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INTEGRAL FORMS AND FUNCTIONAL BOUNDS FOR CERTAIN EXTENDED EXTON'S DOUBLE HYPERGEOMETRIC FUNCTIONS

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Abstract. We extend and systematically investigate some particular Exton's double hypergeometric function $X^{A:B;B'}_{C:D;D'}[x,y]$, which is motivated by the recent integrated version of the Euler's Beta integral form with a Macdonald function $K_\nu(z)$ in the integrand. The newly introduced extended Exton's double hypergeometric functions $X^{A:B;B'}_{C:D;D'}[x,y;p,q,\nu,\lambda]$ is then represented by a number of integral representations of the Euler and Laplace types, including several further representations involving Bessel $J_\nu(z)$ and modified Bessel functions $I_\nu(z)$ of the first kind along with recurrence formulae. Using existing functional bounds for extended Euler's Beta function, various functional upper bounds are derived for particular extended Exton's double hypergeometric functions $X^{A:B;B'}_{C:D;D'}[x,y;p,q,\nu,\lambda]$. Also, plethora of bounding inequalities are established by virtue of Luke's, von Lommel's, Minakshisundaram and Szász and Olenko's bounds. The exposition ends with a newly introduced probability distribution applying extended Kummer and of Horn functions, for which moment inequalities of Turán type are proved.

1. Introduction and preliminaries

The Euler function of the first kind, or in short – the Beta function's integral form [1]

$$\mathrm{B}(s,t)=\int_0^1 x^{s-1}(1-x)^{t-1}\,\mathrm{d}x, \qquad \min\{s,t\}>0, \tag{1.1}$$

becomes the parent function for generalizations by exponential, Kummer's hypergeometric function, Macdonald function, Mittag–Leffler class functions among others of a suitably defined argument $h(t)$, say, by including the kernel $h(x)$ into a parametric integral [26]

$$B_h(s,t)=B(s,t)[h]=\int_0^1 x^{s-1}(1-x)^{t-1}h(x)\,\mathrm{d}x, \qquad \min\{s,t\}>0.$$

The thorough overview of the ancestry and presentation of the Beta-transforms getting B_p, $\mathrm{B}_{p,q}$, $\mathrm{B}_{p,\nu}$, $\mathrm{B}_{p,q;m}$ and others is given in the recent papers [19, 24, 26]. So, we

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refer to these publications and the further appropriate references therein, not repeating unnecessarily the known facts and generalizations.

We begin introducing one of the most used special functions in our exposition, the Macdonald function (modified Bessel function of the second kind) of the order ν which definition reads [22, p. 251, Eq. 10.27.4]

$$K_\nu(z) = \frac{\pi}{2}\csc(\pi\nu)\big(I_{-\nu}(z) - I_\nu(z)\big), \qquad \nu \notin \mathbb{Z},$$

otherwise for any $n \in \mathbb{Z}$, $\lim\limits_{\nu\to n} K_\nu(z) = K_n(z)$ is used. Here [22, p. 249, Eq. 10.25.2]

$$I_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{n\geqslant 0} \frac{\big(\frac{z}{2}\big)^{2n}}{\Gamma(\nu+1+n)\,n!},$$

is the modified Bessel function of the first kind. Next, the extended Beta function [24] is a Beta transform integral with the Macdonald function K_ν for the building block containing the defining kernel $h(x)$. The resulting integral expression turns out to be [24, p. 2, Definition 1]

$$\mathrm{B}^\lambda_{p,q,\nu}(s,t) = \sqrt{\frac{2}{\pi}} \int_0^1 x^{s-1}(1-x)^{t-1}\sqrt{h_\theta(x)}\,K_{\nu+\frac{1}{2}}(h_\theta(x))\,\mathrm{d}x, \tag{1.2}$$

where the modified argument

$$h_\theta(x) = \frac{p}{x^\lambda} + \frac{q}{(1-x)^\lambda}, \qquad \theta = (p,q,\lambda) \tag{1.3}$$

possesses singularities and the endpoints of the unit integration interval. Here the parameters' range $\lambda > 0$, $\min\{\Re(p), \Re(q)\} > 0$; $2\min\{s,t\} > \lambda > 0$, whilst $\nu \in \mathbb{R}$. Now, the modified Bessel $I_\nu(x)$ is real for all $\nu \in \mathbb{R}$ and $\arg(z) = 0$ and have in mind that

$$K_{\nu+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\big(1 + \mathscr{O}(z^{-1})\big), \quad z \to \infty, \tag{1.4}$$

is the reason for including the multiplication factor for $K_{\nu+\frac{1}{2}}$ in the definition (1.2).

The related extended Gauss hypergeometric function Taylor series definition reads [24, p. 3, Definition 2, Eq. (4)]

$$F^\lambda_{p,q,\nu}(a,b;c;z) = \sum_{k\geqslant 0} (a)_k \frac{\mathrm{B}^\lambda_{p,q,\nu}(b+k, c-b)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!}, \tag{1.5}$$

in which the parameters $\lambda > 0$, $\min\{p,q\} \geqslant 0$; $c \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $\Re(c) > \Re(b) > 0$ and $|z| < 1$; the extension of the Kummer function power series form is [24, p. 3, Definition 2, Eq. (5)]

$$\Phi^\lambda_{p,q,\nu}(b;c;z) = \sum_{k\geqslant 0} \frac{\mathrm{B}^\lambda_{p,q,\nu}(b+k, c-b)}{\mathrm{B}(b, c-b)} \frac{z^k}{k!}. \tag{1.6}$$

where $\lambda > 0$, $\min\{p,q\} \geqslant 0$; $\Re(c) > \Re(b) > 0$.

In [24] are used generalized Beta function (1.2) to extend the Gaussian and the Kummer's confluent hypergeometric functions to establish their functional bounding inequalities, Turán inequalities and studied their raw moments and moment inequalities by defining a novel probability Beta distribution. Specifying the values of parameters p,q,λ and ν, the generalized Beta function $\mathrm{B}_{p,q,\nu}^{\lambda}(s,t)$ (1.2) covers a whole spectrum of various well-known forms of extended Beta functions. In fact (1.2) is a so-called Beta function transform and maps a suitable input function ϕ into a multiparameter function [12]

$$\phi \mapsto \int_0^1 x^{s-1}(1-x)^{t-1}\,\phi(x)\,\mathrm{d}x.$$

In defining integral (1.2) we have $\phi(x) = \sqrt{h_\theta(x)}\,K_{\nu+\frac{1}{2}}\big(h_\theta(x)\big)$, being $\sqrt{h_\theta(x)}$ the necessarily implemented correcting factor function (up to the multiplicative constant $\sqrt{2/\pi}$), compare with the relation (1.4). In turn, the constraint $\min\{s,t\} > \frac{\lambda}{2} > 0$ follows immediately by re-writing $\sqrt{h_\theta(x)}$ in (1.2) into a convenient form.

Setting the values of the parameters p,q,ν,λ in (1.2) we get various known and frequently studied members of the Beta functions' family. So, when $\lambda = 1$ and $q = p$ we arrive at the so-called (p,ν)-extended Beta function introduced by Parmar *et al.* [23, p. 93, Eq. (13)]:

$$\mathrm{B}_{p,\nu}(s,t) = \sqrt{\frac{2p}{\pi}}\int_0^1 x^{s-\frac{3}{2}}\,(1-x)^{t-\frac{3}{2}}\,K_{\nu+\frac{1}{2}}\left(\frac{p}{x(1-x)}\right)\mathrm{d}x,$$

where $\Re(p) > 0$; $\min\{\Re(s),\Re(t)\} > 0$, and $\sqrt{p}$ takes its principal value. This kind Beta function is recently considered by Milovanović *et al.* in [19, p. 1433, Eq. (1.1)] for establishing Gautschi–Pinelis type upper bounds for the Macdonald function and the (p,ν)-extended Beta function. For complete details of numerous other special cases, we refer to the recent articles [25, 26]. Finally, if use the fact (1.4) and set $\nu = 0$ and $p,q \searrow 0$, then $\mathrm{B}_{p,q,\nu}^{\lambda}(s,t)$ approaches $\mathrm{B}(s,t)$ giving Euler's integral (1.1).

The research conduction in this paper will be carried out through the following plans: **1.** introducing the four parametric extension of certain Exton's double hypergeometric function $X_{C:D;D'}^{A:B;B'}[x,y,\Theta]$; $\Theta = (p,q,\nu,\lambda)$ by considering the definition of extended Beta function $\mathrm{B}_{p,q,\nu}^{\lambda}(s,t)$ in (1.2) involving the Macdonald kernel $K_{n+\frac{1}{2}}$; **2.** establishing the associated integral representations including Euler's and Laplace-Mellin type, as well as certain integral representations involving Bessel $J_\nu(z)$ and Modified Bessel functions $I_\nu(z)$ along with recurrence formulae; **3.** several functional upper bounds are derived for extended Exton's double hypergeometric functions $X_{C:D;D'}^{A:B;B'}[x,y;\Theta]$. Other fashion bounding inequalities are derived *via* Luke's, von Lommel's, Minakshisundaram and Szász and Olenko bounds. **4.** Finally, we introduce a new probability distribution building the density function in terms of specific extended Exton function. Using related extended Kummer and Horn functions moment inequalities of Turán type are proved.

2. Exton's double hypergeometric function $X^{A:B;B'}_{C:D;D'}[x,y]$

The generalized hypergeometric function with r numerator and s denominator parameters in the power series form reads

$$ {}_rF_s(a_1,\dots,a_r;b_1,\dots,b_s;z) = \sum_{k\geqslant 0} \frac{\prod\limits_{j=1}^{r}(a_j)_k}{\prod\limits_{j=1}^{s}(b_j)_k} \frac{z^k}{k!}, $$

where $(\delta)_n = \delta(\delta+1)\cdots(\delta+n-1) = \Gamma(\delta+n)/\Gamma(\delta)$, $\big(\delta \in \mathbb{C}\setminus\mathbb{Z}_0^-,\, n\in\mathbb{N}_0\big)$ and $(\delta)_0 = 1$ denotes the raising/shifted factorial or Pochhammer symbol, $a_j \in \mathbb{C}$, $j \in \overline{1,r} := \{1, 2, \dots, r\}$ and $b_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $j\in\overline{1,s}$. The symbol Γ being the familiar Euler's Gamma integral

$$ \Gamma(s) = \int_0^\infty \mathrm{e}^{-t}\, t^{s-1}\,\mathrm{d}t, \qquad \Re(s) > 0. $$

Particular cases for $r=2$, $s=1$ and $r=1$, $s=1$ are the Gaussian hypergeometric function and Kummer's confluent hypergeometric function

$$ {}_2F_1\big(a_1,a_2;b_1;z\big) = \sum_{k\geqslant 0} \frac{(a_1)_k(a_2)_k}{(b_1)_k}\frac{z^k}{k!}, \quad \Phi\big(a_1;b_1;z\big) = {}_1F_1\big(a_1;b_1;z\big) = \sum_{k\geqslant 0}\frac{(a_1)_k}{(b_1)_k}\frac{z^k}{k!}, $$

respectively.

In 1921, P. Appell introduced four double hypergeometric functions F_1, F_2, F_3 and F_4, which were unified and generalized by Kampé de Fériet $F^{j:k;l}_{r:s;t}[x,y]$ function (see, for details, [30, p. 27, Eq. (28)]). Presently, the recent authors studied in [25] the extension of the double hypergeometric function

$$ H_4[\,a,b';d,d';x,y] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}(b')_n}{(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!}; \qquad 2\sqrt{|x|}+|y|<1, \tag{2.1} $$

pioneered by Horn in [5], also see [30, p. 24 and p. 59], where $a, b' \in \mathbb{C}$ and $d, d' \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

H. Exton in 1982 introduced the double hypergeometric $X^{A:B;B'}_{C:D;D'}[x,y]$ function which double power series reads [3]

$$ X^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b')\\(c):(d);(d')\end{matrix}\,\middle|\, x,\, y\right] = \sum_{k,n\geqslant 0} \frac{((a))_{2k+n}((b))_k((b'))_n}{((c))_{2k+n}((d))_k((d'))_n}\frac{x^k}{k!}\frac{y^n}{n!}, \tag{2.2} $$

where (a) denotes the sequence of parameters $a_1,\cdots,a_A$, whilst $((a))_n$ stands for $(a_1)_n\cdots(a_A)_n$ and the empty symbol equals 1; consider mutually the other parameter's writing. The convergence conditions of (2.2) are presented in [3] and [29, pp. 153–158].

PROPOSITION 1. *For all* $a, b', d, d' > 0$ *the Exton function*

$$X^{1:0;1}_{0:1;1}\left[\begin{matrix} a:-;b' \\ -:d;d' \end{matrix}\middle| x,\, y\right] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}(b')_n}{(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!}$$

converges for all $(x,y)\in\mathbb{C}^2$ *provided* $2\sqrt{|x|}+|y|<1$.

Proof. Recognize by comparing (2.2) and (2.1) that

$$X^{1:0;1}_{0:1;1}\left[\begin{matrix} a:-;b' \\ -:d;d' \end{matrix}\middle| x,\, y\right] = H_4[a,b';d,d';x,y].$$

Since the convergence domain of the H_4 is $2\sqrt{|x|}+|y|<1$, see [30, p. 59], the claim obviously follows. □

At this point we mention that simultaneously we have in mind that the convergence conditions which hold for the Srivastava–Daoust F function (compare for the detailed definition (5.4)) one reduce to [29, pp. 157–158]

$$\begin{aligned}\Delta_1 &= 1+2\,(C-A)+D-B>0\\ \Delta_2 &= 1+C-A+D'-B'>0;\end{aligned}$$

so (2.2) converges absolutely for all $x,y\in\mathbb{C}$, see [4, Appenix B]. Another cases of convergence conditions related to interiors of a disks in $\mathbb{C}$ can be deduced from [29, pp. 153–157].

Focusing to the Exton X specific cases when $A,B,B',C,D,D'\in\{0,1\}$ which does not harm the generality of considerations and can be straightforwardly unified to the general case (2.2), we distinguish three main forms:

1. $A=C=1$ and the $B\neq D$, $B'\neq D'$ arbitrarily; $\Delta_1=1+D-B\geqslant 0$, $\Delta_2=1+D'-B'\geqslant 0$;

2. $A=C$ and either $B=D=1$ or $B'=D'=1$; $\Delta_1=1$; $\Delta_2=1+D'-B'\geqslant 0$;

3. $A=C$ and $B=D$; $B'=D'$; $\Delta_1=\Delta_2=1$,

pointing out that any X converges in these specified cases.

Moreover, the subcase **2.1.** $A=C=0$, $B=D=0$ means that

$$X^{0:0;1}_{0:0;1}\left[\begin{matrix} -:-;b' \\ -:-;d' \end{matrix}\middle| x,y\right] = \mathrm{e}^x\,\Phi(b';d';y);$$

and using then $B=D=1$ and $B'=D'=0$ we arrive at

$$X^{0:1;0}_{0:1;0}\left[\begin{matrix} -:b;- \\ -:d;- \end{matrix}\middle| x,y\right] = \mathrm{e}^y\,\Phi(b;d;x).$$

Next, **2.2.** $A = C = 1$, $B = D = 1$ results in

$$X^{1:1;0}_{1:1;0}\left[\begin{matrix} a : b; - \\ c : d; - \end{matrix}\,\middle|\, x, y\right] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}(b)_k}{(c)_{2k+n}(d)_k}\frac{x^k}{k!}\frac{y^n}{n!}\,, \tag{2.3}$$

which turns out to be irreducible, according to the best of our knowledge.

It remain the cases **3.1.** $A = C = 0$ and $B = B' = D = D' = 1$ giving a product of two Kummer functions, *viz.*

$$X^{0:1;1}_{0:1;1}\left[\begin{matrix} - : b; b' \\ - : d; d' \end{matrix}\,\middle|\, x, y\right] = \Phi\big(b; d; x\big)\cdot\Phi\big(b'; d'; y\big)\,,$$

and **3.2.** $A = C = B = B' = D = D' = 1$, where the structure of the resulting

$$X^{1:1;1}_{1:1;1}\left[\begin{matrix} a : b; b' \\ c : d; d' \end{matrix}\,\middle|\, x, y\right] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}(b)_k(b')_n}{(c)_{2k+n}(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!} \tag{2.4}$$

depends on the structure of upper and lower constituting parameters. Similarly, by structuring of upper and lower constituting parameters, we get following other forms of Exton functions:

3.3. $A = C = 1$, $B' = D' = 1$, $B = 0$ and $D = 1$

$$X^{1:0;1}_{1:1;1}\left[\begin{matrix} a : -; b' \\ c : d; d' \end{matrix}\,\middle|\, x, y\right] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}(b')_n}{(c)_{2k+n}(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!}\,, \tag{2.5}$$

3.4. $A = C = 1$, $B = D = 1$, $B' = 0$ and $D' = 1$

$$X^{1:1;0}_{1:1;1}\left[\begin{matrix} a : b; - \\ c : d; d' \end{matrix}\,\middle|\, x, y\right] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}(b)_k}{(c)_{2k+n}(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!}\,, \tag{2.6}$$

3.5. $A = C = 1$, $B = B' = 0$ and $D = D' = 1$

$$X^{1:0;0}_{1:1;1}\left[\begin{matrix} a : -; - \\ c : d; d' \end{matrix}\,\middle|\, x, y\right] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}}{(c)_{2k+n}(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!}\,, \tag{2.7}$$

3.6. $A = 1$, $C = 0$, $B = D = 1$ and $B' = D' = 1$

$$X^{1:1;1}_{0:1;1}\left[\begin{matrix} a : b; b' \\ - : d; d' \end{matrix}\,\middle|\, x, y\right] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}(b)_k(b')_n}{(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!}\,, \tag{2.8}$$

3.7. $A = 1$, $C = 0$, $B = 0$, $D = 1$ and $B' = D' = 1$

$$X^{1:0;1}_{0:1;1}\left[\begin{matrix} a : -; b' \\ - : d; d' \end{matrix}\,\middle|\, x, y\right] = \sum_{k,n\geqslant 0} \frac{(a)_{2k+n}(b')_n}{(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!}\,. \tag{2.9}$$

Finally, the most simple case turns out to be

$$X^{0:0;0}_{0:0;0}\left[\begin{matrix} - : -; - \\ - : -; - \end{matrix}\,\middle|\, x, y\right] = \mathrm{e}^{x+y}\,.$$

Now, it is obvious that the transformation of a ratio of two Pochhammer symbols:

$$\frac{(\alpha)_\mu}{(\beta)_\mu} = \frac{\mathrm{B}(\alpha+\mu,\beta-\alpha)}{\mathrm{B}(\alpha,\beta-\alpha)} \tag{2.10}$$

can be successfully applied to the Pochhammer symbols ratio(s) in the summand of Exton functions (2.3) and (2.4), where $\mu \in \mathbb{N}_0$ stands for the summation index. This results in

$$X^{1:1;0}_{1:1;0}\left[\begin{matrix} a:b;- \\ c:d;- \end{matrix}\middle| x,y\right] = \sum_{k,n\geqslant 0} \frac{\mathrm{B}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)} \frac{\mathrm{B}(b+k,d-b)}{\mathrm{B}(b,d-b)} \frac{x^k}{k!}\frac{y^n}{n!}, \tag{2.11}$$

where $\Re(c) > \Re(a) > 0$, $\Re(d) > \Re(b) > 0$, and simultaneously

$$X^{1:1;1}_{1:1;1}\left[\begin{matrix} a:b;b' \\ c:d;d' \end{matrix}\middle| x,y\right] = \sum_{k,n\geqslant 0} \frac{\mathrm{B}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)} \frac{\mathrm{B}(b+k,d-b)}{\mathrm{B}(b,d-b)} \frac{\mathrm{B}(b'+n,d'-b')}{\mathrm{B}(b',d'-b')} \frac{x^k}{k!}\frac{y^n}{n!}, \tag{2.12}$$

in which additionally $\Re(d') > \Re(b') > 0$. Similarly, for the Exton's functions (2.5) and (2.9), we have the transformations of (2.12) by virtue of (2.10) in the following forms:

$$\begin{aligned}
X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\middle| x,y\right] &= \sum_{k,n\geqslant 0} \frac{\mathrm{B}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)} \frac{\mathrm{B}(b'+n,d'-b')}{\mathrm{B}(b',d'-b')} \frac{1}{(d)_k} \frac{x^k}{k!}\frac{y^n}{n!}, \\
X^{1:1;0}_{1:1;1}\left[\begin{matrix} a:b;- \\ c:d;d' \end{matrix}\middle| x,y\right] &= \sum_{k,n\geqslant 0} \frac{\mathrm{B}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)} \frac{\mathrm{B}(b+k,d-b)}{\mathrm{B}(b,d-b)} \frac{1}{(d')_n} \frac{x^k}{k!}\frac{y^n}{n!}, \\
X^{1:0;0}_{1:1;1}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\middle| x,y\right] &= \sum_{k,n\geqslant 0} \frac{\mathrm{B}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)} \frac{1}{(d)_k(d')_n} \frac{x^k}{k!}\frac{y^n}{n!}, \\
X^{1:1;1}_{0:1;1}\left[\begin{matrix} a:b;b' \\ -:d;d' \end{matrix}\middle| x,y\right] &= \sum_{k,n\geqslant 0} (a)_{2k+n} \frac{\mathrm{B}(b+k,d-b)}{\mathrm{B}(b,d-b)} \frac{\mathrm{B}(b'+n,d'-b')}{\mathrm{B}(b',d'-b')} \frac{x^k}{k!}\frac{y^n}{n!}, \\
X^{1:0;1}_{0:1;1}\left[\begin{matrix} a:-;b' \\ -:d;d' \end{matrix}\middle| x,y\right] &= \sum_{k,n\geqslant 0} (a)_{2k+n} \frac{\mathrm{B}(b'+n,d'-b')}{\mathrm{B}(b',d'-b')} \frac{1}{(d)_k} \frac{x^k}{k!}\frac{y^n}{n!}.
\end{aligned} \tag{2.13}$$

The next section deals with some new forms of extended Exton's $X^{A:B;B'}_{C:D;D'}[x,y;\Theta]$ functions by replacing the Beta functions $\mathrm{B}(s,t)$ building the numerator in previous relations (2.11) to (2.13) by the appropriately used extended Beta function $\mathrm{B}^{\lambda}_{p,q,v}(s,t)$.

3. On Exton's extended $X^{A:B;B'}_{C:D;D'}[x,y;\Theta]$ functions

Now, corresponding integral representations are derived by using Beta integral representation (1.2) in relations (2.11) to (2.13). Also, we point out that it is of considerable interest to extend the here established results to other Exton's $X^{A:B;B'}_{C:D;D'}[x,y]$

functions covered by the general case (2.2). First, we introduce the following generalized Exton's double hypergeometric function as the double power series

$$X^{1:1;0}_{1:1;0}\left[\begin{matrix} a:b;-\\ c:d;-\end{matrix}\,\middle|\,x,y;\Theta\right] = \sum_{k,n\geqslant 0} \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)}\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b+k,d-b)}{\mathrm{B}(b,d-b)}\frac{x^k}{k!}\frac{y^n}{n!}, \tag{3.1}$$

$$\begin{aligned} X^{1:1;1}_{1:1;1}\left[\begin{matrix} a:b;b'\\ c:d;d'\end{matrix}\,\middle|\,x,y;\Theta\right] &= \sum_{k,n\geqslant 0} \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)}\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b+k,d-b)}{\mathrm{B}(b,d-b)}\\ &\quad\cdot\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b'+n,d'-b')}{\mathrm{B}(b',d'-b')}\frac{x^k}{k!}\frac{y^n}{n!}, \end{aligned} \tag{3.2}$$

recalling the parameter space for $\Theta = (p,q,\nu,\lambda)$, that is $\lambda > 0$, $\min\{\Re(p),\Re(q)\} > 0$, $\nu \in \mathbb{R}$ and $\min\{u,v\} > \frac{\lambda}{2} > 0$, with $u \in \{a,b,b'\}$, $v \in \{c-a, d-b, d'-b'\}$. Moreover,

$$X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b'\\ c:d;d'\end{matrix}\,\middle|\,x,y;\Theta\right] = \sum_{k,n\geqslant 0} \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)}\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b'+n,d'-b')}{\mathrm{B}(b',d'-b')\,(d)_k}\frac{x^k}{k!}\frac{y^n}{n!}, \tag{3.3}$$

$$X^{1:1;0}_{1:1;1}\left[\begin{matrix} a:b;-\\ c:d;d'\end{matrix}\,\middle|\,x,y;\Theta\right] = \sum_{k,n\geqslant 0} \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)}\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b+k,d-b)}{\mathrm{B}(b,d-b)\,(d')_n}\frac{x^k}{k!}\frac{y^n}{n!} \tag{3.4}$$

$$X^{1:0;0}_{1:1;1}\left[\begin{matrix} a:-;-\\ c:d;d'\end{matrix}\,\middle|\,x,y;\Theta\right] = \sum_{k,n\geqslant 0} \frac{\mathrm{B}^{\lambda}_{p,q,\nu}(a+2k+n,c-a)}{\mathrm{B}(a,c-a)}\frac{1}{(d)_k(d')_n}\frac{x^k}{k!}\frac{y^n}{n!}, \tag{3.5}$$

$$X^{1:1;1}_{0:1;1}\left[\begin{matrix} a:b;b'\\ -:d;d'\end{matrix}\,\middle|\,x,y;\Theta\right] = \sum_{k,n\geqslant 0} (a)_{2k+n}\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b+k,d-b)}{\mathrm{B}(b,d-b)}\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b'+n,d'-b')}{\mathrm{B}(b',d'-b')}\frac{x^k}{k!}\frac{y^n}{n!}, \tag{3.6}$$

$$\begin{aligned} X^{1:0;1}_{0:1;1}\left[\begin{matrix} a:-;b'\\ -:d;d'\end{matrix}\,\middle|\,x,y;\Theta\right] &= \sum_{k,n\geqslant 0} (a)_{2k+n}\frac{\mathrm{B}^{\lambda}_{p,q,\nu}(b'+n,d'-b')}{\mathrm{B}(b',d'-b')\,(d)_k}\frac{x^k}{k!}\frac{y^n}{n!}\\ &= H^{\lambda}_{4,p,q,\nu}[a,b';d,d';x,y], \end{aligned} \tag{3.7}$$

where the extended Horn's function $H^{\lambda}_{4,p,q,\nu}[x,y]$ in (3.7) has been studied recently by the authors, consult [25].

3.1. Integral representations

This section deals with the various integral representations including Euler's and Laplace–Mellin type, as well as certain integral representations including some other representations involving Bessel $J_\nu(z)$ and modified Bessel functions $I_\nu(z)$.

THEOREM 1. *For all* $\lambda > 0$, $\min\{\Re(p), \Re(q)\} > 0$, $\nu \in \mathbb{R}$ *and* $\min\{u, v\} > 0$, *with* $u \in \{a, b, b'\}$, $v \in \{c-a, d-b, d'-b'\}$, *where* $a, b, b', c, d, d' > 0$ *we have*

$$X^{1:1;0}_{1:1;0}\left[\begin{matrix} a:b;- \\ c:d;- \end{matrix}\middle| x,y;\Theta\right] = \frac{2}{\pi}\frac{1}{\mathrm{B}(a,c-a)\mathrm{B}(b,d-b)}\int_{(0,1)^2} t^{a-1}s^{b-1}(1-t)^{c-a-1} \cdot (1-s)^{d-b-1}\,\mathrm{e}^{xst^2+yt}\sqrt{h_\Theta(t)\,h_\Theta(s)}\,K_{\nu+\frac{1}{2}}(h_\Theta(t))\,K_{\nu+\frac{1}{2}}(h_\Theta(s))\,\mathrm{d}t\,\mathrm{d}s\,,$$

and

$$X^{1:1;0}_{1:1;0}\left[\begin{matrix} a:b;- \\ c:d;- \end{matrix}\middle| x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\frac{1}{\mathrm{B}(a,c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\mathrm{e}^{yt}\sqrt{h_\Theta(t)} \cdot K_{\nu+\frac{1}{2}}(h_\Theta(t))\,\Phi^{\lambda}_{p,q,\nu}(b;d;xt^2)\,\mathrm{d}t\,,$$

where h_Θ *is described in* (1.3) *as* $h_\Theta(x) = px^{-\lambda} + q(1-x)^{-\lambda}$.

Proof. Substituting the definition (1.2) of $\mathrm{B}^{\lambda}_{p,q,\nu}(s,t)$ into (3.1), we have

$$\begin{aligned} X^{1:1;0}_{1:1;0}\left[\begin{matrix} a:b;- \\ c:d;- \end{matrix}\middle| x,y;\Theta\right] &= \frac{2}{\pi}\sum_{k,n\geqslant 0}\int_{(0,1)^2}\frac{t^{a+2k+n-1}(1-t)^{c-a-1}}{\mathrm{B}(a,c-a)}\frac{s^{b+k-1}(1-s)^{d-b-1}}{\mathrm{B}(b,d-b)} \\ &\quad\cdot\sqrt{h_\Theta(t)h_\Theta(s)}\,K_{\nu+\frac{1}{2}}(h_\Theta(t))K_{\nu+\frac{1}{2}}(h_\Theta(s))\,\mathrm{d}t\,\mathrm{d}s\,\frac{x^k}{k!}\,\frac{y^n}{n!} \\ &= \frac{2}{\pi}\int_{(0,1)^2}\frac{t^{a-1}(1-t)^{c-a-1}}{\mathrm{B}(a,c-a)}\frac{s^{b-1}(1-s)^{d-b-1}}{\mathrm{B}(b,d-b)}\sqrt{h_\Theta(t)h_\Theta(s)} \\ &\quad\cdot K_{\nu+\frac{1}{2}}(h_\Theta(t))K_{\nu+\frac{1}{2}}(h_\Theta(s))\sum_{k,n\geqslant 0}\frac{(xst^2)^k}{k!}\frac{(yt)^n}{n!}\,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{2}{\pi}\int_{(0,1)^2}\frac{t^{a-1}(1-t)^{c-a-1}}{\mathrm{B}(a,c-a)}\frac{s^{b-1}(1-s)^{d-b-1}}{\mathrm{B}(b,d-b)}\sqrt{h_\Theta(t)\,h_\Theta(s)} \\ &\quad\cdot K_{\nu+\frac{1}{2}}(h_\Theta(t))K_{\nu+\frac{1}{2}}(h_\Theta(s))\,\mathrm{e}^{xst^2+yt}\,\mathrm{d}t\,\mathrm{d}s, \end{aligned}$$

which confimrs the first claim. As to the second integral, observe [24, Theorem 1, Eq. (7)]

$$\Phi^{\lambda}_{p,q,\nu}(b;c;z) = \frac{\sqrt{2/\pi}}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\mathrm{e}^{zt}\sqrt{h_\Theta(t)}\,K_{\nu+\frac{1}{2}}\big(h_\Theta(t)\big)\,\mathrm{d}t,$$

provided $p > 0$ and $|\arg(1-z)| < \pi$ or, for $p = 0$, $\Re(c) > \Re(b) > 0$, which implies

$$X^{1:1;0}_{1:1;0}\left[\begin{matrix} a:b;- \\ c:d;- \end{matrix}\middle| x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\frac{1}{\mathrm{B}(a,c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\mathrm{e}^{yt}\sqrt{h_\Theta(t)} \cdot K_{\nu+\frac{1}{2}}\big(h_\Theta(t)\big)\,\Phi^{\lambda}_{p,q,\nu}(b;d;xt^2)\,\mathrm{d}t,$$

and completes the proof. □

In the next seven theorems we present the integral representations for special extended Exton's X functions: single integral forms (3.2)–(3.7) double integral expressions ((3.2) in two different ways) and a triple integral representation for (3.2). The proving methodology is a routine one, substituting (1.2) of $\mathrm{B}^{\lambda}_{p,q,\nu}(a,b)$ into the extended series definitions (3.2)–(3.7) and after legitimate summation–integration order changes we implement into obtained expressions' integrands either the exponential or another appropriate special functions, for instance, the ordinary confluent hypergeometric and/or the extended Kummer hypergeometric functions ${}_0F_1, \Phi^{\lambda}_{p,q,\nu}$, or their products. We omit the detalied proof of these results.

THEOREM 2. *For all* $\lambda > 0$, $\min\{\Re(p),\Re(q)\} > 0$, $\nu \in \mathbb{R}$ *and* $\min\{u,v\} > 0$, *with* $u \in \{a,b,b'\}$, $v \in \{c-a, d-b, d'-b'\}$, *where* $a,b,b',c,d,d' > 0$ *the following integral expressions exist for* $X^{1:1;1}_{1:1;1}[x,y;\Theta]$ *in* (3.2) *reads as*

$$X^{1:1;1}_{1:1;1}\left[\begin{matrix} a:b;b' \\ c:d;d' \end{matrix}\middle| x,y,\Theta\right] = \frac{(2/\pi)^{3/2}}{\mathrm{B}(a,c-a)\mathrm{B}(b,d-b)\mathrm{B}(b',d'-b')} \int_{(0,1)^3} t^{a-1}s^{b-1}r^{b'-1}$$
$$\cdot (1-t)^{c-a-1}(1-s)^{d-b-1}(1-r)^{d'-b'-1}\mathrm{e}^{xst^2+ytr}\sqrt{h_\theta(t)h_\theta(s)h_\theta(r)}$$
$$\cdot K_{\nu+\frac{1}{2}}(h_\theta(t))K_{\nu+\frac{1}{2}}(h_\theta(s))\,K_{\nu+\frac{1}{2}}(h_\theta(r))\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}r\,.$$

THEOREM 3. *For all* $\lambda > 0$, $\min\{\Re(p),\Re(q)\} > 0$, $\nu \in \mathbb{R}$ *and* $\min\{a,c-a\}$, $\min\{b,d-b\}$, $\min\{b',d'-b'\} > 0$ *we have the integral expressions for* $X^{1:1;1}_{1:1;1}[x,y;\Theta]$ *in* (3.2) *as*

$$X^{1:1;1}_{1:1;1}\left[\begin{matrix} a:b;b' \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right] = \frac{\sqrt{2/\pi}}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}\sqrt{h_\theta(t)}$$
$$\cdot K_{\nu+\frac{1}{2}}(h_\theta(t))\,\Phi^{\lambda}_{p,q,\nu}(b;d;xt^2)\,\Phi^{\lambda}_{p,q,\nu}(b';d';yt)\,\mathrm{d}t\,,$$
$$X^{1:1;1}_{1:1;1}\left[\begin{matrix} a:b;b' \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right] = \frac{2/\pi}{\mathrm{B}(a,c-a)\mathrm{B}(b,d-b)} \int_{(0,1)^2} \frac{t^{a-1}}{(1-t)^{a-c+1}}\frac{s^{b-1}}{(1-s)^{b-d+1}}\,\mathrm{e}^{xst^2}$$
$$\cdot\sqrt{h_\theta(t)h_\theta(s)}K_{\nu+\frac{1}{2}}(h_\theta(t))\,K_{\nu+\frac{1}{2}}(h_\theta(s))\,\Phi^{\lambda}_{p,q,\nu}(b';d';yt)\mathrm{d}t\,\mathrm{d}s,$$

and

$$X^{1:1;1}_{1:1;1}\left[\begin{matrix} a:b;b' \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right] = \frac{2/\pi}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \int_{(0,1)^2} \frac{t^{a-1}}{(1-t)^{a-c+1}}\frac{r^{b'-1}}{(1-r)^{b'-d'+1}}$$
$$\cdot\mathrm{e}^{ytr}\sqrt{h_\theta(t)h_\theta(r)}K_{\nu+\frac{1}{2}}(h_\theta(t))K_{\nu+\frac{1}{2}}(h_\theta(r))\Phi^{\lambda}_{p,q,\nu}(b;d;xt^2)\,\mathrm{d}t\,\mathrm{d}r.$$

THEOREM 4. *The function* $X^{1:0;1}_{1:1;1}[x,y;\Theta]$ *in* (3.3) *possesses the integral form:*

$$X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\frac{1}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}\sqrt{h_\theta(t)}$$
$$\cdot K_{\nu+\frac{1}{2}}(h_\theta(t))\,{}_0F_1(-;d;xt^2)\,\Phi^{\lambda}_{p,q,\nu}(b';d';yt)\,\mathrm{d}t, \quad (3.8)$$

provided $\min\{a,c-a\} > 0$.

The Kummer's confluent hypergeometric function ${}_0F_1$ is related to the Bessel functions $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$, both of the first kind of the order ν with the formulae [22, p. 228, Entries 10.16.9, 10.39.9]

$$ {}_0F_1\left(-;\nu+1;-\frac{z^2}{4}\right) = \Gamma(\nu+1)\left(\frac{2}{z}\right)^\nu J_\nu(z), \tag{3.9}$$

$$ {}_0F_1\left(-;\nu+1;\frac{z^2}{4}\right) = \Gamma(\nu+1)\left(\frac{2}{z}\right)^\nu I_\nu(z), \tag{3.10}$$

where $-\nu \notin \mathbb{N}$ in both cases. Thus, using (3.9) and (3.10) in (3.8), (3.13) and (3.14), we yield integral expressions for these extended Exton's double hypergeometric functions.

COROLLARY 4.1. *Let the parameter space the same as in* Theorem 4*. Then integral representations hold for* $X_{1:1;1}^{1:0;1}[x,y;\Theta]$ *in* (3.3) *as*

$$ X_{1:1;1}^{1:0;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, -x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(d)x^{\frac{1-d}{2}}}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-d}(1-t)^{c-a-1}\sqrt{h_\theta(t)} \\ \cdot K_{\nu+\frac12}(h_\theta(t))\, J_{d-1}(2\sqrt{x}\,t)\,\Phi_{p,q,\nu}^{\lambda}(b';d';yt)\,\mathrm{d}t \tag{3.11}$$

$$ X_{1:1;1}^{1:0;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(d)x^{\frac{1-d}{2}}}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-d}(1-t)^{c-a-1}\sqrt{h_\theta(t)} \\ \cdot K_{\nu+\frac12}(h_\theta(t))\, I_{d-1}(2\sqrt{x}\,t)\,\Phi_{p,q,\nu}^{\lambda}(b';d';yt)\,\mathrm{d}t \tag{3.12}$$

THEOREM 5. *For* $\lambda, \Re(p), \Re(q) > 0$, $\nu \in \mathbb{R}$ *and* $\min\{a, c-a\} > 0$*, we have the integral expression for* $X_{1:1;1}^{1:1;0}[x,y;\Theta]$ *in* (3.4)*, namely*

$$ X_{1:1;1}^{1:1;0}\left[\begin{matrix} a:b;- \\ c:d;d' \end{matrix}\,\middle|\, x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}\sqrt{h_\theta(t)} \\ \cdot K_{\nu+\frac12}(h_\theta(t))\, {}_0F_1(-;d';yt)\,\Phi_{p,q,\nu}^{\lambda}(b;d;xt^2)\,\mathrm{d}t\,. \tag{3.13}$$

COROLLARY 5.1. *For* $\lambda, \Re(p), \Re(q) > 0$, $\nu \in \mathbb{R}$ *and* $a, c, d' > 0$ *that* $\min\{a, c-a\} > 0$*. Then we have*

$$ X_{1:1;1}^{1:1;0}\left[\begin{matrix} a:b;- \\ c:d;d' \end{matrix}\,\middle|\, x,-y;\Theta\right] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(d')y^{\frac{1-d'}{2}}}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-\frac{d'-1}{2}}(1-t)^{c-a-1}\sqrt{h_\theta(t)} \\ \cdot K_{\nu+\frac12}(h_\theta(t))\, J_{d'-1}(2\sqrt{yt})\,\Phi_{p,q,\nu}^{\lambda}(b;d;xt^2)\,\mathrm{d}t$$

$$ X_{1:1;1}^{1:1;0}\left[\begin{matrix} a:b;- \\ c:d;d' \end{matrix}\,\middle|\, x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(d')y^{\frac{1-d'}{2}}}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-\frac{d'-1}{2}}(1-t)^{c-a-1}\sqrt{h_\theta(t)} \\ \cdot K_{\nu+\frac12}(h_\theta(t))\, I_{d'-1}(2\sqrt{yt})\,\Phi_{p,q,\nu}^{\lambda}(b;d;xt^2)\,\mathrm{d}t\,.$$

THEOREM 6. *For* $\lambda, \Re(p), \Re(q) > 0$, $\nu \in \mathbb{R}$ *and* $\min\{a, c-a\} > 0$ *we have the integral*

$$X^{1:0;0}_{1:1;1}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\,\middle|\, x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mathrm{B}(a,c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\,\sqrt{h_\theta(t)} \cdot K_{\nu+\frac12}(h_\theta(t))\,{}_0F_1(-;d;xt^2)\,{}_0F_1(-;d';yt)\,\mathrm{d}t\,. \tag{3.14}$$

COROLLARY 6.1. *For* $d, d' > 0$ *and the same parameters as in* Theorem 6 *the integrals hold*

$$X^{1:0;0}_{1:1;1}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(d)\,\Gamma(d')\,x^{\frac{1-d}{2}}y^{\frac{1-d'}{2}}}{\mathrm{B}(a,c-a)}\int_0^1 t^{a-d-\frac{d'-1}{2}}(1-t)^{c-a-1} \cdot \sqrt{h_\theta(t)}K_{\nu+\frac12}(h_\theta(t))J_{d-1}(2\sqrt{xt})J_{d'-1}(2\sqrt{yt})\mathrm{d}t, \tag{3.15}$$

$$X^{1:0;0}_{1:1;1}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\,\middle|\, x,y;\Theta\right] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(d)\,\Gamma(d')\,x^{\frac{1-d}{2}}y^{\frac{1-d'}{2}}}{\mathrm{B}(a,c-a)}\int_0^1 t^{a-d-\frac{d'-1}{2}}(1-t)^{c-a-1} \cdot \sqrt{h_\theta(t)}K_{\nu+\frac12}(h_\theta(t))I_{d-1}(2\sqrt{xt})\,I_{d'-1}(2\sqrt{yt})\mathrm{d}t.$$

THEOREM 7. *For* $\lambda, \Re(p), \Re(q) > 0$, $\nu \in \mathbb{R}$ *and* $a > 0$ *we infer*

$$X^{1:1;1}_{0:1;1}\left[\begin{matrix} a:b;b' \\ -:d;d' \end{matrix}\,\middle|\, x,y;\Theta\right] = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-t}t^{a-1}\Phi^\lambda_{p,q,\nu}(b;d;xt^2)\Phi^\lambda_{p,q,\nu}(b';d';yt)\,\mathrm{d}t. \tag{3.16}$$

THEOREM 8. *For* $\lambda, \Re(p), \Re(q) > 0$, $\nu \in \mathbb{R}$ *and* $a > 0$ *there follows*

$$X^{1:0;1}_{0:1;1}\left[\begin{matrix} a:-;b' \\ -:d;d' \end{matrix}\,\middle|\, x,y;\Theta\right] = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-t}\,t^{a-1}\,{}_0F_1(-;d;xt^2)\,\Phi^\lambda_{p,q,\nu}(b';d';yt)\,\mathrm{d}t\,.$$

3.2. Recurrence relations

Next, we establish three recurrence formulae for the Exton's functions $X^{1:0;1}_{1:1;1}[x,y;\Theta]$, $X^{1:1;0}_{1:1;1}[x,y;\Theta]$ and $X^{1:0;0}_{1:1;1}[x,y;\Theta]$ by using the contiguous relation for the confluent hypergeometric ${}_0F_1$, see [28, p. 19, Eq. (2.2.2)] or [28, p. 20, Eq. (2.2.7)].

LEMMA 1. *We have the following contiguous relation*

$${}_0F_1(-;\gamma-1;x) - {}_0F_1(-;\gamma;x) - \frac{x}{\gamma(\gamma-1)}\,{}_0F_1(-;\gamma+1;x) = 0. \tag{3.17}$$

THEOREM 9. *There holds the recurrence relation for* $X^{1:0;1}_{1:1;1}[x,y,\Theta]$ *in* (3.3)*:*

$$X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, x,y,\Theta\right] = X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b' \\ c:d-1;d' \end{matrix}\,\middle|\, x,y;\Theta\right] + \frac{a(a+1)x}{c(c+1)d(1-d)}X^{1:0;1}_{1:1;1}\left[\begin{matrix} a+2:-;b' \\ c+2:d+1;d' \end{matrix}\,\middle|\, x,y;\Theta\right].$$

Proof. Applying the contiguous relation in the Lemma 1 to the integral (3.8) after simplification we obtain the statement. □

THEOREM 10. *The recurrence relation holds true:*

$$X^{1:1;0}_{1:1;1}\left[\begin{matrix} a:b;- \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right] = X^{1:1;0}_{1:1;1}\left[\begin{matrix} a:b;- \\ c:d;d'-1 \end{matrix}\middle| x,y;\Theta\right] + \frac{ay}{cd'(1-d')} X^{1:1;0}_{1:1;1}\left[\begin{matrix} a+1:b;- \\ c+1:d;d'+1 \end{matrix}\middle| x,y;\Theta\right].$$

By virtue of the relation (3.17) applied to (3.13) we arrive at the claim of the theorem. So, no need is there for further detailed explanations. Moreover, the relation (3.17) to used in the integral expression (3.14) results in the next result. We also skip the straightforward steps in proving this recurrence.

THEOREM 11. *The following recurrence relation for* $X^{1:0;0}_{1:1;1}[x,y,\Theta]$ *in* (3.5) *holds true:*

$$\begin{aligned} X^{1:0;0}_{1:1;1}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right] &= X^{1:0;0}_{1:1;1}\left[\begin{matrix} a:-;- \\ c:d-1;d'-1 \end{matrix}\middle| x,y;\Theta\right] \\ &+ \frac{ay}{cd'(1-d')} X^{1:0;0}_{1:1;1}\left[\begin{matrix} a+1:-;- \\ c+1:d-1;d'+1 \end{matrix}\middle| x,y;\Theta\right] \\ &+ \frac{a(a+1)x}{c(c+1)d(1-d)} X^{1:0;0}_{1:1;1}\left[\begin{matrix} a+2:-;- \\ c+2:d+1;d'-1 \end{matrix}\middle| x,y;\Theta\right] \\ &+ \frac{a(a+1)(a+2)xy}{c(c+1)(c+2)d(1-d)d'(1-d')} X^{1:0;0}_{1:1;1}\left[\begin{matrix} a+3:-;- \\ c+3:d+1;d'+1 \end{matrix}\middle| x,y,\Theta\right]. \end{aligned}$$

4. Functional bounds for extended Exton's X functions

This section explores bounding inequalities for the extended Exton's double hypergeometric function $X^{A:B;B'}_{C:D;D'}[x,y;\Theta]$. Our main aim in this section are to find bounding inequalities for the above considered cases of this function.

4.1. Functional bounds extablished *via* series representations

In this introductory part of this subsection we present sharp estimates for the extended Beta, hypergeometric and confluent hypergeometric functions $\mathrm{B}^{\lambda}_{p,q,\nu}$, $F^{\lambda}_{p,q,\nu}$ and $\Phi^{\lambda}_{p,q,\nu}$.

The first auxiliary lemma describes an estimate for $\mathrm{B}^{\lambda}_{p,q,\nu}$ defined in (1.2).

LEMMA 2. [24, p. 5, Theorem 2, Eq. (8)] *Let* $p,q>0$, $\lambda\in(0,1)\cup(1,\infty)$, $\nu\in\mathbb{R}$. *Then for all* $2\min\{s,t\}>\lambda>0$ *we have*

$$\begin{aligned}\mathrm{B}^{\lambda}_{p,q,\nu}(s,t) &\leqslant \frac{\sqrt{2pq}\,K_{\nu+\frac12}\Big(\big(p^{\frac{1}{\lambda+1}}+q^{\frac{1}{\lambda+1}}\big)^{\lambda+1}\Big)}{\sqrt{\pi}\,\big(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}}\big)^{\frac{\lambda-1}{2}}}\,\mathrm{B}\big(s-\tfrac{\lambda}{2},t-\tfrac{\lambda}{2}\big)\\ &=:\Omega^{\lambda}_{\nu}(p,q)\,\mathrm{B}\big(s-\tfrac{\lambda}{2},t-\tfrac{\lambda}{2}\big).\end{aligned} \tag{4.1}$$

The second auxiliary lemma gives an estimate for $F^{\lambda}_{p,q,\nu}$ and $\Phi^{\lambda}_{p,q,\nu}$ presented in (1.5) and (1.6)

LEMMA 3. [25, p. 7, Theorem 3.2] *For all* $p,q>0$ *and* $\lambda\in(0,1)\cup(1,\infty)$, $\nu\in\mathbb{R}$, *or when* $p=0=q$, $\Re(t)>\Re(s)>0$ *we have*

$$\begin{aligned}\big|F^{\lambda}_{p,q,\nu}(a,s;t;z)\big| &\leqslant \Omega^{\lambda}_{\nu}(p,q)\,\frac{\mathrm{B}\big(s-\frac{\lambda}{2},t-s-\frac{\lambda}{2}\big)}{\mathrm{B}(s,t-s)}\,{}_2F_1\big(a,s-\tfrac{\lambda}{2};t-\lambda;|z|\big),\\ \big|\Phi^{\lambda}_{p,q,\nu}(s;t;z)\big| &\leqslant \Omega^{\lambda}_{\nu}(p,q)\,\frac{\mathrm{B}\big(s-\frac{\lambda}{2},t-s-\frac{\lambda}{2}\big)}{\mathrm{B}\big(s,t-s\big)}\,\Phi\big(s-\tfrac{\lambda}{2};t-\lambda;|z|\big),\end{aligned} \tag{4.2}$$

provided $2\min\{s,t-s\}>\lambda$ *and* ${}_2F_1$ *and* Φ *denote the Gauss hypergeometric function and the Kummer confluent hypergeometric function, respectively.*

The following theorem provides bounding inequalities for following Exton's double hypergeometric function $X^{1:1;0}_{1:1;0}[x,y;\Theta]$, $X^{1:1;1}_{1:1;1}[x,y;\Theta]$ $X^{1:0;1}_{1:1;1}[x,y;\Theta]$, $X^{1:1;0}_{1:1;1}[x,y;\Theta]$ and $X^{1:0;0}_{1:1;1}[x,y,\Theta]$ by using their series representations given in the previous section 3.

THEOREM 12. *Assume that* $p,q>0$ *and* $\lambda\in(0,1)\cup(1,\infty)$, $\nu\in\mathbb{R}$ *and the constraints*

$$\min\{a,c-a\}>\frac{\lambda}{2};\quad \min\{b,d-b\}>\frac{\lambda}{2};\quad \min\{b',d'-b'\}>\frac{\lambda}{2}.$$

Then we have the following functional bounds

$$\begin{aligned}\left|X^{1:1;0}_{1:1;0}\left[\begin{matrix}a:b;-\\c:d;-\end{matrix}\,\middle|\,x,y;\Theta\right]\right| &\leqslant \Big[\Omega^{\lambda}_{\nu}(p,q)\Big]^2\,\frac{\mathrm{B}\big(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\big)\mathrm{B}\big(b-\frac{\lambda}{2},d-b-\frac{\lambda}{2}\big)}{\mathrm{B}(a,c-a)\mathrm{B}(b,d-b)}\\ &\quad\cdot X^{1:1;0}_{1:1;0}\left[\begin{matrix}a-\frac{\lambda}{2}:b-\frac{\lambda}{2};-\\c-\lambda:d-\lambda;-\end{matrix}\,\middle|\,|x|,|y|\right],\end{aligned} \tag{4.3}$$

$$\begin{aligned}\left|X^{1:1;1}_{1:1;1}\left[\begin{matrix}a:b;b'\\c:d;d'\end{matrix}\,\middle|\,x,y;\Theta\right]\right| &\leqslant \Big[\Omega^{\lambda}_{\nu}(p,q)\Big]^3\,\frac{\mathrm{B}\big(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\big)\mathrm{B}\big(b-\frac{\lambda}{2},d-b-\frac{\lambda}{2}\big)}{\mathrm{B}(a,c-a)\mathrm{B}(b,d-b)B(b',d'-b')}\\ &\quad\cdot\mathrm{B}\big(b'-\tfrac{\lambda}{2},d'-b'-\tfrac{\lambda}{2}\big)X^{1:1;1}_{1:1;1}\left[\begin{matrix}a-\frac{\lambda}{2}:b-\frac{\lambda}{2};b'-\frac{\lambda}{2}\\c-\lambda:d-\lambda;d'-\lambda\end{matrix}\,\middle|\,|x|,|y|\right],\end{aligned} \tag{4.4}$$

$$\left|X_{1:1;1}^{1:0;1}\left[\begin{matrix}a:-;b'\\c:d;d'\end{matrix}\middle|x,y;\Theta\right]\right| \leqslant \left[\Omega_\nu^\lambda(p,q)\right]^2 \frac{\mathrm{B}\left(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\right)\mathrm{B}\left(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\right)}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \cdot X_{1:1;1}^{1:0;1}\left[\begin{matrix}a-\frac{\lambda}{2}:-;b'-\frac{\lambda}{2}\\c-\lambda:d;d'-\lambda\end{matrix}\middle||x|,|y|\right], \tag{4.5}$$

$$\left|X_{1:1;1}^{1:1;0}\left[\begin{matrix}a:b;-\\c:d;d'\end{matrix}\middle|x,y;\Theta\right]\right| \leqslant \left[\Omega_\nu^\lambda(p,q)\right]^2 \frac{\mathrm{B}\left(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\right)\mathrm{B}\left(b-\frac{\lambda}{2},d-b-\frac{\lambda}{2}\right)}{\mathrm{B}(a,c-a)\mathrm{B}(b,d-b)} \cdot X_{1:1;1}^{1:1;0}\left[\begin{matrix}a-\frac{\lambda}{2}:b-\frac{\lambda}{2};-\\c-\lambda:d-\lambda;d'\end{matrix}\middle||x|,|y|\right], \tag{4.6}$$

$$\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix}a:-;-\\c:d;d'\end{matrix}\middle|x,y;\Theta\right]\right| \leqslant \Omega_\nu^\lambda(p,q)\frac{\mathrm{B}\left(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\right)}{\mathrm{B}(a,c-a)} X_{1:1;1}^{1:0;0}\left[\begin{matrix}a-\frac{\lambda}{2}:-;-\\c-\lambda:d;d'\end{matrix}\middle||x|,|y|\right]. \tag{4.7}$$

Proof. We first prove the functional bound for (4.3). Applying the bound of $\mathrm{B}_{p,q,\nu}^\lambda(s,t)$ in (4.1) to the extended Exton function (3.1) we get

$$\begin{aligned}\left|X_{1:1;0}^{1:1;0}\left[\begin{matrix}a:b;-\\c:d;-\end{matrix}\middle|x,y;\Theta\right]\right| &= \left|\sum_{k,n\geqslant 0}\frac{\mathrm{B}_{p,q,\nu}^\lambda(a+2k+n,c-a)}{\mathrm{B}(a,c-a)}\frac{\mathrm{B}_{p,q,\nu}^\lambda(b+k,d-b)}{\mathrm{B}(b,d-b)}\frac{x^k}{k!}\frac{y^n}{n!}\right|\\
&\leqslant \left[\Omega_\nu^\lambda(p,q)\right]^2\sum_{k,n\geqslant 0}\frac{\mathrm{B}\left(a+2k+n-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\right)}{\mathrm{B}(a,c-a)}\frac{\mathrm{B}\left(b+k-\frac{\lambda}{2},d-b-\frac{\lambda}{2}\right)}{\mathrm{B}(b,d-b)}\frac{|x|^k}{k!}\frac{|y|^n}{n!}\\
&= \frac{\mathrm{B}\left(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\right)\mathrm{B}\left(b-\frac{\lambda}{2},d-b-\frac{\lambda}{2}\right)}{\left[\Omega_\nu^\lambda(p,q)\right]^{-2}\mathrm{B}(a,c-a)\mathrm{B}(b,d-b)} X_{1:1;0}^{1:1;0}\left[\begin{matrix}a-\frac{\lambda}{2}:b-\frac{\lambda}{2};-\\c-\lambda:d-\lambda;-\end{matrix}\middle||x|,|y|\right].\end{aligned}$$

This proves the inequality (4.3). Similar arguments as in this proof verify (4.4), (4.5), (4.6) and (4.7). The details are omitted here. □

4.2. Functional bounds *via* integral representations

In this subsection, we investigate the bounds of some members from the class of extended Exton's double hypergeometric function $X_{C:D;D'}^{A:B;B'}[x,y;\Theta]$ having integral representation formulae. To accomplish this goal we review and recall certain inequalities pertaining to the generalized hypergeometric function, Bessel function and modified Bessel functions of the first kind as follows:

- For $\beta_j \geqslant \alpha_j > 0,\ j=\overline{1,r}$ and $t \geqslant 0$, there exist Luke's bilateral functional inequalities for the generalized hypergeometric function ${}_rF_r$ [17, Theorem 16, Eq. (5.6)]

  $$\mathrm{e}^{\omega t} \leqslant {}_rF_r\left(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r;t\right) \leqslant 1-\omega\left(1-\mathrm{e}^t\right),$$

  where

  $$\omega = \frac{\max\limits_{1\leqslant j\leqslant r}\alpha_j}{\min\limits_{1\leqslant j\leqslant r}\beta_j},$$

and the equality holds for $t = 0$. In the case when $r = 1$ we have the Kummer confluent hypergeometric function $\Phi(\alpha;\beta;t) = {}_1F_1(\alpha;\beta;t)$; in that case when $\beta \geqslant \alpha > 0$ we have

$$\mathrm{e}^{\frac{\alpha}{\beta}t} \leqslant \Phi(\alpha;\beta;t) \leqslant 1 - \frac{\alpha}{\beta}\big(1-\mathrm{e}^t\big), \qquad t \geqslant 0. \tag{4.8}$$

- The set of bounding inequalities for J_ν and I_ν read as

  (i) von Lommel's bounds [31, pp. 31 and 406], [14], [15, pp. 548–549]

$$|J_\nu(t)| \leqslant 1, \qquad |J_{\nu+1}(t)| \leqslant \frac{1}{\sqrt{2}}, \qquad \nu \in \mathbb{R}_+,\ t \in \mathbb{R}; \tag{4.9}$$

  (ii) Minakshisundaram and Szász bound [6, Eq. (1.8)], [20, pp. 36-37]; cf. [31, p. 16]

$$|J_\nu(t)| \leqslant \frac{1}{\Gamma(\nu+1)}\left(\frac{|t|}{2}\right)^\nu, \qquad \nu \geqslant 0,\ t \in \mathbb{R}; \tag{4.10}$$

  (iii) For $\nu \geqslant 0$ and $t \in \mathbb{R}$ there are the bounds by Landau [13]

$$|J_\nu(t)| \leqslant b_L\, \nu^{-1/3}, \qquad b_L = \sqrt[3]{2}\sup_{t\geqslant 0}\mathrm{Ai}(t), \tag{4.11}$$

$$|J_\nu(t)| \leqslant c_L\, |t|^{-1/3}, \qquad c_L = \sup_{t\geqslant 0} t^{1/3} J_0(t), \tag{4.12}$$

  where $\mathrm{Ai}(\cdot)$ stands for the Airy function

$$\mathrm{Ai}(t) = \frac{\pi}{2}\sqrt{\frac{t}{3}}\left[J_{-1/3}\left\{2\left(\frac{t}{3}\right)^{3/2}\right\} + J_{-1/3}\left\{2\left(\frac{t}{3}\right)^{3/2}\right\}\right]. \tag{4.13}$$

  (iv) Olenko's bound [21, Theorem 2.1]

$$\sup_{t\geqslant 0}\sqrt{t}\,|J_\nu(t)| \leqslant b_L\sqrt{\nu^{1/3} + \frac{\tau_1}{\nu^{1/3}} + \frac{3\tau_1^2}{10\nu}} =: d_O, \qquad \nu > 0, \tag{4.14}$$

  where τ_1 is the smallest positive zero of the Airy-function Ai in (4.13) and b_L is the Landau's constant in (4.11). This bound is asymptotically precise and the constant b_L is the best possible.

  (v) Luke [17, p. 55, Eq. (6.25)] obtained the following result

$$I_\mu(t) < \frac{\left(\frac{t}{2}\right)^\mu}{\Gamma(\mu+1)}\cosh t, \qquad t > 0,\ \mu > -\tfrac{1}{2}. \tag{4.15}$$

The following theorem states our second set of bounding inequalities.

THEOREM 13. *Letting* $p,q>0$ *and* $\lambda\in(0,1)\cup(1,\infty)$, $\nu\in\mathbb{R}$ *to hold the following constraints*

$$\min\{a,c-a\}>\frac{\lambda}{2};\,\min\{b,d-b\}>\frac{\lambda}{2};\,\min\{b',d'-b'\}>\frac{\lambda}{2}.$$

Then we have the following bounds

$$\left|X^{1:0;1}_{1:1;1}\left[\begin{matrix}a:-;b'\\c:d;d'\end{matrix}\,\middle|\,-x,y,\Theta\right]\right|\leqslant\left[\Omega^{\lambda}_{\nu}(p,q)\right]^2\frac{|x|^{\frac{1-d}{2}}\Gamma(d)\,\mathrm{B}\big(a-d-\frac{\lambda}{2}+1,c-a-\frac{\lambda}{2}\big)}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')}$$
$$\cdot\,\mathrm{B}\big(b'-\tfrac{\lambda}{2},d'-b'-\tfrac{\lambda}{2}\big)\left\{1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left[1-\Phi\left(a-d-\tfrac{\lambda}{2}+1;c-d-\lambda+1;|y|\right)\right]\right\}.$$

Next, there holds

$$\left|X^{1:0;1}_{1:1;1}\left[\begin{matrix}a:-;b'\\c:d;d'\end{matrix}\,\middle|\,-x,y,\Theta\right]\right|\leqslant\frac{b_L\,|x|^{\frac{1-d}{2}}\,\Gamma(d)\,\mathrm{B}\big(a-d-\frac{\lambda}{2}+1,c-a-\frac{\lambda}{2}\big)}{\sqrt[3]{d-1}\,\left[\Omega^{\lambda}_{\nu}(p,q)\right]^{-2}\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')}$$
$$\cdot\,\mathrm{B}\big(b'-\tfrac{\lambda}{2},d'-b'-\tfrac{\lambda}{2}\big)\left\{1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left[(1-\Phi\left(a-d-\tfrac{\lambda}{2}+1;c-d-\lambda+1;|y|\right)\right]\right\},\tag{4.16}$$

where $b_L:=\sqrt[3]{2}\,\sup_{t\geqslant0}\mathrm{Ai}(2\sqrt{x}t)$ *denotes the first Landau's constant.*

Proof. First, apply the estimate (4.2) to the integral representation (3.11) to obtain

$$\left|X^{1:0;1}_{1:1;1}\left[\begin{matrix}a:-;b'\\c:d;d'\end{matrix}\,\middle|\,-x,y;\Theta\right]\right|\leqslant\Omega^{\lambda}_{\nu}(p,q)\sqrt{\frac{2}{\pi}}\,\frac{|x|^{\frac{1-d}{2}}\,\Gamma(d)\,\mathrm{B}\big(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}(a,c-a)\mathrm{B}\big(b',d'-b'\big)}$$
$$\cdot\int_0^1\frac{t^{a-d}\sqrt{h_\theta(t)}}{(1-t)^{a-c+1}}K_{\nu+\frac12}(h_\theta(t))\big|J_{d-1}(2\sqrt{x}t)\big|\Phi(b'-\tfrac{\lambda}{2};d'-\lambda;|y|t)\mathrm{d}t$$
$$\leqslant\Omega^{\lambda}_{\nu}(p,q)\sqrt{\frac{2}{\pi}}\,\frac{|x|^{\frac{1-d}{2}}\,\Gamma(d)\,\mathrm{B}\big(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}(a,c-a)\mathrm{B}\big(b',d'-b'\big)}$$
$$\cdot\max_{0\leqslant t\leqslant1}\sqrt{p(1-t)^{\lambda}+qt^{\lambda}}\sup_{0<t<1}K_{\nu+\frac12}\left(\frac{p}{t^{\lambda}}+\frac{q}{(1-t)^{\lambda}}\right)$$
$$\cdot\int_0^1 t^{a-d-\frac{\lambda}{2}}(1-t)^{c-a-\frac{\lambda}{2}-1}\big|J_{d-1}(2\sqrt{x}t)\big|\,\Phi(b'-\tfrac{\lambda}{2};d'-\lambda;|y|t)\,\mathrm{d}t$$
$$\leqslant\Omega^{\lambda}_{\nu}(p,q)\sqrt{\frac{2}{\pi}}\,\frac{|x|^{\frac{1-d}{2}}\,\Gamma(d)\,\mathrm{B}\big(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}(a,c-a)\mathrm{B}\big(b',d'-b'\big)}$$
$$\cdot\sqrt{\max_{0\leqslant t\leqslant1}\{p(1-t)^{\lambda}+qt^{\lambda}\}}\,K_{\nu+\frac12}\left(\inf_{0<t<1}\left\{\frac{p}{t^{\lambda}}+\frac{q}{(1-t)^{\lambda}}\right\}\right)$$
$$\cdot\int_0^1 t^{a-d-\frac{\lambda}{2}}(1-t)^{c-a-\frac{\lambda}{2}-1}\big|J_{d-1}(2\sqrt{x}t)\big|\,\Phi(b'-\tfrac{\lambda}{2};d'-\lambda;|y|t)\,\mathrm{d}t.\tag{4.17}$$

Both estimated functions in the integrand are positive in the declared range of its parameters, and the Macdonald function $K_\mu(z)$ monotone decreases and it is continuous in $z>0$. For $h_1(t)=p(1-t)^\lambda+qt^\lambda$, we have $h_1'(t)=\lambda\big[-p(1-t)^{\lambda-1}+qt^{\lambda-1}\big]$, and the stationary point becomes

$$t_0=\frac{1}{1+\left(\frac{q}{p}\right)^{\frac{1}{\lambda-1}}}\in(0,1).$$

Furthermore

$$h_1''(t)=\lambda(\lambda-1)\big[p(1-t)^{\lambda-2}+qt^{\lambda-2}\big]<0,\qquad \lambda\in(0,1),$$

therefore t_0 is the abscissa of maximum for $h_1(t)$, consequently

$$h_1(t_0)=\frac{pq}{\big(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}}\big)^{\lambda-1}}, \tag{4.18}$$

Now, we analyse the argument function of the Macdonald function $h_\theta(t)=pt^{-\lambda}+q(1-t)^{-\lambda}$. The stationary point t_1 is the solution of $h_\theta'(t)=-\lambda\big[pt^{-\lambda-1}-q(1-t)^{-\lambda-1}\big]=0$ in t *viz.*

$$t_1=\frac{1}{1+\left(\frac{q}{p}\right)^{\frac{1}{\lambda+1}}}\in(0,1).$$

Being $h_\theta''(t)=\lambda(\lambda+1)\big[pt^{-\lambda-2}+q(1-t)^{-\lambda-2}\big]>0$ for all $t\in(0,1)$, so is $h_\theta(t_1)$ the global minimum; here

$$h_\theta(t_1)=\min_{0<t<1}h_\theta(t)=\big(p^{\frac{1}{\lambda+1}}+p^{\frac{1}{\lambda+1}}\big)^{\lambda+1},$$

accordingly

$$K_{\nu+\frac12}(h_\theta(t_1))=K_{\nu+\frac12}\Big(\big(p^{\frac{1}{\lambda+1}}+q^{\frac{1}{\lambda+1}}\big)^{\lambda+1}\Big). \tag{4.19}$$

Put (4.18) and (4.19) into (4.17) and abbreviate

$$\Omega_\nu^\lambda(p,q)=\frac{\sqrt{2pq}\,K_{\nu+\frac12}\Big(\big(p^{\frac{1}{\lambda+1}}+q^{\frac{1}{\lambda+1}}\big)^{\lambda+1}\Big)}{\sqrt{\pi}\,\big(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}}\big)^{\frac{\lambda-1}{2}}},$$

which gives

$$\begin{aligned}\left|X_{1:1;1}^{1:0;1}\left[\begin{matrix}a:-;b'\\ c:d;d'\end{matrix}\,\middle|\,-x,y;\Theta\right]\right|&\leqslant\Big[\Omega_\nu^\lambda(p,q)\Big]^2\,\frac{\Gamma(d)\,|x|^{\frac{1-d}{2}}}{\mathrm{B}(a,c-a)}\,\frac{\mathrm{B}\big(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}\big(b',d'-b'\big)}\\ &\quad\cdot\int_0^1 t^{a-d-\frac{\lambda}{2}}(1-t)^{c-a-\frac{\lambda}{2}-1}\big|J_{d-1}(2\sqrt{xt})\big|\Phi(b'-\tfrac{\lambda}{2};d'-\lambda;|y|t)\mathrm{d}t.\end{aligned} \tag{4.20}$$

By virtue of Luke's upper bound (4.8) for Kummer's confluent hypergeometric function $\Phi(\cdot)$ in (4.20) we conclude

$$\left|X_{1:1;1}^{1:0;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\middle| -x,y;\Theta\right]\right| \leqslant \left[\Omega_\nu^\lambda(p,q)\right]^2 \frac{\Gamma(d)\,|x|^{\frac{1-d}{2}}}{\mathrm{B}(a,c-a)} \frac{\mathrm{B}\big(b'-\frac{\lambda}{2}, d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}\big(b',d'-b'\big)} \cdot \int_0^1 t^{a-d-\frac{\lambda}{2}}(1-t)^{c-a-\frac{\lambda}{2}-1}\,\big|J_{d-1}(2\sqrt{xt})\big|\left[1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\big(1-\mathrm{e}^{|y|t}\big)\right]\mathrm{d}t\,. \tag{4.21}$$

Using the first von Lommel's bound in (4.9) and evaluating the (4.21) we find

$$\left|X_{1:1;1}^{1:0;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\middle| -x,y;\Theta\right]\right| \leqslant \left[\Omega_\nu^\lambda(p,q)\right]^2 \frac{\Gamma(d)\,|x|^{\frac{1-d}{2}}}{\mathrm{B}(a,c-a)} \frac{\mathrm{B}\big(b'-\frac{\lambda}{2}, d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}\big(b',d'-b'\big)} \cdot \int_0^1 t^{a-d-\frac{\lambda}{2}}(1-t)^{c-a-\frac{\lambda}{2}-1}\left[1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\big(1-\mathrm{e}^{|y|t}\big)\right]\mathrm{d}t\,.$$

By utilizing the first Landau's result (4.11) we readily derive the inequality (4.16) in a similar manner. □

As to the counterpart result in which the modified Bessel function of the first kind I_{d-1} has important role we should start with the integral representation formula (3.12) of Corollary 4.1.

THEOREM 14. *For all* $p,q>0$ *and* $\lambda\in(0,1)\cup(1,\infty)$, $\nu\in\mathbb{R}$ *and to hold*

$$\min\{a,c-a\}>\frac{\lambda}{2};\quad \min\{b',d'-b'\}>\frac{\lambda}{2},$$

we have for $x>0$

$$\left|X_{1:1;1}^{1:0;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right]\right| \leqslant \left[\Omega_\nu^\lambda(p,q)\right]^2 \frac{\mathrm{B}\big(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\big)\,\mathrm{B}(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2})}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \cdot\cosh(2\sqrt{x})\left[1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}+\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\Phi\big(a-\tfrac{\lambda}{2},c-\lambda;|y|\big)\right]. \tag{4.22}$$

Moreover, for the same parametric range and for $0<x<\frac14$, $2\sqrt{x}+|y|<1$, *it is*

$$\left|X_{1:1;1}^{1:0;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right]\right| \leqslant \left[\Omega_\nu^\lambda(p,q)\right]^2 \frac{\mathrm{B}\big(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\big)\mathrm{B}\big(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \cdot\left\{\big(1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\big)\Phi\big(a-\tfrac{\lambda}{2};c-\lambda;2\sqrt{x}\big)+\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\Phi\big(a-\tfrac{\lambda}{2};c-\lambda;2\sqrt{x}+|y|\big)\right\}. \tag{4.23}$$

Proof. As $x>0$, estimating the integral (3.12) with the Luke's bound (4.15) we get

$$\left|X_{1:1;1}^{1:0;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\middle| x,y;\Theta\right]\right| \leqslant \sqrt{\frac{2}{\pi}}\frac{\Gamma(d)\,x^{\frac{1-d}{2}}}{\mathrm{B}(a,c-a)}\int_0^1 t^{a-d}(1-t)^{c-a-1}\sqrt{h_\Theta(t)} \cdot K_{\nu+\frac12}(h_\Theta(t))\,|I_{d-1}(2\sqrt{x}\,t)|\,|\Phi_{p,q,\nu}^\lambda(b';d';yt)|\,\mathrm{d}t$$

$$\leqslant \sqrt{\frac{2}{\pi}} \frac{1}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \sqrt{h_\theta(t)} \cdot K_{\nu+\frac{1}{2}}(h_\theta(t)) \cosh(2\sqrt{x}t)\, |\Phi^{\lambda}_{p,q,\nu}(b';d';yt)|\, \mathrm{d}t.$$

Apply now (4.2) to the extended Kummer function's modulus, and treat the resulting Kummer function by another Luke's bound (4.8). This results in:

$$\begin{aligned}
\left|X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b' \\ c:d;d'\end{matrix}\,\middle|\, x,y;\Theta\right]\right| &\leqslant \frac{\left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \mathrm{B}(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2})}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \\
&\quad\cdot \int_0^1 t^{a-\frac{\lambda}{2}-1}(1-t)^{c-a-\frac{\lambda}{2}-1} \cosh(2\sqrt{x}t)\, \Phi(b'-\tfrac{\lambda}{2};d'-\lambda;|y|t)\, \mathrm{d}t \\
&\leqslant \frac{\left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \mathrm{B}(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2})}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \\
&\quad\cdot \int_0^1 t^{a-\frac{\lambda}{2}-1}(1-t)^{c-a-\frac{\lambda}{2}-1} \cosh(2\sqrt{x}t) \left[1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left(1-\mathrm{e}^{|y|t}\right)\right] \mathrm{d}t \qquad (4.24)\\
&\leqslant \frac{\left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \mathrm{B}(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2})\cosh(2\sqrt{x})}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \\
&\quad\cdot \int_0^1 t^{a-\frac{\lambda}{2}-1}(1-t)^{c-a-\frac{\lambda}{2}-1} \left[1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left(1-\mathrm{e}^{|y|t}\right)\right] \mathrm{d}t \\
&= \frac{\left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \mathrm{B}(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2})\cosh(2\sqrt{x})}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \mathrm{B}\left(a-\tfrac{\lambda}{2},c-a-\tfrac{\lambda}{2}\right) \\
&\quad\cdot \left[1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}+\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\Phi\left(a-\tfrac{\lambda}{2},c-\lambda;|y|\right)\right].
\end{aligned}$$

Here, we use the obvious estimate

$$\sup_{0<t<1} \cosh(2\sqrt{x}t) = \cosh(2\sqrt{x}), \qquad x>0,$$

whilst the last equality follows by

$$\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{e}^{rx}\,\mathrm{d}x = \mathrm{B}(p,q)\,\Phi(p;p+q;r), \qquad \min\{p,q\}>0,$$

which completes the proof of the first bound in (4.22).

Next, by using the inequality $\cosh(t)\leqslant \mathrm{e}^t$, that is, in our setting $\cosh(2\sqrt{x}t)\leqslant \mathrm{e}^{2\sqrt{x}t}$ for $t\geqslant 0$ in (4.24) for the same parametric range and simplifying, we get the desired second bound (4.23). □

THEOREM 15. *Assume that* $p,q>0$, $\lambda\in(0,1)\cup(1,\infty)$, $\nu\in\mathbb{R}$, *and* $\min\{a,c,b,d-1,b',d'\}>0$ *for which hold the following constraints*

$$\min\{a,c-a\}>\frac{\lambda}{2};\ \min\{b,d-b\}>\frac{\lambda}{2};\ \min\{b',d'-b'\}>\frac{\lambda}{2}.$$

Then, the following bounded inequality holds true for $x \geqslant 0$, $y \in \mathbb{R}$:

$$\left|X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, -x,y;\Theta\right]\right| \leqslant \left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \frac{\mathrm{B}\left(a-\frac{\lambda}{2},c-a-\frac{\lambda}{2}\right)\mathrm{B}\left(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\right)}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')} \cdot\left\{1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left[1-\Phi\left(a-\tfrac{\lambda}{2};c-\lambda;|y|\right)\right]\right\}.$$

Moreover,

$$\left|X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, -x,y,\Theta\right]\right| \leqslant \left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \frac{\Gamma(d)\,\mathrm{B}\left(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\right)}{\mathrm{B}(a,c-a)\mathrm{B}(b',d'-b')}$$
$$\cdot\begin{cases} \dfrac{c_L}{\sqrt[3]{2}\,|x|^{\frac{3d-2}{6}}}\,\mathrm{B}\left(a-d-\frac{\lambda}{2}+\frac{2}{3},c-a-\frac{\lambda}{2}\right) \\ \qquad\cdot\left\{1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left[1-\Phi\left(a-d-\frac{\lambda}{2}+\frac{2}{3};c-d-\lambda+\frac{2}{3};|y|\right)\right]\right\} \\ \dfrac{d_O}{\sqrt{2}\,|x|^{\frac{2d-1}{4}}}\,\mathrm{B}\left(a-d-\frac{\lambda-1}{2},c-a-\frac{\lambda}{2}\right) \\ \qquad\cdot\left\{1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left[1-\Phi\left(a-d-\frac{\lambda-1}{2};c-d-\lambda+\frac{1}{2};|y|\right)\right]\right\} \end{cases},$$

where for Landau's second constant c_L *see* (4.12)*,* d_O *denotes Olenko's constant* (4.14) *and* b_L *stands for the Landau's first constant* (4.11).

Proof. First, we point out that the estimates of Bessel function in (4.10), (4.12) and (4.14) are of the magnitude $|J_{d-1}(t)| \leqslant \mathfrak{C} t^{\kappa}$ where $\mathfrak{C} \in \{[2^{d-1}\Gamma(d)]^{-1}, c_L, d_O\}$ and $\kappa \in \{d-1, -\frac{1}{3}, -\frac{1}{2}\}$, respectively. We also point out that the domain of (4.10) is $t \in \mathbb{R}$, whilst for other estimates holds $t \geqslant 0$. Now, the application of the bound (4.10) to the integrand in (4.21) results in

$$\begin{aligned}\left|X^{1:0;1}_{1:1;1}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, -x,y;\Theta\right]\right| &\leqslant \left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \frac{|x|^{\frac{1-d}{2}}\,\Gamma(d)\,\mathrm{B}\left(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\right)}{\mathrm{B}(a,c-a)\,\mathrm{B}\left(b',d'-b'\right)} \\ &\quad\cdot\int_0^1 t^{a-d-\frac{\lambda}{2}}\,(1-t)^{c-a-\frac{\lambda}{2}-1}\,\left|J_{d-1}(2\sqrt{xt})\right|\left[1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left(1-\mathrm{e}^{|y|t}\right)\right]\mathrm{d}t \\ &\leqslant \mathfrak{C}\left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \frac{\Gamma(d)|x|^{\frac{1-d+\kappa}{2}}}{\mathrm{B}(a,c-a)}\,\frac{\mathrm{B}\left(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\right)}{\mathrm{B}\left(b',d'-b'\right)} \\ &\quad\cdot\int_0^1 t^{a+\kappa-d-\frac{\lambda}{2}}\,(1-t)^{c-a-\frac{\lambda}{2}-1}\left[1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\left(1-\mathrm{e}^{|y|t}\right)\right]\mathrm{d}t \\ &= \mathfrak{C}\left[\Omega^{\lambda}_{\nu}(p,q)\right]^2 \frac{\Gamma(d)|x|^{\frac{1-d+\kappa}{2}}}{\mathrm{B}(a,c-a)}\,\frac{\mathrm{B}\left(b'-\frac{\lambda}{2},d'-b'-\frac{\lambda}{2}\right)}{\mathrm{B}\left(b',d'-b'\right)}\left\{\left(1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\right)\right. \\ &\quad\cdot\int_0^1 t^{a+\kappa-d-\frac{\lambda}{2}}\,(1-t)^{c-a-\frac{\lambda}{2}-1}\,\mathrm{d}t \\ &\quad\left.+\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\int_0^1 t^{a+\kappa-d-\frac{\lambda}{2}}\,(1-t)^{c-a-\frac{\lambda}{2}-1}\,\mathrm{e}^{|y|t}\,\mathrm{d}t\right\}\end{aligned}$$

$$
\begin{aligned}
&= \mathfrak{C}\left[\Omega_\nu^\lambda(p,q)\right]^2 \frac{\Gamma(d)|x|^{\frac{1-d+\kappa}{2}}}{\mathrm{B}(a,c-a)} \frac{\mathrm{B}\big(b'-\frac{\lambda}{2}, d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}\big(b',d'-b'\big)} \\
&\qquad \cdot \mathrm{B}\big(a+\kappa-d-\tfrac{\lambda}{2}+1, c-a-\tfrac{\lambda}{2}\big) \\
&\qquad \cdot \left\{1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}+\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\Phi\big(a+\kappa-d-\tfrac{\lambda}{2}+1; c+\kappa-d-\lambda+1; |y|\big)\right\} \\
&= \mathfrak{C}\left[\Omega_\nu^\lambda(p,q)\right]^2 \frac{\Gamma(d)|x|^{\frac{1-d+\kappa}{2}}}{\mathrm{B}(a,c-a)} \frac{\mathrm{B}\big(b'-\frac{\lambda}{2}, d'-b'-\frac{\lambda}{2}\big)}{\mathrm{B}\big(b',d'-b'\big)} \\
&\qquad \cdot \mathrm{B}\big(a+\kappa-d-\tfrac{\lambda}{2}+1, c-a-\tfrac{\lambda}{2}\big) \\
&\qquad \cdot \left\{1-\frac{b'-\frac{\lambda}{2}}{d'-\lambda}\big(1-\Phi\big(a+\kappa-d-\tfrac{\lambda}{2}+1; c+\kappa-d-\lambda+1; |y|\big)\big)\right\}.
\end{aligned}
$$

Than, inserting $(\mathfrak{C},\kappa) \in \{(1/[2^{d-1}\Gamma(d)], d-1), (c_L, -\frac{1}{3}), (d_O, -\frac{1}{2})\}$, respectively, we get the bounds affiliated to the Minakshisundaram and Szász, the second Landau's and Olenko's estimates. □

THEOREM 16. *Following bounded inequalities hold true:*

$$
\begin{aligned}
\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| &\leqslant \Omega_\nu^\lambda(p,q)\frac{|x|^{\frac{1-d}{2}}\,|y|^{\frac{1-d'}{2}}\,\Gamma(d)\,\Gamma(d')}{\mathrm{B}(a,c-a)} \\
&\qquad \cdot \mathrm{B}\Big(a-d-\tfrac{d'+\lambda-3}{2},\, c-a-\tfrac{\lambda}{2}\Big) \\
\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| &\leqslant \Omega_\nu^\lambda(p,q)\frac{b_L'\,b_L''\,|x|^{\frac{1-d}{2}}\,|y|^{\frac{1-d'}{2}}\,\Gamma(d)\,\Gamma(d')}{\sqrt[3]{d-1}\,\sqrt[3]{d'-1}\,\mathrm{B}(a,c-a)} \\
&\qquad \cdot \mathrm{B}\Big(a-d-\tfrac{d'+\lambda-3}{2},\, c-a-\tfrac{\lambda}{2}\Big),
\end{aligned} \tag{4.25}
$$

as $a,c>0$*;* $d,d'>1$*;* $2(a-d)+3>d'+\lambda$*,* $2(c-a)>\lambda$*. For* $\min\{a,c-a\}>\frac{\lambda}{2}>0$ *it is*

$$
\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| \leqslant \Omega_\nu^\lambda(p,q)\frac{\mathrm{B}\big(a-\frac{\lambda}{2},\, c-a-\frac{\lambda}{2}\big)}{\mathrm{B}(a,c-a)},
$$

whilst when $2(a-d)>d'+\lambda-2, 2(c-a)-\lambda>0$*, we have*

$$
\begin{aligned}
\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| &\leqslant \Omega_\nu^\lambda(p,q)\frac{\Gamma(d)\,\Gamma(d')}{\mathrm{B}(a,c-a)}\frac{c_L^2\,|x|^{\frac{2-3d}{6}}\,|y|^{\frac{2-3d'}{6}}}{\sqrt[3]{4}} \\
&\qquad \cdot \mathrm{B}\big(a-d-\tfrac{d'+\lambda}{2}+1,\, c-a-\tfrac{\lambda}{2}\big),
\end{aligned}
$$

and $\min\{4(a-d)-2(d'+\lambda)+3, 2(c-a)-\lambda\}>0$ *implies the bound*

$$
\begin{aligned}
\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;- \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| &\leqslant \Omega_\nu^\lambda(p,q)\frac{d_O^2\,\Gamma(d)\,\Gamma(d')}{2\,\mathrm{B}(a,c-a)}\,|x|^{\frac{1-2d}{4}}\,|y|^{\frac{1-2d'}{4}} \\
&\qquad \cdot \mathrm{B}\big(a-d-\tfrac{d'+\lambda}{2}+\tfrac{3}{4},\, c-a-\tfrac{\lambda}{2}\big).
\end{aligned}
$$

In this theorem throughout $\min\{a,c-a\},d,d'>0$*, unless otherwise stated.*

Proof. Taking modulus of the both side on the integral representations (3.15), we obtain

$$\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| \leqslant \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(d)\,\Gamma(d')\,|x|^{\frac{1-d}{2}}|y|^{\frac{1-d'}{2}}}{\mathrm{B}(a,c-a)}\int_0^1 \frac{t^{a-d-\frac{d'+\lambda-1}{2}}}{(1-t)^{a-c+\frac{\lambda}{2}+1}} \cdot \sqrt{h_1(t)}\,K_{\nu+\frac{1}{2}}\big(h_\theta(t)\big)\,|J_{d-1}(2\sqrt{xt})\,J_{d'-1}(2\sqrt{yt})|\,\mathrm{d}t. \tag{4.26}$$

Now estimating $h_1(t)$ and $h_\theta(t)$ in the integrand similarly to Theorem 13 and substituting (4.18) and (4.19) in (4.26), writing shorthand

$$\Omega_\nu^\lambda(p,q) = \frac{\sqrt{2pq}\,K_{\nu+\frac{1}{2}}\left(\left(p^{\frac{1}{\lambda+1}}+q^{\frac{1}{\lambda+1}}\right)^{\lambda+1}\right)}{\sqrt{\pi}\left(p^{\frac{1}{\lambda-1}}+q^{\frac{1}{\lambda-1}}\right)^{\frac{\lambda-1}{2}}},$$

we obtain

$$\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| \leqslant \Omega_\nu^\lambda(p,q)\frac{\Gamma(d)\,\Gamma(d')\,|x|^{\frac{1-d}{2}}|y|^{\frac{1-d'}{2}}}{\mathrm{B}(a,c-a)} \cdot\int_0^1 t^{a-d-\frac{d'+\lambda-1}{2}}\,(1-t)^{c-a-\frac{\lambda}{2}-1}\,|J_{d-1}(2\sqrt{xt})\,J_{d'-1}(2\sqrt{yt})|\,\mathrm{d}t. \tag{4.27}$$

Using the first one of von Lommel's bounds in (4.9) we find that

$$\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| \leqslant \Omega_\nu^\lambda(p,q)\frac{\Gamma(d)\Gamma(d')\,|x|^{\frac{1-d}{2}}|y|^{\frac{1-d'}{2}}}{\mathrm{B}(a,c-a)} \cdot\mathrm{B}\left(a-d-\tfrac{d'-\lambda}{2}+\tfrac{3}{2},c-a-\tfrac{\lambda}{2}\right).$$

Next, by utilizing the first Landau's result (4.9) we deduce the inequality (4.25) in a similar manner.

Now, re-calling that the estimates of Bessel function in (4.10), (4.12) and (4.14) have magnitude $|J_{d-1}(t)| \leqslant \mathfrak{C}|t|^\kappa, |J_{d'-1}(t)| \leqslant \mathfrak{C}_1|t|^{\kappa_1}$ where

$$\mathfrak{C} \in \{[2^{d-1}\Gamma(d)]^{-1}, c_L, d_O\}; \quad \mathfrak{C}_1 \in \{[2^{d'-1}\Gamma(d')]^{-1}, c_L, d_O\};$$

and

$$\kappa \in \{d-1,-\tfrac{1}{3},-\tfrac{1}{2}\}; \quad \kappa_1 \in \{d'-1,-\tfrac{1}{6},-\tfrac{1}{4}\},$$

respectively. Now, the application of these estimates (4.10) to the integral (4.27) results gives

$$\left|X_{1:1;1}^{1:0;0}\left[\begin{matrix} a:-;b' \\ c:d;d' \end{matrix}\,\middle|\, -x,-y;\Theta\right]\right| \leqslant \Omega_\nu^\lambda(p,q)\frac{\Gamma(d)\,\Gamma(d')\,|x|^{\frac{1-d}{2}}|y|^{\frac{1-d'}{2}}}{\mathrm{B}(a,c-a)} \cdot\int_0^1 t^{a-d-\frac{d'+\lambda-1}{2}}\,(1-t)^{c-a-\frac{\lambda}{2}-1}\,|J_{d-1}(2\sqrt{xt})\,J_{d'-1}(2\sqrt{yt})|\,\mathrm{d}t$$

$$\leqslant \mathfrak{C}\,\mathfrak{C}_1\,\Omega_\nu^\lambda(p,q)\,\frac{\Gamma(d)\,\Gamma(d')\,|x|^{\frac{1-d+\kappa}{2}}\,|y|^{\frac{1-d'+\kappa_1}{2}}}{\mathrm{B}(a,c-a)}\int_0^1 \frac{t^{a+\kappa+\frac{\kappa_1}{2}-d-\frac{d'+\lambda-1}{2}}}{(1-t)^{a-c+\frac{\lambda}{2}+1}}\,\mathrm{d}t$$

$$= \mathfrak{C}\,\mathfrak{C}_1\,\Omega_\nu^\lambda(p,q)\,\frac{\Gamma(d)\Gamma(d')\,|x|^{\frac{1-d+\kappa}{2}}\,|y|^{\frac{1-d'+\kappa_1}{2}}}{\mathrm{B}(a,c-a)} \cdot \mathrm{B}\big(a+\kappa+\tfrac{\kappa_1}{2}-d-\tfrac{d'+\lambda-1}{2}+1, c-a-\tfrac{\lambda}{2}\big).$$

Than, taking either $\mathfrak{C} = [2^{d-1}\Gamma(d)]^{-1}$, $\mathfrak{C}_1 = [2^{d'-1}\Gamma(d')]^{-1}$, c_L or d_O and $\kappa \in \{d-1, -\frac{1}{3}, -\frac{1}{2}\}$, $\kappa_1 \in \{d'-1, -\frac{1}{6}, -\frac{1}{4}\}$ mutually, we realize the bounds affiliated to the Minakshisundaram and Szász, the second Landau's and Olenko's estimates, respectively. $\square$

5. Applications to statistical distribution

Special functions are important in studying probability distribution and statistical inference (see for instance [2, Chapter 17], [18, Chapters 6 and 8], [4, 7–9, 11]). Recently, researchers have been studying McKay Bessel-type distributions, which are related to special functions, such as Horn's confluent functions (see [4, 7, 8, 16]). The extended Exton's double hypergeometric function (3.16) is expected to have many applications, similar to the generalized Beta and Gamma functions. One potential application is in statistics, and it can also be applied in inequality theory to derive novel bilateral bounds for the generalized Exton's function $X_{0:1;1}^{1:1;1}\left[\begin{matrix} a:b;b' \\ -:d;d' \end{matrix}\middle| x,y;\Theta\right]$ using probabilistic methods.

Consider the random variable ξ defined on a standard probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, where Ω is a sample space, $\mathfrak{F}$ is the related sigma algebra on Ω, and $\mathbf{P}$ is a probability function characterized by the following probability density function throughout (*abbr.* density):

$$f_\xi(u) = \begin{cases} C_{p,q}(\kappa,a)\,u^{a-1}\mathrm{e}^{-\kappa u}\,\Phi_{p,q,\nu}^\lambda(b;d;xu^2)\,\Phi_{p,q,\nu}^\lambda(b';d';yu), & u>0 \\ 0 & \text{elsewhere}, \end{cases}$$

where it is assumed that $\Re(\kappa)>0$, $\Re(a)>0$, the positive arguments (x,y), and the parameters p,q,ν,λ and u,v,v' are suitably constrained so that $f_\xi(u)$ remains non-negative. By the Theorem 7, that is, Eq. (3.17) the normalization constant reads

$$C_{p,q}(\kappa,a) = \frac{\kappa^a}{\Gamma(a)\,X_{0:1;1}^{1:1;1}\left[\begin{matrix} a:b;b' \\ -:d;d' \end{matrix}\middle| \frac{x}{\kappa^2},\frac{y}{\kappa};\Theta\right]}.$$

We define the generalized Horn's gamma distribution of the random variable (*abbr.* r.v.) ξ as $\mathrm{GHG}(\theta)$, where $\theta = (p,q,\nu,\lambda;a,b,b';-,d,d',\kappa;x,y)$ is the parameter vector. Alternatively, we denote this as $\xi \sim f_\xi(u)$. Hereafter, we will derive some statistical functions for the r.v. $\xi \sim \mathrm{GHG}(\theta)$.

5.1. Raw moments and Turán inequalities

The sth fractional-order moments $m_s,\, s>0$ equal

$$m_s = \mathsf{E}\xi^s = \int_0^\infty u^s f_\xi(u)\,\mathrm{d}u = \frac{(a)_s}{\kappa^s}\,\frac{X^{1:1;1}_{0:1;1}\left[\begin{matrix} a+s : b; b' \\ - : d; d'\end{matrix}\,\middle|\, \frac{x}{\kappa^2}, \frac{y}{\kappa}; \Theta\right]}{X^{1:1;1}_{0:1;1}\left[\begin{matrix} a : b; b' \\ - : d; d'\end{matrix}\,\middle|\, \frac{x}{\kappa^2}, \frac{y}{\kappa}; \Theta\right]}\,. \tag{5.1}$$

As the first application of (5.1), we derive a Turán-type inequality for the extended Exton's double hypergeometric function $X^{1:1;1}_{0:1;1}[\cdot]$ by virtue of the moment inequality, which holds for the nonnegative r.v. $\xi \sim f_\xi(u)$. Lukacs reported on the moment inequality [16, p. 28, Equation (1.4.6)]

$$m^2_{s+r} \leqslant m_s\, m_{s+2r}, \qquad \min\{s,r\} > 0\,. \tag{5.2}$$

By inserting the expression (5.1) in (5.2), we obtain for all $2s > -a$, $s+2r > -a$ the bounding inequality

$$\big\{X^{1:1;1}_{0:1;1}[s+r]\big\}^2 \leqslant \frac{\Gamma(a+s)\,\Gamma(a+s+2r)}{\Gamma^2(a+s+r)}\, X^{1:1;1}_{0:1;1}[s]\cdot X^{1:1;1}_{0:1;1}[s+2r],$$

where the shorthand

$$X^{1:1;1}_{0:1;1}[\sigma] = X^{1:1;1}_{0:1;1}\left[\begin{matrix} a+\sigma : b; b' \\ - : d; d'\end{matrix}\,\middle|\, \frac{x}{\kappa^2}, \frac{y}{\kappa}; \Theta\right]$$

is used. Also, another statement by Lukacs [16, p. 393, a)] asserts that for $0 < r \leqslant s$, the moment inequality $m^2_{s+r} \leqslant m_{2s}\, m_{2r}$ holds, which can be inferred using the Cauchy–Bunyakovsky–Schwarz inequality. This inequality implies a variant of the Turán-type inequality, *viz.*

$$\big\{X^{1:1;1}_{0:1;1}[s+r]\big\}^2 \leqslant \frac{\Gamma(a+2s)\,\Gamma(a+2r)}{\Gamma^2(a+s+r)}\, X^{1:1;1}_{0:1;1}[2s]\cdot X^{1:1;1}_{0:1;1}[2r]\,, \qquad 2\min\{s,r\} > -a.$$

5.2. Characteristic function

The Fourier transform of the density $f_\xi(t)$ is the characteristic function (ch.f.) $\varphi_\xi(t)$ of the r.v. ξ. Hence,

$$\begin{aligned}\varphi_\xi(t) &= \mathsf{E}\mathrm{e}^{\mathrm{i}t\xi} = \int_0^\infty \mathrm{e}^{\mathrm{i}tu} f_\xi(u)\,\mathrm{d}u \\ &= C_{p,q}(\kappa,a)\int_0^\infty \mathrm{e}^{-(\kappa-\mathrm{i}t)u}\, u^{a-1}\, \Phi^\lambda_{p,q,\nu}(b;d;xu^2)\, \Phi^\lambda_{p,q,\nu}(b';d';yu)\,\mathrm{d}u\,.\end{aligned}$$

Therefore, again by virtue of Theorem 7, the ch.f. becomes

$$\varphi_\xi(t) = \frac{\kappa^a X^{1:1;1}_{0:1;1}\left[\begin{matrix} a : b; b' \\ - : d; d'\end{matrix}\,\middle|\, \frac{x}{(\kappa-\mathrm{i}t)^2}, \frac{y}{\kappa-\mathrm{i}t}; \Theta\right]}{(\kappa-\mathrm{i}t)^a X^{1:1;1}_{0:1;1}\left[\begin{matrix} a : b; b' \\ - : d; d'\end{matrix}\,\middle|\, \frac{x}{\kappa^2}, \frac{y}{\kappa}; \Theta\right]}\,. \tag{5.3}$$

The next summation result establishes a connection between the density and the ch.f. through the corresponding integer-order moments. For this result we need the definition of the multiple Srivastava–Daoust generalized Lauricella $F_D^{(n)}$ series [30, p. 37, Eq. (21)]

$$F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\left(\begin{matrix}[(a):\theta^{(1)},\cdots,\theta^{(n)}]:[(b^{(1)}):\varphi^{(1)}];\cdots;[(b^{(n)}):\varphi^{(n)}]\\ [(c):\psi^{(1)};\cdots;\psi^{(n)}]:[(d^{(1)}):\delta^{(1)}];\cdots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\,\middle|\, x_1,\ldots,x_n\right)$$
$$= \sum_{\boldsymbol{m}\geqslant 0} \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\varphi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\varphi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}, \tag{5.4}$$

where $\boldsymbol{m} := (m_1,\cdots,m_n)$ and the parameters satisfy

$$\theta_1^{(1)},\cdots,\theta_A^{(1)},\cdots,\delta_1^{(n)},\cdots,\delta_{D^{(n)}}^{(n)} > 0.$$

We write (a) for the sequence of A parameters $a_1,\cdots,a_A$, with similar interpretations for $(b^{(1)}),\cdots,(c),\cdots,(d^{(n)})$. Empty products should be interpreted as unity.

The transformation formula of the specific Srivastava–Daoust triple generalized Lauricella hypergeometric function into the Exton's double hypergeometric series follows.

THEOREM 17. *For any positive parameter vector $\boldsymbol{\Theta} = (p,q,\nu,\lambda;a,b,b';d,d',\kappa;x,y)$ it is*

$$\frac{\kappa^a}{(\kappa-\mathrm{i}t)^a}X^{1:1;1}_{0:1;1}\left[\begin{matrix}a:b;b'\\ -:d;d'\end{matrix}\,\middle|\,\frac{x}{(\kappa-\mathrm{i}t)^2},\frac{y}{\kappa-\mathrm{i}t};\Theta\right]$$
$$= F^{1:1;1}_{0:1;1}\left(\begin{matrix}[a:1,2,1]:[b:1];[b':1]\\ -:[d:1];[d':1]\end{matrix}\,\middle|\,\frac{\mathrm{i}t}{\kappa},\frac{x}{\kappa^2},\frac{y}{\kappa}\right),$$

where $F^{1:1;1}_{0:1;1}$ stands for the Srivastava–Daoust triple generalized Lauricella hypergeometric F function.

Proof. The Maclaurin series of the ch.f. reads [16, p. 41]

$$\varphi_\xi(t) = \sum_{n\geqslant 0} m_n\frac{(\mathrm{i}t)^n}{n!}.$$

Inserting (5.1) into this expansion routine steps lead to the assertion. Indeed, we have

$$\varphi_\xi(t)\cdot X^{1:1;1}_{0:1;1}\left[\begin{matrix}a:b;b'\\ -:d;d'\end{matrix}\,\middle|\,\frac{x}{\kappa^2},\frac{y}{\kappa};\Theta\right] = \sum_{n\geqslant 0}\frac{(a)_n}{\kappa^n}X^{1:1;1}_{0:1;1}\left[\begin{matrix}a+n:b;b'\\ -:d;d'\end{matrix}\,\middle|\,\frac{x}{\kappa^2},\frac{y}{\kappa};\Theta\right]\frac{(\mathrm{i}t)^n}{n!}$$
$$= \sum_{n,k,m\geqslant 0}\frac{(a)_n(a+n)_{2k+m}(b)_k(b')_m}{(d)_k(d')_m}\frac{\left(\frac{\mathrm{i}t}{\kappa}\right)^n}{n!}\frac{\left(\frac{x}{\kappa^2}\right)^k}{k!}\frac{\left(\frac{y}{\kappa}\right)^m}{m!}$$

$$= \sum_{n,k,m\geqslant 0} \frac{(a)_{n+2k+m}(b)_k(b')_m}{(d)_k(d')_m} \frac{\left(\frac{\mathrm{i}t}{\kappa}\right)^n}{n!} \frac{\left(\frac{x}{\kappa^2}\right)^k}{k!} \frac{\left(\frac{y}{\kappa}\right)^m}{m!}$$

$$= F^{1:1;1}_{0:1;1}\left(\begin{matrix} [a:1,2,1]:[b:1];[b':1] \\ -:[d:1];[d':1] \end{matrix} \middle| \frac{\mathrm{i}t}{\kappa}, \frac{x}{\kappa^2}, \frac{y}{\kappa} \right),$$

where we applied the property $(a)_j(a+j)_l = (a)_{j+l}$ of the product of two Pochhammer symbols. Putting (5.3) into the left-hand-side expression, we complete the proof. □

6. Concluding remarks and observations

Present research outcomes introducing the four parametric extension of certain Exton's double hypergeometric function $X^{A:B;B'}_{C:D;D'}[x,y,\Theta]$; $\Theta = (p,q,\nu,\lambda)$ by utilizing the definition of extended Beta function $\mathrm{B}^{\lambda}_{p,q,\nu}(s,t)$ in (1.2) which involves the Macdonald kernel $K_{n+\frac{1}{2}}$ in the kernel of integral. Then we systematically developed the associated integral representations including Euler's and Laplace-Mellin type, as well as certain integral representations involving Bessel $J_\nu(z)$ and modified Bessel functions $I_\nu(z)$ along with some recurrence formulae. By applying several functional upper bounds such as Luke's, von Lommel's, Minakshisundaram and Szász and Olenko bounds, we derived several bounds for defined extended Exton's double hypergeometric functions $X^{A:B;B'}_{C:D;D'}[x,y;\Theta]$. Finally, as an application, we introduced a new probability distribution building the density function in terms of specific extended Exton's function and studied moments and characteristic function. Using related extended Kummer and Horn functions moment inequalities of Turán type are also proved. It is worth to mention here the recent articles by Jankov Maširević and Pogány [10] and Pogány [27] in which another type functional inequality results are presented for the Exton's X functions, inferred by probabilistic considerations.

We observe that for Exton's double hypergeometric function $X^{1:0;0}_{1:1;1}[x,y;\Theta]$, we can derive similar results parallel to Theorem 16 for (3.15) involving product of modified Bessel function of the first kind I_{d-1}; however, this study we left to the interested readers. Further, for above defined certain Exton's double hypergeometric function $X^{A:B;B'}_{C:D;D'}[x,y,\Theta]$, monotonicity properties, log convexity and generating function are under investigations.

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