of (2.1))}\\&=\left(\|\mathrm{Re}(A)\|_p^p+\|\mathrm{Im}(A)\|_p^p\right)^{1/p}\\&\leqslant 2^{1/p}w_p(A)\ \text{(by the relations (1.11) and (1.12))},\end{aligned}$$

as required. □

COROLLARY 4.8. *Let $A,B\in\mathbb{B}_p(\mathbb{H})$, and let $T=\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}$. Then*

$$\mathfrak{n}_p^2(T)\geqslant\frac{c_p^2}{2}\max\left(\left\|A^2+B^2\right\|_p,\left\|A^2-B^2\right\|_p\right)+\left|\,\|A\|_p^2-\|B\|_p^2\right| \tag{4.13}$$

for $1\leqslant p\leqslant\infty$, and

$$\mathfrak{n}_p^2(T)\geqslant\frac{c_p^2}{2}\left(\left\|\,|A|^2+|B|^2\right\|_{p/2}+\left|\,\|A\|_p^2-\|B\|_p^2\right|\right) \tag{4.14}$$

for $2\leqslant p\leqslant\infty$, where c_p is defined in the relation (1.14).

Proof. Let $1 \leqslant p \leqslant \infty$. Then

$$\begin{aligned} \mathfrak{n}_p^2(T) &\geqslant c_p^2 \max\left(\|A\|_p^2, \|B\|_p^2\right) \text{ (by Theorem 4.2 } (a)) \\ &= \frac{c_p^2}{2}\left(\|A\|_p^2 + \|B\|_p^2 + \left| \, \|A\|_p^2 - \|B\|_p^2 \right|\right) \qquad (4.15) \\ &\geqslant \frac{c_p^2}{2}\left(\left\|A^2\right\|_p + \left\|B^2\right\|_p + \left| \, \|A\|_p^2 - \|B\|_p^2 \right|\right) \\ &\geqslant \frac{c_p^2}{2}\left(\left\|A^2 \pm B^2\right\|_p + \left| \, \|A\|_p^2 - \|B\|_p^2 \right|\right), \qquad (4.16) \end{aligned}$$

which proves the inequality (4.13). For the inequality (4.14), let $2 \leqslant p \leqslant \infty$ and by using the facts that

$$\|A\|_p^2 = \left\| \, |A|^2 \right\|_{p/2} \quad \text{and} \quad \|B\|_p^2 = \left\| \, |B|^2 \right\|_{p/2},$$

the inequality (4.15) implies that

$$\begin{aligned} \mathfrak{n}_p^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\geqslant \frac{c_p^2}{2}\left(\left\| \, |A|^2 \right\|_{p/2} + \left\| \, |B|^2 \right\|_{p/2} + \left| \, \|A\|_p^2 - \|B\|_p^2 \right|\right) \\ &\geqslant \frac{c_p^2}{2}\left(\left\| \, |A|^2 + |B|^2 \right\|_{p/2} + \left| \, \|A\|_p^2 - \|B\|_p^2 \right|\right), \qquad (4.17) \end{aligned}$$

which proves the inequality (4.14). □

COROLLARY 4.9. *Let $A, B \in \mathbb{B}_p(\mathbb{H})$, and let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Then*

$$\begin{aligned} \mathfrak{n}_p^2(T) \geqslant\ & 2^{-2+2/p} \max\left(\left\|A^2 + B^{*2}\right\|_p, \|AB^* + B^*A\|_p\right) \\ & + 2^{-3+2/p}\left| \, \|A + B^*\|_p^2 - \|A - B^*\|_p^2 \right| \qquad (4.18) \end{aligned}$$

for $1 \leqslant p \leqslant \infty$, and

$$\begin{aligned} \mathfrak{n}_p^2(T) \geqslant\ & 2^{-2+2/p} \max\left(\left\| \, |A|^2 + |B^*|^2 \right\|_{p/2}, \|A^*B^* + BA\|_{p/2}\right) \\ & + 2^{-3+2/p}\left| \, \|A + B^*\|_p^2 - \|A - B^*\|_p^2 \right| \qquad (4.19) \end{aligned}$$

for $2 \leqslant p \leqslant \infty$.

Proof. Let $1 \leqslant p \leqslant \infty$. Then

$$\begin{aligned} \mathfrak{n}_p^2(T) &\geqslant 2^{-2+2/p} \max\left(\|A + B^*\|_p^2, \|A - B^*\|_p^2\right) \text{ (by Theorem 4.2 } (a)) \\ &\geqslant 2^{-3+2/p}\left(\left\|(A + B^*)^2\right\|_p + \left\|(A - B^*)^2\right\|_p + \left| \, \|A + B^*\|_p^2 - \|A - B^*\|_p^2 \right|\right) \\ &\geqslant 2^{-3+2/p}\left(\left\|(A + B^*)^2 \pm (A - B^*)^2\right\|_p + \left| \, \|A + B^*\|_p^2 - \|A - B^*\|_p^2 \right|\right). \end{aligned}$$

It follows that

$$\mathfrak{n}_p^2(T) \geqslant 2^{-2+2/p}\max\left(\left\|A^2+B^{*2}\right\|_p, \|AB^*+B^*A\|_p\right) \\ +2^{-3+2/p}\left|\ \|A+B^*\|_p^2-\|A-B^*\|_p^2\right|,$$

which proves the inequality (4.18). For the inequality (4.19), let $2 \leqslant p \leqslant \infty$ and by using the facts that

$$\|A+B^*\|_p^2 = \left\|\ |A+B^*|^2\right\|_{p/2} \text{ and } \|A-B^*\|_p^2 = \left\|\ |A-B^*|^2\right\|_{p/2}, \tag{4.20}$$

we have

$$\begin{aligned} \mathfrak{n}_p^2(T) &\geqslant 2^{-2+2/p}\max\left(\|A+B^*\|_p^2, \|A-B^*\|_p^2\right) \text{ (by Theorem 4.2 (a))} \\ &= 2^{-3+2/p}\left(\left\|\ |A+B^*|^2\right\|_{p/2}+\left\|\ |A-B^*|^2\right\|_{p/2}+\left|\ \|A+B^*\|_p^2-\|A-B^*\|_p^2\right|\right) \\ &\qquad\text{(by the relations (4.20))} \\ &\geqslant 2^{-3+2/p}\left(\left\|\ |A+B^*|^2 \pm |A-B^*|^2\right\|_{p/2}+\left|\ \|A+B^*\|_p^2-\|A-B^*\|_p^2\right|\right). \end{aligned}$$

It follows that

$$\mathfrak{n}_p^2(T) \geqslant 2^{-2+2/p}\max\left(\left\|\ |A|^2+|B^*|^2\right\|_{p/2}, \|A^*B^*+BA\|_{p/2}\right) \\ +2^{-3+2/p}\left|\ \|A+B^*\|_p^2-\|A-B^*\|_p^2\right|,$$

as required. □

REMARK 4.10. The inequalities (4.10) and (4.19) imply that if $A,B \in \mathbb{B}_p(\mathbb{H})$ and $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, then

$$2^{-2+2/p}\left\|\ |A|^2+|B^*|^2\right\|_{p/2} \leqslant \mathfrak{n}_p^2(T) \leqslant 2\left\|\ |A|^2+|B^*|^2\right\|_{p/2} \tag{4.21}$$

for $2 \leqslant p \leqslant \infty$.

An application of Corollary 4.9 can be stated as follows. This application relates the new p-numerical radius $\mathfrak{n}_p(\cdot)$ of an off-diagonal part of a 2×2 operator matrix with the old p-numerical radius $w_p(\cdot)$ of the products of the off-diagonal parts.

COROLLARY 4.11. *Let $A,B \in \mathbb{B}_p(\mathbb{H})$, and let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Then*

$$\mathfrak{n}_p^2(T) \geqslant 2^{-1+2/p}\max\left(w_p(AB), w_p(BA)\right).$$

for $2 \leqslant p \leqslant \infty$.

Proof. Let $\theta \in \mathbb{R}$. Then we have

$$\begin{aligned} \mathfrak{n}_p^2(T) &= \mathfrak{n}_p^2\left(\begin{bmatrix} 0 & e^{i\theta}A \\ B & 0 \end{bmatrix}\right) \text{ (by Theorem 3.1 } (d)) \\ &\geqslant 2^{-2+2/p}\left\| e^{-i\theta}A^*B^* + e^{i\theta}BA \right\|_p \text{ (by the inequality (4.19))} \\ &= 2^{-1+2/p}\left\| \operatorname{Re}\left(e^{i\theta}BA\right)\right\|_p . \end{aligned} \tag{4.22}$$

By taking the supremum in both sides of the inequality (4.22) over all real numbers θ, we have

$$\mathfrak{n}_p^2(T) \geqslant 2^{-1+2/p} w_p(BA). \tag{4.23}$$

Also, interchanging A and B and observing that

$$\mathfrak{n}_p^2(T) = \mathfrak{n}_p^2\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right) \text{ (by Theorem 3.1 } (b))$$

imply that

$$\mathfrak{n}_p^2(T) \geqslant 2^{-1+2/p} w_p(AB) \text{ (by the inequality (4.23)).} \tag{4.24}$$

Now, the result follows from the inequalities (4.23) and (4.24). □

We end this section with the following remark.

REMARK 4.12. It is natural to ask for explicit formulas for $\mathfrak{n}_p\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$ and $\mathfrak{n}_p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$ for $0 < p < \infty$.

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REFERENCES

[1] A. ABU-OMAR AND F. KITTANEH, *A generalization of the numerical radius*, Linear Algebra Appl. **569** (2019), 323–334.

[2] S. AICI, A. FARKIS, AND F. KITTANEH, *Further p-numerical radius inequalities*, J. Anal. **33** (2025), 1471–1489.

[3] S. AICI, A. FARKIS, AND F. KITTANEH, *On p-numerical radius inequalities for sums and products of operators*, J. Anal. **33** (2025), 673–689.

[4] A. BENMAKHLOUF, A. FRAKIS, F. KITTANEH, AND A. MENNOUNI, *Novel p-numerical radius inequalities for Hilbert space operators*, Ann. Univ. Ferrara (2025) **71**: 42, https://doi.org/10.1007/s11565-025-00598-w.

[5] A. BENMAKHLOUF, O. HIRZALLAH, AND F. KITTANEH, *On the p-numerical radii of Hilbert space operators*, Linear Multilinear Algebra **69** (2021), 2813–2829.

[6] R. BHATIA AND F. KITTANEH, *Norm inequalities for positive operators*, Lett. Math. Phys. **43** (1998), 225–231.

[7] P. BHUNIA AND S. SAHOO, *Schatten p-norm and numerical radius inequalities with applications*, Results Math. (2024) **80**: 15, https://doi.org/10.1007/s00025-024-02314-0.

[8] A. M. BIKCHENTAEV, F. KITTANEH, M. S. MOSLEHIAN, Y. SEO, *Trace inequalities for matrices and Hilbert space operators*, in: Forum for Interdisciplinary Mathematics, Springer, ISBN 978-981-97-6519-5, 2024, 332+xiii pp.

[9] T. BOTTAZZI AND C. CONDE, *Generalized numerical radius and related inequalities*, Oper. Matrices **15** (2021), 1289–1308.

[10] A. FRAKIS AND F. KITTANEH, *On p-numerical radius inequalities for commutators of operators*, Quaest. Math. **48** (2025), 417–428.

[11] A. FRAKIS, F. KITTANEH, AND S. SOLTANI, *On the p-numerical radii of 2×2 operator matrices*, J. Appl. Math. Comput. **70** (2024), 335–350.

[12] A. FARKIS, F. KITTANEH, AND S. SOLTANI, *Upper and lower bounds for the p-numerical radii of operators*, Results Math. (2024) **79**: 59, https://doi.org/10.1007/s00025-023-02090-3.

[13] I. C. GOHBERG AND M. G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs, **18**, Amer. Math. Soc., Providence, RI, 1969, MR 39 #7447, Zbl 0181.13504.

[14] J. HAMZA AND H. ISSA, *Generalized numerical radius inequalities for the Schatten p-norms*, arXiv:2204.0269.

[15] O. HIRZALLAH, F. KITTANEH, AND K. SHEBRAWI, *Numerical radius inequalities for certain 2×2 operator matrices*, Integr. Equ. Oper. Theory **71** (2011), 129–147.

[16] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math. **178** (2007), 83–89.

[17] B. SIMON, *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, 1979.

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Mohammad El-Haddad
Amman, Jordan
e-mail: m_el_haddad@yahoo.com

Omar Hirzallah
Department of mathematics
Faculty of Science, The Hashemite University
Zarqa, Jordan
e-mail: o.hirzal@hu.edu.jo

Fuad Kittaneh
Department of mathematics
The University of Jordan
Amman, Jordan
e-mail: fkitt@ju.edu.jo
and
Department of mathematics
Korea University
Seoul 02841, South Korea