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ON BERNSTEIN–TYPE ESTIMATES FOR LINEAR OPERATORS ACTING ON POLYNOMIALS

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Abstract. We establish new Bernstein-type inequalities for a class of linear operator T_g acting on complex polynomials, generalizing classical results in geometric function theory. Notably, we introduce a family of operators T_g induced by polynomials g with all zeros in the open unit disk, whose action is governed by the Schur-Szegő composition theorem and derive sharp bounds relating the maximum norms of $T_g[(P\circ f_R)(z)-\alpha(P\circ f_r)(z)]$, where $f_a(z)=az+S$, $a\in\{r,R\}$, and P on the unit disk. Our framework offers a unified extension of foundational inequalities due to Bernstein, Lax, Ankeny-Rivlin, and Aziz-Dawood, specifically for polynomials with restricted zeros. The results not only encapsulate known theorems as special cases but also introduce new bounds for a wider class of operators. This study enhances the connection between operator theory and polynomial inequalities, furnishing tools with potential applications in approximation theory and related domains.

1. Introduction

Polynomial inequalities occupy a central position in both geometric function theory and approximation theory, with Bernstein-type inequalities representing some of the most foundational results in the field. Over time, considerable effort has been devoted to refining and generalizing these inequalities, resulting in numerous important contributions (see, e.g., [2,4,10,14,15,17–19,21]). In this paper, we derive new Bernstein-type inequalities for a class of linear operators known as B_n-operators, emphasising their connection to the maximum norm of polynomials on the unit disk.

Let $\mathcal{P}_n$ denote the class of complex polynomials $p(z):=\sum_{j=0}^{n}a_jz^j$ of degree n. The classical Bernstein inequality [3] states that for any $p\in\mathcal{P}_n$,

$$\max_{|z|=1}|p'(z)|\leqslant n\max_{|z|=1}|p(z)|, \tag{1}$$

with equality attained for $p(z)=\zeta z^n$, where $\zeta\neq 0$. A related result, derived from the Maximum Modulus Principle, gives for $R>1$,

$$\max_{|z|=1}|p(Rz)|\leqslant R^n\max_{|z|=1}|p(z)|. \tag{2}$$

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Equality in (2) again holds for $p(z) = \zeta z^n$, where $\zeta \neq 0$. Both the inequalities can be improved if we restrict ourselves to the class of polynomials that have no zeros in $|z| < 1$. In fact, if $p \in \mathcal{P}_n$ and does not vanish in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)| \tag{3}$$

and for $R > 1$,

$$\max_{|z|=1} |p(Rz)| \leqslant \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|. \tag{4}$$

The inequality (3) was conjectured by Erdős and proved by Lax [9], while (4) was established by Ankeny and Rivlin [1]. Further improvements were obtained by Aziz and Dawood [2] who proved:

If $p \in \mathcal{P}_n$ having no zero in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \left[\max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right] \tag{5}$$

and for $R > 1$,

$$\max_{|z|=1} |p(Rz)| \leqslant \frac{R^n + 1}{2} \max_{|z|=1} |p(z)| - \frac{R^n - 1}{2} \min_{|z|=1} |p(z)|. \tag{6}$$

Over the past several decades, many mathematicians have contributed to the development of various versions and generalizations of the aforementioned inequalities. These extensions have taken multiple directions, encompassing different norms and diverse classes of functions. Notable contributions in this area include the works of Milovanović et al. [13] and Rahman and Schmeisser [21], among others.

As a generalization of inequality (2), we have the following inequality for reference see [16].

THEOREM A. *If $p \in \mathcal{P}_n$, then for every $|\beta| \leqslant 1$ and $R \geqslant r \geqslant 1$, we have for $|z| \geqslant 1$,*

$$|p(Rz) - \beta p(rz)| \leqslant |R^n - \beta r^n| |z|^n \max_{|z|=1} |p(z)|. \tag{7}$$

Equality in (7) holds for $p(z) = \gamma z^n$, $\gamma \neq 0$.

A key observation in the study of Bernstein-type inequalities is that linear operators preserving certain zero-distribution properties – termed B_n-operators – also satisfy analogous inequalities. Rahman and Schmeisser [21, p. 538] demonstrated that any linear operator T mapping polynomials in $\mathcal{P}_n$ to polynomials in $\mathcal{P}_n$ while preserving zeros in the closed unit disk must obey a Bernstein-type inequality. Examples of such operators include:

- The derivative operator (by the Gauss-Lucas Theorem),
- The polar derivative $D_\alpha f(z) := nf(z) - (z - \alpha) f'(z)$ (by Laguerre's Theorem [8, p. 38]),

- The operator $T[p(z)] = zp'(z) + \frac{n\beta}{2}p(z)$ for $|\beta| \leqslant 1$ (introduced by Jain [7]).

It is natural to investigate characterizations of operators that are both linear and zero-preserving, along with the corresponding Bernstein-type inequalities they satisfy. In this paper, we focus on a particular class of such operators, known as B_n-operators, and establish several Bernstein-type inequalities associated with them.

Let $g(z) := \sum_{\nu=0}^{n} \binom{n}{\nu} \lambda_\nu z^\nu$ be a polynomial of degree n having all zeros in the open unit disk. By a consequence of Schur-Szegő composition Theorem (Lemma 1), the operator $T_g[f]$, which associates with the polynomial $f(z) := \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ the polynomial

$$T_g[f(z)] := \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu \lambda_\nu z^\nu \tag{8}$$

is a B_n operator. If $p \in \mathcal{P}_n$ is a nth degree polynomial, one can easily see that for $g(z) = \sum_{j=0}^{n} \binom{n}{j} j z^j$, $T_g[p(z)] = zp'(z)$ and for $g(z) = \sum_{j=0}^{n} \binom{n}{j} z^j$, $T_g[p(Rz)] = p(Rz)$.

It is known that $T_g[f]$ is a linear operator (for reference see [20]).

Gulzar and Rather [6] extended inequalities (1), (2), (3) and (4) to the operator T_g. Recently, Qasim and Mir [20] extended few results including Theorem A to the operator T_g. As an extension of Theorem A they proved:

THEOREM B. *If $p \in \mathcal{P}_n$, then for $R > r \geqslant 1$, $|\alpha| \leqslant 1$ and $|z| \geqslant 1$, we have*

$$\left|T_g[p(Rz)] - \alpha T_g[p(rz)]\right| \leqslant |R^n - \alpha r^n| \, |T_g[z^n]| \max_{|z|=1} |p(z)|. \tag{9}$$

In this paper, we look at the bound of operator T_g operating on more general polynomial $p(Rz+S) - \alpha p(rz+S)$. Our results encompass many well-known outcomes as special cases, highlighting the broader applicability of the derived inequalities. Many known results are obtained as a special case from our results. The paper is organized as follows. Auxiliary results are given in Section 2, and the main ones in Section 3. Proofs of the main results are presented in Section 4, while a brief conclusion is given in the last section.

2. Auxiliary results

To establish our main theorems, we require the following lemmas. Our first result follows directly from the Schur-Szegő composition theorem [12, p. 66].

LEMMA 1. *Let f and g are two polynomials of degree n. If all the zeros of f lie in $|z| \leqslant r$ and all the zeros of g lie in $|z| \leqslant s$, then all the zeros of $T_g[f]$ lie in $|z| \leqslant rs$.*

LEMMA 2. *If $p \in \mathcal{P}_n$ has all its zeros in $|z| \leqslant 1$, then for $R \geqslant r \geqslant 1 + |S|$ and $|z| = 1$,*

$$|(p \circ f_R)(z)| \geqslant |(p \circ f_r)(z)|,$$

where $f_a(z) = az + S$, $a \in \{r, R\}$.

Proof. Let $z_j = r_j e^{i\theta_j}$, $j = 1, 2, \ldots, n$, be the n zeros of $p(z)$. Then by hypothesis $|r_j| \leqslant 1$ for all j. Also, for $|z| = 1$,

$$\left|(p \circ f_R)(z)\right| = \prod_{j=1}^{n} \left|Re^{i\theta} + S - r_j e^{i\theta_j}\right|$$

and

$$\left|(p \circ f_r)(z)\right| = \prod_{j=1}^{n} \left|re^{i\theta} + S - r_j e^{i\theta_j}\right|.$$

We define $f(x) = |xz + A|$, where $A = S - te^{i\phi}$ with $t \leqslant 1$. To proves the lemma it suffices to show that $f(x)$ is an increasing function of x for $x \geqslant r$ and $|z| = 1$. In fact, for $|z| = 1$, we can write $f(x)^2 = x^2 + 2x\,\mathrm{Re}\,(z\overline{A}) + |A|^2$. Then,

$$\frac{d}{dx} f(x)^2 = 2x + 2\,\mathrm{Re}\,(z\overline{A}) = 2(x + \mathrm{Re}\,(z\overline{A})).$$

So $f(x)^2$, and hence $f(x)$, is increasing if

$$x + \mathrm{Re}\,(z\overline{A}) \geqslant 0.$$

We have for $|z| = 1$,

$$z\overline{A} = e^{i\theta}\overline{S} - te^{i(\theta-\phi)} \;\Rightarrow\; \mathrm{Re}\,(z\overline{A}) \geqslant -|S| - t \geqslant -(|S| + 1).$$

So that for $x > 0$, we have

$$x + \mathrm{Re}\,(z\overline{A}) \geqslant x - (|S| + 1).$$

Now, we are assuming

$$r \geqslant 1 + |S| \;\Rightarrow\; r - (|S| + 1) \geqslant 0.$$

Therefore,

$$x + \mathrm{Re}\,(z\overline{A}) \geqslant 0 \quad \text{for all } x \geqslant r.$$

So $f(x)$ is increasing for $x \geqslant r$, and thus for $R \geqslant r \geqslant 1 + |S|$, we obtain

$$|Rz + A| \geqslant |rz + A|,$$

i.e.,

$$\left|Re^{i\theta} + S - te^{i\phi}\right| \geqslant \left|re^{i\theta} + S - te^{i\phi}\right|.$$

This completes the proof of the lemma. □

We also need the following lemma.

LEMMA 3. *If $p(z)$ and $q(z)$ are two polynomials of degree n, then for any R and S, we have*

$$T_g[p(Rz + S) + q(Rz + S)] = T_g[p(Rz + S)] + T_g[q(Rz + S)].$$

Proof. Let

$$p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu, \quad q(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$$

Expand the compositions we have

$$p(Rz+S) = \sum_{\nu=0}^{n} a_\nu (Rz+S)^\nu, \quad q(Rz+S) = \sum_{\nu=0}^{n} b_\nu (Rz+S)^\nu,$$

so

$$p(Rz+S) + q(Rz+S) = \sum_{\nu=0}^{n} (a_\nu + b_\nu)(Rz+S)^\nu.$$

Now expand each term using the binomial theorem

$$(Rz+S)^\nu = \sum_{k=0}^{\nu} \binom{\nu}{k} R^k S^{\nu-k} z^k.$$

Thus,

$$p(Rz+S) = \sum_{\nu=0}^{n} a_\nu \sum_{k=0}^{\nu} \binom{\nu}{k} R^k S^{\nu-k} z^k = \sum_{k=0}^{n} \left(\sum_{\nu=k}^{n} a_\nu \binom{\nu}{k} R^k S^{\nu-k} \right) z^k,$$

and similarly,

$$q(Rz+S) = \sum_{k=0}^{n} \left(\sum_{\nu=k}^{n} b_\nu \binom{\nu}{k} R^k S^{\nu-k} \right) z^k.$$

Therefore,

$$p(Rz+S) + q(Rz+S) = \sum_{k=0}^{n} \left(\sum_{\nu=k}^{n} (a_\nu + b_\nu) \binom{\nu}{k} R^k S^{\nu-k} \right) z^k.$$

Now apply T_g to the above expression

$$\begin{aligned}
T_g[p(Rz+S) + q(Rz+S)] &= \sum_{k=0}^{n} \left(\sum_{\nu=k}^{n} (a_\nu + b_\nu) \binom{\nu}{k} R^k S^{\nu-k} \right) \lambda_k z^k \\
&= \sum_{k=0}^{n} \lambda_k z^k \left(\sum_{\nu=k}^{n} a_\nu \binom{\nu}{k} R^k S^{\nu-k} + \sum_{\nu=k}^{n} b_\nu \binom{\nu}{k} R^k S^{\nu-k} \right) \\
&= \sum_{k=0}^{n} \lambda_k z^k \left(\sum_{\nu=k}^{n} a_\nu \binom{\nu}{k} R^k S^{\nu-k} \right) + \sum_{k=0}^{n} \lambda_k z^k \left(\sum_{\nu=k}^{n} b_\nu \binom{\nu}{k} R^k S^{\nu-k} \right) \\
&= T_g[p(Rz+S)] + T_g[q(Rz+S)].
\end{aligned}$$

This proves the lemma. □

3. Main results

Now, we present our main results.

THEOREM 1. *If $p, q \in \mathcal{P}_n$ such that*

(i) $|p(z)| \leqslant |q(z)|$ *for* $|z| = 1$,

(ii) $q(z)$ *has all zeros in* $|z| \leqslant 1$,

then, for $R > r \geqslant 1 + |S|$, $|\alpha| \leqslant 1$ *and* $|z| \geqslant 1$,

$$\left|T_g[(p\circ f_R)(z)] - \alpha T_g[(p\circ f_r)(z)]\right| \leqslant \left|T_g[(q\circ f_R)(z)] - \alpha T_g[(q\circ f_r)(z)]\right|, \tag{10}$$

where $f_a(z)$ *is defined in Lemma 2. Equality in (10) holds for* $p(z) = \mathrm{e}^{\mathrm{i}\alpha} q(z)$, $\alpha \in \mathbb{R}$.

REMARK 1. For $S = 0$, Theorem 1 reduces to the result proved in [20].

Taking $q(z) = Mz^n$, where $M := \max_{|z|=1} |p(z)|$, we get the following result from Theorem 1.

COROLLARY 1. *If* $p \in \mathcal{P}_n$, *then for* $R > r \geqslant 1 + |S|$, $|\alpha| \leqslant 1$ *and* $|z| \geqslant 1$, *we have*

$$\begin{aligned}&\left|T_g[(p\circ f_R)(z)] - \alpha T_g[(p\circ f_r)(z)]\right| \\ &\qquad\leqslant \sum_{\nu=0}^{n} \binom{n}{\nu} \left|R^\nu - \alpha r^\nu\right| \left|S\right|^{n-\nu} \left|\lambda_\nu\right| \left|z^\nu\right| \max_{|z|=1} |p(z)|.\end{aligned} \tag{11}$$

Equality in (11) holds for $p(z) = az^n$, $a \neq 0$.

REMARK 2. The inequality (11) generalizes and improves upon (9) in three significant aspects. First, when $S = 0$, (11) reduces to (9) since

$$\begin{aligned}\left|T_g[p(Rz)] - \alpha T_g[p(rz)]\right| &\leqslant |R^n - \alpha r^n|\,|\lambda_n|\,|z|^n \max_{|z|=1} |p(z)| \\ &= |R^n - \alpha r^n|\,|T_g[z^n]| \max_{|z|=1} |p(z)|.\end{aligned}$$

Second, (11) extends the scope of (9) by replacing the scaling transformation $z \mapsto Rz$ (respectively $z \mapsto rz$) with the *affine transformation* $z \mapsto Rz + S$ (resp. $z \mapsto rz + S$). This generalization is essential for applications involving shifted polynomial evaluations. Third, (11) provides a sharper bound than the direct application of (9) to shifted polynomials. Specifically, applying (9) naively to $Q(w) = P(w+S)$ yields

$$\begin{aligned}\left|T_g[Q(Rz)] - \alpha T_g[Q(rz)]\right| &\leqslant |R^n - \alpha r^n|\,|T_g[w^n]| \max_{|w|=1} |Q(w)| \\ &= |R^n - \alpha r^n|\,|\lambda_n|\,|z|^n \max_{|w|=1} |P(w+S)|.\end{aligned}$$

This bound depends on $\max_{|w|=1}|P(w+S)|$, which scales with $|S|$ (e.g., as $|S|^n$ for $p(z)=z^n$). In contrast, (11) depends only on $\max_{|z|=1}|p(z)|$ (independent of S) and incorporates the decaying factors $|S|^{n-\nu}$, leading to tighter estimates for large $|S|$ or polynomials dominated by lower-degree terms. Moreover, (11) is sharp, achieving equality for $p(z)=z^n$, demonstrating optimality.

REMARK 3. For $\alpha=0$, we get the following inequality from Corollary 1 independently proved by Manzoor and Shah [11],

$$\left|T_g[(p\circ f_R)(z)]\right| \leqslant \sum_{\nu=0}^{n}\binom{n}{\nu}R^{\nu}|S|^{n-\nu}|\,|\lambda_\nu|\,|z^{\nu}|\max_{|z|=1}|p(z)|. \tag{12}$$

If we assume all zeros of p lie in $|z|\geqslant 1$, then, $Q(z):=z^n\overline{p(1/\overline{z})}$ has all zeros in $|z|\leqslant 1$. Also, $|p(z)|=|Q(z)|$ for $|z|=1$. Therefore, from Theorem 1, we have the following result:

COROLLARY 2. *If $p\in\mathcal{P}_n$ and $p(z)\neq 0$ in $|z|<1$, then for $R>r\geqslant 1+|S|$, $|\alpha|\leqslant 1$ and $|z|\geqslant 1$,*

$$\left|T_g[(p\circ f_R)(z)]-\alpha T_g[(p\circ f_r)(z)]\right| \leqslant \left|T_g[(Q\circ f_R)(z)]-\alpha T_g[(Q\circ f_r)(z)]\right|,$$

where $Q(z):=z^n\overline{p(1/\overline{z})}$.

Next, we prove the following result:

THEOREM 2. *If $p\in\mathcal{P}_n$, then for every complex number α with $|\alpha|\leqslant 1$, $R>r\geqslant 1+|S|$ and $|z|\geqslant 1$,*

$$\left|T_g[(p\circ f_R)(z)]-\alpha T_g[(p\circ f_r)(z)]\right| + \left|T_g[(Q\circ f_R)(z)]-\alpha T_g[(Q\circ f_r)(z)]\right|$$
$$\leqslant \left\{\sum_{\nu=0}^{n}\binom{n}{\nu}\left|R^{\nu}-\alpha r^{\nu}\right|\,|S|^{n-\nu}\left|\lambda_\nu\right|\left|z^{\nu}\right| + |1-\alpha|\left|\lambda_0\right|\right\}M,$$

where $M:=\max_{|z|=1}|p(z)|$ and $Q(z):=z^n\overline{p(1/\overline{z})}$.

We now prove the following result for polynomials having all zeros in $|z|\leqslant 1$. The result gives analog of Corollary 1 for minimum modulus of a polynomial $p(z)$ on $|z|=1$.

THEOREM 3. *If $p\in\mathcal{P}_n$ and p has all its zeros in $|z|\leqslant 1$, then for every real or complex number α with $|\alpha|\leqslant 1$, $R>r\geqslant 1+|S|$ and $|z|\geqslant 1$,*

$$\left|T_g[(p\circ f_R)(z)]-\alpha T_g[(p\circ f_r)(z)]\right| \geqslant \sum_{\nu=0}^{n}\binom{n}{\nu}\left|R^{\nu}-\alpha r^{\nu}\right|\,|S|^{n-\nu}\left|\lambda_\nu\right|\left|z^{\nu}\right|\min_{|z|=1}|p(z)|. \tag{13}$$

Equality in (13) holds for $p(z)=az^n$, $a\neq 0$.

Next, we have a result for polynomials which does not vanish in $|z|<1$.

THEOREM 4. *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ in $|z|<1$, then for $R>r \geqslant 1+|S|$, $|\alpha| \leqslant 1$ and $|z| \geqslant 1$,*

$$\begin{aligned}\big|T_g[(p\circ f_R)(z)]-\alpha T_g[(p\circ f_r)(z)]\big| &\leqslant \frac{M}{2}\left\{\sum_{\nu=0}^{n}\binom{n}{\nu}\left|R^{\nu}-\alpha r^{\nu}\right||S|^{n-\nu}\left|\lambda_{\nu}\right||z^{\nu}|+|1-\alpha|\,|\lambda_0|\right\}\\ &\quad -\frac{m}{2}\left\{\left|\sum_{\nu=0}^{n}\binom{n}{\nu}\left(R^{\nu}-\alpha r^{\nu}\right)S^{n-\nu}\lambda_{\nu}z^{\nu}\right|-|1-\alpha|\,|\lambda_0|\right\}, \qquad (14)\end{aligned}$$

where $M:=\max\limits_{|z|=1}|p(z)|$ and $m:=\min\limits_{|z|=1}|p(z)|$. Equality in (14) holds for $p(z)=az^n+b$, with $|a|=|b|\neq 0$.

For $\alpha=0$, we have the following corollary which gives refinement of a result due to Manzoor and Shah [11].

COROLLARY 3. *If $p \in \mathcal{P}_n$ having no zero in $|z|<1$, then for $R>1+|S|$ and $|z| \geqslant 1$,*

$$\begin{aligned}|T_g[(p\circ f_R)(z)]| \leqslant \frac{1}{2}\Bigg\{&\left(\sum_{\nu=0}^{n}\binom{n}{\nu}R^{\nu}|S|^{n-\nu}\left|\lambda_{\nu}\right||z^{\nu}|+|\lambda_0|\right)\max_{|z|=1}|p(z)|\\ &-\left(\left|\sum_{\nu=0}^{n}\binom{n}{\nu}R^{\nu}S^{n-\nu}\lambda_{\nu}z^{\nu}\right|-|\lambda_0|\right)\min_{|z|=1}|p(z)|\Bigg\}. \qquad (15)\end{aligned}$$

Equality in (15) holds for $p(z)=az^n+b$, with $|a|=|b|\neq 0$.

REMARK 4. For $g(z)=\sum\limits_{j=0}^{n}\binom{n}{j}jz^j$, $R\to 1$, $S=0$, and $|z|=1$, Corollary 3 reduces to inequality (5). Also if we take $g(z)=\sum\limits_{j=0}^{n}\binom{n}{j}z^j$, $S=0$, and $|z|=1$ in Corollary 3, we get inequality (6).

By taking $g(z)=z^n+z^k$, $0\leqslant k\leqslant n-1$ and $|z|=1$ in Corollary 3, we obtain the following result which gives an improvement upon Corollary 1.25 in [11].

COROLLARY 4. *If $p(z)=\sum\limits_{j=0}^{n}a_jz^j$ is a polynomial of degree n which does not vanish in $|z|<1$, then for $R>1+|S|$ and $|z|=1$,*

$$R^n|a_n|+\frac{R^k}{\binom{n}{k}}\sum_{j=k}^{n}\binom{j}{k}|S|^{j-k}|a_j| \leqslant \frac{1}{2}\left[\Lambda\max_{|z|=1}|p(z)|-\zeta\min_{|z|=1}|p(z)|\right], \quad 0\leqslant k\leqslant n-1,$$

where

$$\Lambda = \begin{cases} R^n + \binom{n}{k} R^k |S|^{n-k}, & \text{if } 1 \leqslant k \leqslant n-1 \\ R^n + |S|^n + 1, & \text{if } k = 0, \end{cases}$$

and

$$\zeta = \begin{cases} \left|R^n + \binom{n}{k} R^k S^{n-k}\right|, & \text{if } 1 \leqslant k \leqslant n-1 \\ \left|R^n + S^n\right| - 1, & \text{if } k = 0. \end{cases}$$

The result is sharp.

Applying Corollary 4 to the polynomial $z^n p(1/z)$, we obtain the following corollary which provides an improvement upon Corollary 1.7 in [6].

COROLLARY 5. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $R > 1 + |S|$ and $|z| = 1$,*

$$|a_0| R^n + \frac{R^k}{\binom{n}{k}} \sum_{j=k}^{n} \binom{j}{k} |a_{n-j}| |S|^{j-k} \leqslant \frac{1}{2}\left[\Lambda_1 \max_{|z|=1} |p(z)| - \zeta_1 \min_{|z|=1} |p(z)|\right], \quad 1 \leqslant k \leqslant n,$$

where

$$\Lambda_1 = \begin{cases} R^n + \binom{n}{k} R^k |S|^{n-k}, & \text{if } 1 \leqslant k \leqslant n-1 \\ R^n + |S^n| + 1, & \text{if } k = n, \end{cases}$$

and

$$\zeta_1 = \begin{cases} \left|R^n + \binom{n}{k} R^k S^{n-k}\right|, & \text{if } 1 \leqslant k \leqslant n-1 \\ \left|R^n + S^n\right| - 1, & \text{if } k = n. \end{cases}$$

A polynomial $p \in \mathcal{P}_n$ is said to be self-inversive if $p(z) = \zeta Q(z)$, $|\zeta| = 1$, where $Q(z) = z^n \overline{p(1/\overline{z})}$. Finally, we prove the following result for sef-inversive polynomials.

THEOREM 5. *If $p \in \mathcal{P}_n$ is a self-inversive polynomial, then for $R > r \geqslant 1$, $|\alpha| \leqslant 1 + |S|$ and $|z| \geqslant 1$,*

$$|T_g[(p \circ f_R)(z)] - \alpha T_g[(p \circ f_r)(z)]| \leqslant \frac{1}{2}\left[\sum_{\nu=0}^{n} \binom{n}{\nu} |R^\nu - \alpha r^\nu| |S^{n-\nu}| |\lambda_\nu| |z^\nu| + |1-\alpha| |\lambda_0|\right] \max_{|z|=1} |p(z)|. \quad (16)$$

Equality in (16) holds for $p(z) = z^n + 1$.

REMARK 5. A variety of interesting results can easily be deduced from Theorem 5 in the same way as we have deduced from Theorem 4.

4. Proofs of main results

Proof of Theorem 1. Since $q(z)$ has all zeros in $|z| \leqslant 1$ and $|p(z)| \leqslant |q(z)|$ for $|z| = 1$, it follows by Rouché's theorem that for every real or complex number β with $|\beta| > 1$, the polynomial $\Phi(z) = p(z) - \beta q(z)$ has all its zeros in $|z| \leqslant 1$. Applying Lemma 2 to $\Phi(z)$, we get for $R > r \geqslant 1 + |S|$ and $|z| = 1$,

$$|(\Phi \circ f_r)(z)| < |(\Phi \circ f_R)(z)|.$$

Also, all zeros of $(\Phi \circ f_R)(z)$ lie in $|z| \leqslant \dfrac{1+|S|}{|R|} < 1$. Using Rouché's theorem we conclude that the polynomial

$$\psi(z) = (\Phi \circ f_R)(z) - \alpha(\Phi \circ f_r)(z)$$

has all the zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leqslant 1$. Applying Lemmas 1 and 3, it follows that

$$\begin{aligned}
\phi(z) &= T_g[\psi(z)] \\
&= T_g[(\Phi \circ f_R)(z) - \alpha(\Phi \circ f_r)(z)] \\
&= T_g[((p-\beta q) o f_R)(z)] - \alpha T_g[((p-\beta q) o f_r)(z)] \\
&= T_g[(p \circ f_R)(z) - \beta T_g[(q \circ f_R)(z)] - \alpha \left\{ T_g[(p \circ f_r)(z) - \beta T_g[(q \circ f_r)(z)]] \right\} \\
&= T_g[(p \circ f_R)(z)] - \alpha T_g[(p \circ f_r)(z)] - \beta \left\{ T_g[(q \circ f_R)(z)] - \alpha T_g[(q \circ f_r)(z)] \right\} \quad (17)
\end{aligned}$$

has all zeros in $|z| < 1$. We claim that for $|z| \geqslant 1$,

$$\left| T_g[(p \circ f_R)(z)] - \alpha T_g[(p \circ f_r)(z)] \right| \leqslant \left| T_g[(q \circ f_R)(z)] - \alpha T_g[(q \circ f_r)(z)] \right|. \quad (18)$$

For if inequality (18) is not true, then there exists $z_0 \in \mathbb{C}$ with $|z_0| \geqslant 1$ such that

$$\left| T_g[(p \circ f_R)(z_0)] - \alpha T_g[(p \circ f_r)(z_0)] \right| > \left| T_g[(q \circ f_R)(z_0)] - \alpha T_g[(q \circ f_r)(z_0)] \right|. \quad (19)$$

Let

$$\beta = \frac{\left| T_g[(p \circ f_R)(z_0)] - \alpha T_g[(p \circ f_r)(z_0)] \right|}{\left| T_g[(q \circ f_R)(z_0)] - \alpha T_g[(q \circ f_r)(z_0)] \right|},$$

then by (19), $|\beta| > 1$. For this value of β it follows from equation (17) that $\phi(z_0) = 0$, which is a contradiction to the fact that all zeros of $\phi(z)$ lie in $|z| < 1$. This proves the theorem. □

Proof of Corollary 1. For $q(z) = Mz^n$, we have

$$\begin{aligned}
(q \circ f_R)(z) - \alpha(q \circ f_r)(z) &= [(Rz+S)^n - \alpha(rz+S)^n] M \\
&= \sum_{\nu=0}^{n} \binom{n}{n-\nu} (R^\nu - \alpha r^\nu) S^{n-\nu} z^\nu M.
\end{aligned}$$

Therefore

$$T_g[(q\circ f_R)(z)]-\alpha T_g[\alpha(q\circ f_r)(z)]=\sum_{\nu=0}^{n}\binom{n}{n-\nu}(R^\nu-\alpha r^\nu)S^{n-\nu}z^\nu\lambda_\nu M,$$

which on using in (10) and invoking triangle inequality gives the required result. □

Proof of Theorem 2. We have

$$|p(z)|\leqslant M \quad \text{for} \quad |z|=1.$$

By Rouché's theorem all the zeros of $\mathscr{F}(z)=p(z)-\mu M$ lie in $|z|\geqslant 1$ for some $\mu\in\mathbb{C}$ with $|\mu|>1$. By Corollary 2, it follows that for $|z|\geqslant 1$ and $|\alpha|\leqslant 1$,

$$|T_g[(\mathscr{F}\circ f_R)(z)]-\alpha T_g[(\mathscr{F}\circ f_r)(z)]|\leqslant|T_g[(H\circ f_R)(z)]-\alpha T_g[(H\circ f_r)(z)]|,$$

where $H(z)=z^n\overline{\mathscr{F}(1/\overline{z})}=Q(z)-\overline{\mu}z^nM$. Using Lemma 3, this implies:

$$\begin{aligned}&\left|T_g[(p\circ f_R)(z)]-\alpha T_g[(p\circ f_r)(z)]-\mu(1-\alpha)\lambda_0 M\right|\\&\leqslant\left|T_g[(Q\circ f_R)(z)]-\alpha T_g[(Q\circ f_r)(z)]-\overline{\mu}\sum_{\nu=0}^{n}\binom{n}{\nu}(R^\nu-\alpha r^\nu)S^{n-\nu}\lambda_\nu z^\nu M\right|. \qquad (20)\end{aligned}$$

Now choosing the argument of μ on the right side which is possible by Corollary 1 and triangle inequality on the left hand side of (20), we get for $R>r\geqslant 1+|S|$, $|\alpha|\leqslant 1$ and $|z|\geqslant 1$,

$$\begin{aligned}&\left|T_g[(p\circ f_R)(z)]-\alpha T_g[(p\circ f_r)(z)]\right|-|\mu||(1-\alpha)\lambda_0 M|\\&\leqslant|\overline{\mu}|\left|\sum_{\nu=0}^{n}\binom{n}{\nu}(R^\nu-\alpha r^\nu)S^{n-\nu}\lambda_\nu z^\nu M\right|-\left|T_g[(Q\circ f_R)(z)]-\alpha T_g[(Q\circ f_r)(z)]\right|.\end{aligned}$$

Letting $|\mu|\to 1$, the theorem follows. □

Proof of Theorem 3. Let $m=\min_{|z|=1}|p(z)|$, then

$$m|z|^n\leqslant|p(z)| \quad \text{for} \quad |z|=1.$$

If $p(z)$ has a zero on $|z|=1$, then $m=0$. Assume instead that $p(z)$ has all its zeros in $|z|<1$. Then, by Rouché's theorem, for some $\beta\in\mathbb{C}$ with $|\beta|<1$ the polynomial $F(z)=p(z)-\beta mz^n$ has all its zeros in $|z|<1$. Applying Lemma 2, we get for $R>r\geqslant 1+|S|$ and $|z|=1$,

$$|(F\circ f_R)(z)|>|(F\circ f_r)(z)|.$$

Again by Rouché's theorem:

$$G(z)=(F\circ f_R)(z)-\alpha(F\circ f_r)(z)$$

has all zeros in $|z| < 1$ for $|\alpha| \leqslant 1$. On applying Lemmas 1 and 3 it follows that the zeros of

$$\begin{aligned} T_g[G(z)] &= T_g[(F \circ f_R)(z) - \alpha(F \circ f_r)(z)] \\ &= T_g[(F \circ f_R)(z)] - \alpha T_g[(F \circ f_r)(z)] \\ &= T_g[(p \circ f_R)(z) - \beta m(Rz+S)^n] - \alpha T_g[(p \circ f_r)(z) - \beta m(rz+S)^n] \\ &= T_g[(p \circ f_R)(z)] - \alpha T_g[(p \circ f_r)(z)] - \beta \sum_{\nu=0}^{n} \binom{n}{\nu} (R^\nu - \alpha r^\nu) S^{n-\nu} \lambda_\nu z^\nu m \end{aligned}$$

lie in $|z| < 1$. On the similar lines as in the proof of Theorem 1, this implies for $|z| \geqslant 1$,

$$|T_g[(p \circ f_R)(z)] - \alpha T_g[(p \circ f_r)(z)]| \geqslant \sum_{\nu=0}^{n} \binom{n}{\nu} |R^\nu - \alpha r^\nu| |S|^{n-\nu} |\lambda_\nu| |z^\nu| m.$$

This completes the proof of the theorem. □

Proof of Theorem 4. If $p(z)$ has a zero on $|z| = 1$, then $m = 0$, so we suppose that all the zeros of $p(z)$ lie in $|z| > 1$, so that $m > 0$ and

$$m \leqslant |p(z)| \quad \text{for} \quad |z| = 1. \tag{21}$$

Therefore, for every complex number β with $|\beta| < 1$, it follows by Rouché's theorem that all the zeros of $F(z) = p(z) - m\beta$ lie in $|z| > 1$. We note that $F(z)$ has no zero on $|z| = 1$, because for some $z = z_0$ with $|z_0| = 1$

$$F(z_0) = p(z_0) - m\beta = 0,$$

then

$$|p(z_0)| = m|\beta| < m,$$

which is a contradiction to (21). Now if

$$\begin{aligned} G(z) &= z^n \overline{F(1/\overline{z})} \\ &= z^n \overline{p(1/\overline{z})} - \overline{\beta} m z^n \\ &= Q(z) - \overline{\beta} m z^n, \end{aligned}$$

then all the zeros of $G(z)$ lie in $|z| < 1$ and

$$|G(z)| = |F(z)| \quad \text{for} \quad |z| = 1.$$

Therefore, by Theorem 1, we get for $R > r \geqslant 1 + |S|$, $|\alpha| \leqslant 1$ and $|z| \geqslant 1$,

$$\left| T_g[(F \circ f_R)(z)] - \alpha T_g[(F \circ f_r)(z)] \right| \leqslant \left| T_g[(G \circ f_R)(z)] - \alpha T_g[(G \circ f_r)(z)] \right|.$$

Substituting back the value of F and G, we get for $|z| \geqslant 1$,

$$\begin{aligned} &\left| T_g[(p \circ f_R)(z) - m\beta] - \alpha T_g[(p \circ f_r)(z) - m\beta] \right| \\ &\qquad \leqslant \left| T_g[(Q \circ f_R)(z) - \overline{\beta} m(Rz+S)^n] - \alpha T_g[(Q \circ f_r)(z) - \overline{\beta} m(rz+S)^n] \right|. \end{aligned}$$

Using Lemma 3, we obtain for $|z| \geqslant 1$,

$$\left|T_g[(p\circ f_R)(z)] - \alpha T_g[(p\circ f_r)(z)] - m\beta\lambda_0(1-\alpha)\right| \leqslant \Bigg|T_g[(Q\circ f_R)(z)] - \alpha T_g[(Q\circ f_r)(z)] - \overline{\beta} m \sum_{\nu=0}^{n} \binom{n}{\nu}\left(R^\nu - \alpha r^\nu\right) S^{n-\nu}\lambda_\nu z^\nu\Bigg|. \quad (22)$$

In inequality (22), choosing argument of β suitably on the right hand side (which is possible by Theorem 3), using the triangle inequality on the left side and then letting $|\beta| \to 1$, we obtain

$$\begin{aligned} &\left|T_g[(p\circ f_R)(z)] - \alpha T_g[(p\circ f_r)(z)]\right| - m|\beta||\lambda_0||1-\alpha| \\ &\leqslant \left|T_g[(Q\circ f_R)(z)] - \alpha T_g[(Q\circ f_r)(z)]\right| - |\overline{\beta}| m \left|\sum_{\nu=0}^{n} \binom{n}{\nu}\left(R^\nu - \alpha r^\nu\right) S^{n-\nu}\lambda_\nu z^\nu\right|. \end{aligned} \quad (23)$$

Adding $\left|T_g[(p\circ f_R)(z)] - \alpha T_g[(p\circ f_r)(z)]\right|$ on both sides of inequality (23) and using Theorem 2, the required result follows. □

Proof of Theorem 5. By the given condition it easily follows that for $R > r \geqslant 1+|S|$, $|\alpha| \leqslant 1$ and for all z, we have

$$|T_g[(p\circ f_R)(z)] - \alpha T_g[(p\circ f_r)(z)]| = |T_g[(q\circ f_R)(z)] - \alpha T_g[(q\circ f_r)(z)]|.$$

The above inequality when combined with Theorem 2 gives the required result. □

5. Conclusion

Polynomial inequalities form an important area of research in both approximation theory and geometric function theory. In this paper, we extend several of these inequalities to the operator $T_g[p\circ f_R]$. Many known inequalities exhibit structural patterns similar to those considered and generalized here. Although not all such inequalities lend themselves to meaningful extensions, a considerable number can indeed be effectively generalized using the operator $T_g[p\circ f_R]$. A detailed compilation of related inequalities, which may provide a basis for further generalizations, is available in [21].

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