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A NOVEL PARAMETRIC METHOD FOR PROVING SOME ANALYTIC INEQUALITIES AND DETERMINING MINIMAX APPROXIMATIONS

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Abstract. In this paper, we give a novel method for proving inequalities and determining minimax approximants of the stratified families of functions. In the theory of analytic inequalities, there are numerous inequalities based on which one can form a family of functions that is stratified, i.e. monotonic with respect to the introduced parameter. The introduced method is based on identifying those functions from the family that have a stationary point on the observed interval. The applications of this method are demonstrated to the Cusa-Huygens, Mitrinović-Adamović-type and Jordan-type inequalities.

1. Introduction and preliminaries

In the theory of analytic inequalities, the concept of stratified families of functions was recently introduced [28, 31]. Let $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ be a family of functions that we consider for values of the argument $x\in\mathbb{S}\subseteq\mathbb{R}$ and values of the parameter $p\in\mathbb{P}\subseteq\mathbb{R}$. The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is *increasingly stratified at the point* $x_0\in\mathbb{S}$ *with respect to the parameter* $p\in\mathbb{P}$ if it holds that

$$(\forall p_1,p_2\in\mathbb{P})\ p_1<p_2 \Longleftrightarrow \varphi_{p_1}(x_0)<\varphi_{p_2}(x_0),$$

i.e. *decreasingly stratified at the point* $x_0\in\mathbb{S}$ *with respect to the parameter* $p\in\mathbb{P}$ if it holds that

$$(\forall p_1,p_2\in\mathbb{P})\ p_1<p_2 \Longleftrightarrow \varphi_{p_1}(x_0)>\varphi_{p_2}(x_0).$$

The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is *increasingly* (i.e. *decreasingly*) *stratified on the set* $\mathbb{S}$ *with respect to the parameter* $p\in\mathbb{P}$ if it is increasingly (i.e. decreasingly) stratified at every point in the set $\mathbb{S}$ with respect to the parameter $p\in\mathbb{P}$.

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It has been shown in [5, 24, 26–33] that many inequalities could be proved, improved and generalised using the concept of stratification.

According to [31], it is also significant to determine, if it exists, *the minimax approximant* $\varphi_{p_0}(x)$, for some $p_0 \in \mathbb{P}$, as a function with the following property

$$\inf_{p\in\mathbb{P}}\sup_{x\in\mathbb{S}}|\varphi_p(x)| = \sup_{x\in\mathbb{S}}\left|\varphi_{p_0}(x)\right|.$$

The function $d^{(p)} = \sup_{x\in\mathbb{S}}\left|\varphi_p(x)\right|$, we call *the error function*. The approximation $\varphi_{p_0}(x) \approx 0$, we call *the minimax approximation on the set* $\mathbb{S}$ for a given family, and the number $d_0 = d^{(p_0)} = \sup_{x\in\mathbb{S}}\left|\varphi_{p_0}(x)\right|$, we call *the approximation error*.

In the following, we consider the case when $\mathbb{S}$ is a bounded real interval.

A parametric method. The method for proving some analytic inequalities from [28] is based on the introduction and analysis of the function $g : \mathbb{S} \longrightarrow \mathbb{P}$ such that

$$g(x) = p \quad \Longleftrightarrow \quad \varphi_p(x) = 0,$$

if such a function exists. In particular, in [28], the cases when the function g is strictly monotonic or when it has exactly one local extremum on the observed interval were analysed.

In this paper, we consider families $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ of differentiable functions with respect to x and we give one method for proving some analytic inequalities that is based on the introduction of the function $g_1 : \mathbb{S} \longrightarrow \mathbb{P}$ such that

$$g_1(x) = p \quad \Longleftrightarrow \quad \frac{\partial\varphi_p(x)}{\partial x} = 0,$$

if such a function exists. We show that based on the monotonicity of the function g_1, some inequalities could be proved and the minimax approximant of the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ could be determined.

Compression at the point. Let $\mathbb{S}$ be a bounded interval with endpoints $a \in \mathbb{R}$ and $b \in \mathbb{R}$, $a < b$. We say that the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is *compressed at the point* a if $\varphi_p(a+) = 0$ for each $p \in \mathbb{P}$, i.e. that the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is *compressed at the point* b if $\varphi_p(b-) = 0$ for each $p \in \mathbb{P}$.

Determination of the minimax approximant. To determine the minimax approximants in [5, 24, 31, 32], Theorem 1 and Theorem 1' from [31] were used. Based on those theorems, the following theorem holds.

THEOREM 1.1. *Let* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$*, for* $x \in \mathbb{S} = [a,b]$ *and* $p \in \mathbb{P} = [c,d]$*, be a family of continuous functions with respect to the argument* x*, which is compressed at the point* a *and increasingly* (*i.e. decreasingly*) *stratified on the interval* $(a,b]$ *with respect to the parameter* $p \in [c,d]$*. If it holds*:

(a) $\varphi_c(x) < 0$ *(i.e.* $\varphi_c(x) > 0$ *) and* $\varphi_d(x) > 0$ *(i.e.* $\varphi_d(x) < 0$ *) for each* $x \in (a,b)$ *and* $\varphi_c(b) = 0$ *(i.e.* $\varphi_d(b) = 0$ *),*

(b) *the functions* $\varphi_p(x)$ *are continuous with respect to* $p \in (c,d)$ *for each* $x \in (a,b]$*,*

(c) *for each* $p \in (c,d)$*, there exists a right neighbourhood of the point* a *in which* $\varphi_p(x) < 0$ *holds,*

(d) *for each* $p \in (c,d)$*, the function* $\varphi_p(x)$ *has exactly one extremum* $t^{(p)}$ *on the interval* (a,b)*, which is a minimum,*

then there exists exactly one solution $p = p_0 \in (c,d)$ *of the following equation*

$$\left|\varphi_p\left(t^{(p)}\right)\right| = \varphi_p(b)$$

and for $d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}(b)$*, we have*

$$d_0 = \inf_{p \in \mathbb{P}} \sup_{x \in (a,b)} |\varphi_p(x)|.$$

In the given theorem, (compared to Theorem 1 and Theorem 1' from [31]) we additionally require the stratification of the family at the point b, which ensures that the functions $\varphi_c(x)$ and $\varphi_d(x)$ are determined uniquely.

A method for proving MTP inequalities. In applications of the novel parametric method in this paper, to prove the monotonicity of the function g_1, we will use the method for proving mixed trigonometric polynomial (MTP) inequalities from [5, 25]. MTP function is a function of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x,$$

where $\alpha_i \in \mathbb{R} \backslash \{0\}, p_i, q_i, r_i \in \mathbb{N}_0$ and $n \in \mathbb{N}$, for $x \in \mathbb{R}$. MTP inequality is an inequality of the form $f(x) > 0$ for $x \in \mathbb{S}$, see [5, 8–10, 19, 21–23, 25, 29, 38, 45].

According to the method for proving MTP inequalities from [5, 25], we first transform the MTP function $f(x)$ in the form $f(x) = \sum_{i=1}^{m} \beta_i x^{s_i} \mathrm{trig}_i(k_i x)$, where $\beta_i \in \mathbb{R} \backslash \{0\}, s_i, k_i \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\mathrm{trig}_i = \cos$ or $\mathrm{trig}_i = \sin$. Then, by approximating each function trig_i with the corresponding Taylor expansion of that function, we determine a downward polynomial approximation $P(x)$ of the function $f(x)$. If $P(x) > 0$ for $x \in \mathbb{S}$, then the MTP inequality $f(x) > 0$ for $x \in \mathbb{S}$ is proved. To prove polynomial inequalities, we use Sturm's theorem [14, 44].

2. Main results

If there exists a function $g_1 : \mathbb{S} \longrightarrow \mathbb{P}$ such that

$$g_1(x) = p \quad \Longleftrightarrow \quad \frac{\partial \varphi_p(x)}{\partial x} = 0, \qquad (*)$$

then the function g_1 determines values of the parameter $p \in \mathbb{P}$ for which functions $\varphi_p(x)$ have a stationary point on the observed interval $\mathbb{S}$. Therefore, on the interval $\mathbb{S}$, the functions $\varphi_p(x)$ for $p \in g_1(\mathbb{S})$ have a stationary point, while all other functions from the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ are monotonic.

In Theorems 2.1, 2.2 and 2.3, based on the monotonicity of the function g_1 and based on the properties of stratification of the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, we form corresponding inequalities for $p \in \mathbb{P}$ and determine the minimax approximant of the family.

THEOREM 2.1. *Let $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, for $x \in \mathbb{S} = [a,b]$ and $p \in \mathbb{P} \subseteq \mathbb{R}$ $(\mathbb{P} \neq \emptyset)$, be a family of functions that are continuous with respect to p and with respect to x for each $x \in [a,b]$ and for each $p \in \mathbb{P}$ such that the following conditions hold*:

(1) *the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is compressed at the point a and increasingly (i.e. decreasingly) stratified on the interval $(a,b]$ with respect to the parameter $p \in \mathbb{P}$,*

(2) *the functions $\varphi_p(x)$, for $p \in \mathbb{P}$, are differentiable on the interval (a,b) and there exists a continuous monotonically decreasing function $g_1 : (a,b) \longrightarrow \mathbb{P}$ that satisfies $(*)$,*

(3) *there exist limits $\lim\limits_{x\to a+} g_1(x) = C \in \mathbb{R}$ and $\lim\limits_{x\to b-} g_1(x) = A \in \mathbb{R}$ such that $(A,C) \subseteq \mathbb{P}$,*

(4) *there exists a value of the parameter $p = B \in (A,C)$ such that $\varphi_B(b) = 0$,*

(5) *there exist a right neighbourhood of the point a in which it holds that $\varphi_p(x) < 0$ (i.e. $\varphi_p(x) > 0$) for each $p \in (A,C)$.*

Then, it holds:

(i) *If $p \leqslant B$, it holds that*

$$(\forall x \in (a,b))\ \varphi_p(x) \leqslant \varphi_B(x) < 0 \qquad (i.e.\ (\forall x \in (a,b))\ \varphi_p(x) \geqslant \varphi_B(x) > 0)$$

and the constant B is the best possible.

(ii) *If $p \in (B,C)$, then the equation $\varphi_p(x) = 0$ has a unique solution $x_0^{(p)} \in (a,b)$ and it holds that*

$$\left(\forall x \in \left(a, x_0^{(p)}\right)\right)\ \varphi_p(x) < 0 \qquad \left(i.e.\ \left(\forall x \in \left(a, x_0^{(p)}\right)\right)\ \varphi_p(x) > 0\right)$$

and

$$\left(\forall x \in \left(x_0^{(p)}, b\right)\right)\ \varphi_p(x) > 0 \qquad \left(i.e.\ \left(\forall x \in \left(x_0^{(p)}, b\right)\right)\ \varphi_p(x) < 0\right).$$

(iii) *If $p \geqslant C$, it holds that*

$$(\forall x \in (a,b))\ \varphi_p(x) \geqslant \varphi_C(x) > 0 \qquad (i.e.\ (\forall x \in (a,b))\ \varphi_p(x) \leqslant \varphi_C(x) < 0)$$

and the constant C is the best possible.

(iv) *Each function $\varphi_p(x)$, for $p\in(B,C)$, has exactly one local extremum, which is a minimum (i.e. maximum), on the interval (a,b) at the point $\left(t^{(p)},\varphi_p\left(t^{(p)}\right)\right)$.*

(v) *There exists exactly one solution $p=p_0\in(B,C)$ of the equation*

$$\left|\varphi_p\left(t^{(p)}\right)\right|=\varphi_p(b) \qquad \left(\text{i.e. } \varphi_p\left(t^{(p)}\right)=\left|\varphi_p(b)\right|\right).$$

The function $\varphi_{p_0}(x)$ is the minimax approximant of the family on the interval (a,b). The approximation error is

$$d_0=\left|\varphi_{p_0}\left(t^{(p_0)}\right)\right|=\varphi_{p_0}(b) \qquad \left(\text{i.e. } d_0=\varphi_{p_0}\left(t^{(p_0)}\right)=\left|\varphi_{p_0}(b)\right|\right).$$

Proof. Let us consider the case when $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is an increasingly stratified family of functions. The case when $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is a decreasingly stratified family of functions could be considered analogously.

Based on conditions **(2)** and **(3)**, the functions $\varphi_p(x)$, for $p\in(A,C)$, have exactly one stationary point on the interval (a,b), while the functions $\varphi_p(x)$, for $p\in\mathbb{P}\backslash(A,C)$, are monotonic on the interval (a,b).

(i) The function $\varphi_B(x)$ has exactly one stationary point on the interval (a,b). Since $\varphi_B(a)=0$ (condition **(1)**), $\varphi_B(x)<0$ in a right neighbourhood of the point a (condition **(5)**) and $\varphi_B(b)=0$ (condition **(4)**), we conclude that the function $\varphi_B(x)$ has exactly one minimum on the interval (a,b). Additionally, we conclude that that $\varphi_B(x)<0$ on the interval (a,b) and that, based on the stratification, the stated inequalities hold.

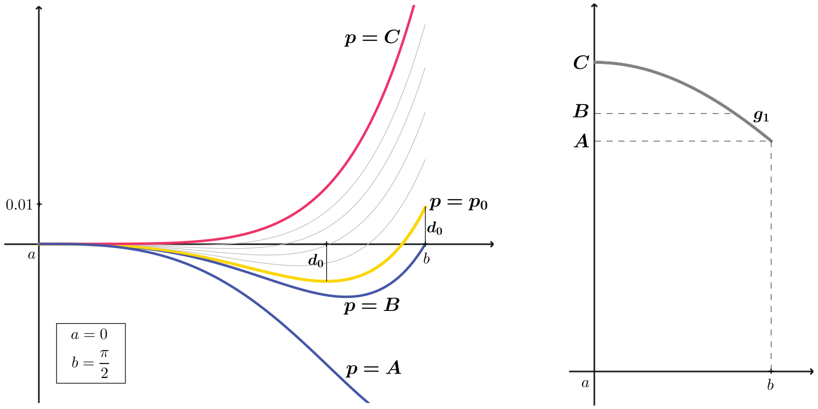

Figure 1: *Some functions from the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ (see Application 1) that satisfy the conditions of Theorem 2.1 and the corresponding function g_1.*

(ii), ***(iv)*** Since $\varphi_B(b) = 0$ (condition **(4)**) and the family is increasingly stratified at the point b, it holds that $(\forall p \in (B,C))\ \varphi_p(b) > 0$. Additionally, from $\varphi_p(a) = 0$ for $p \in (B,C)$ (condition **(1)**) and the fact that $\varphi_p(x) < 0$, for $p \in (B,C)$, in a right neighbourhood of the point a (condition **(5)**), it follows that the functions $\varphi_p(x)$, for $p \in (B,C)$, are decreasing and negative in a right neighbourhood of the point a. Hence, we conclude that the functions $\varphi_p(x)$, for $p \in (B,C)$, have exactly one local extremum, which is a minimum, and exactly one zero on the interval (a,b) and that the stated inequalities hold.

(iii) The function $\varphi_C(x)$ is monotonic on the interval (a,b). Since $\varphi_B(b) = 0$ (condition **(4)**), based on the increasing stratification of the family, we conclude that $\varphi_C(b) > 0$. Additionally, since it holds that $\varphi_C(a) = 0$ (condition **(1)**), we conclude that $\varphi_C(x) > 0$ on the interval (a,b) and that, based on the stratification, the stated inequalities hold.

(v) Based on Theorem 1.1. □

THEOREM 2.2. *Let $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, for $x \in \mathbb{S} = [a,b]$ and $p \in \mathbb{P} \subseteq \mathbb{R}$ $(\mathbb{P} \neq \emptyset)$, be a family of functions that are continuous with respect to p and with respect to x for each $x \in [a,b]$ and for each $p \in \mathbb{P}$ such that the following conditions hold:*

(1) *the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is compressed at the point a and increasingly (i.e. decreasingly) stratified on the interval $(a,b]$ with respect to the parameter $p \in \mathbb{P}$,*

(2) *the functions $\varphi_p(x)$, for $p \in \mathbb{P}$, are differentiable on the interval (a,b) and there exists a continuous monotonically increasing function $g_1 : (a,b) \longrightarrow \mathbb{P}$ that satisfies $(*)$,*

(3) *there exist limits $\lim\limits_{x\to a+} g_1(x) = A \in \mathbb{R}$ and $\lim\limits_{x\to b-} g_1(x) = C \in \mathbb{R}$, such that $(A,C) \subseteq \mathbb{P}$,*

(4) *there exists a value of the parameter $p = B \in (A,C)$ such that $\varphi_B(b) = 0$,*

(5) *there exist a right neighbourhood of the point a in which it holds that $\varphi_p(x) > 0$ (i.e. $\varphi_p(x) < 0$) for each $p \in (A,C)$.*

Then, it holds:

(i) *If $p \leqslant A$, it holds that*

$$(\forall x \in (a,b))\ \varphi_p(x) \leqslant \varphi_A(x) < 0 \qquad (i.e.\ (\forall x \in (a,b))\ \varphi_p(x) \geqslant \varphi_A(x) > 0)$$

and the constant A is the best possible.

(ii) *If $p \in (A,B)$, then the equation $\varphi_p(x) = 0$ has a unique solution $x_0^{(p)} \in (a,b)$ and it holds that*

$$\left(\forall x \in \left(a, x_0^{(p)}\right)\right)\ \varphi_p(x) > 0 \qquad \left(i.e.\ \left(\forall x \in \left(a, x_0^{(p)}\right)\right)\ \varphi_p(x) < 0\right)$$

and

$$\left(\forall x\in\left(x_0^{(p)},b\right)\right)\ \varphi_p(x)<0 \qquad \left(i.e.\ \left(\forall x\in\left(x_0^{(p)},b\right)\right)\ \varphi_p(x)>0\right).$$

(*iii*) *If* $p\geqslant B$, *it holds that*

$$(\forall x\in(a,b))\ \varphi_p(x)\geqslant\varphi_B(x)>0 \qquad (i.e.\ (\forall x\in(a,b))\ \varphi_p(x)\leqslant\varphi_B(x)<0)$$

and the constant B *is the best possible.*

(*iv*) *Each function* $\varphi_p(x)$, *for* $p\in(A,B)$, *has exactly one local extremum, which is a maximum* $($*i.e. minimum*$)$, *on the interval* (a,b) *at the point* $\left(t^{(p)},\varphi_p\left(t^{(p)}\right)\right)$.

(*v*) *There exists exactly one solution* $p=p_0\in(A,B)$ *of the equation*

$$\varphi_p\left(t^{(p)}\right)=\left|\varphi_p(b)\right| \qquad \left(i.e.\ \left|\varphi_p\left(t^{(p)}\right)\right|=\varphi_p(b)\right).$$

The function $\varphi_{p_0}(x)$ *is the minimax approximant of the family on the interval* (a,b). *The approximation error is*

$$d_0=\varphi_{p_0}\left(t^{(p_0)}\right)=\left|\varphi_{p_0}(b)\right| \qquad \left(i.e.\ d_0=\left|\varphi_{p_0}\left(t^{(p_0)}\right)\right|=\varphi_{p_0}(b)\right).$$

Proof. It is analogous to the proof of Theorem 2.1. □

In the case when the family of functions is compressed only at the point b, the corresponding theorems could be formulated analogously. For example, Theorem 2.3 is one such theorem (and it is used in Application 3).

THEOREM 2.3. *Let* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, *for* $x\in\mathbb{S}=[a,b]$ *and* $p\in\mathbb{P}\subseteq\mathbb{R}$ $(\mathbb{P}\neq\emptyset)$, *be a family of functions that are continuous with respect to* p *and with respect to* x *for each* $x\in[a,b]$ *and for each* $p\in\mathbb{P}$ *such that the following conditions hold*:

(1) *the family of functions* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ *is compressed at the point* b *and increasingly* $($*i.e. decreasingly*$)$ *stratified on the interval* $[a,b)$ *with respect to the parameter* $p\in\mathbb{P}$,

(2) *the functions* $\varphi_p(x)$, *for* $p\in\mathbb{P}$, *are differentiable on the interval* (a,b) *and there exists a continuous monotonically decreasing function* $g_1:(a,b)\longrightarrow\mathbb{P}$ *that satisfies* $(*)$,

(3) *there exist limits* $\lim\limits_{x\to a+}g_1(x)=C\in\mathbb{R}$ *and* $\lim\limits_{x\to b-}g_1(x)=A\in\mathbb{R}$, *such that* $(A,C)\subseteq\mathbb{P}$,

(4) *there exists a value of the parameter* $p=B\in(A,C)$ *such that* $\varphi_B(a)=0$,

(5) *there exist a left neighbourhood of the point* b *in which it holds that* $\varphi_p(x)>0$ $($*i.e.* $\varphi_p(x)<0$ $)$ *for each* $p\in(A,C)$.

Then, it holds:

(*i*) *If* $p \leqslant A$, *it holds that*

$$(\forall x \in (a,b))\ \varphi_p(x) \leqslant \varphi_A(x) < 0 \qquad (i.e.\ (\forall x \in (a,b))\ \varphi_p(x) \geqslant \varphi_A(x) > 0)$$

and the constant A *is the best possible.*

(*ii*) *If* $p \in (A,B)$, *then the equation* $\varphi_p(x) = 0$ *has a unique solution* $x_0^{(p)} \in (a,b)$ *and it holds that*

$$\left(\forall x \in \left(a, x_0^{(p)}\right)\right)\ \varphi_p(x) < 0 \qquad \left(i.e.\ \left(\forall x \in \left(a, x_0^{(p)}\right)\right)\ \varphi_p(x) > 0\right)$$

and

$$\left(\forall x \in \left(x_0^{(p)}, b\right)\right)\ \varphi_p(x) > 0 \qquad \left(i.e.\ \left(\forall x \in \left(x_0^{(p)}, b\right)\right)\ \varphi_p(x) < 0\right).$$

(*iii*) *If* $p \geqslant B$, *it holds that*

$$(\forall x \in (a,b))\ \varphi_p(x) \geqslant \varphi_B(x) > 0 \qquad (i.e.\ (\forall x \in (a,b))\ \varphi_p(x) \leqslant \varphi_B(x) < 0)$$

and the constant B *is the best possible.*

(*iv*) *Each function* $\varphi_p(x)$, *for* $p \in (A,B)$, *has exactly one local extremum, which is a maximum* (*i.e. minimum*), *on the interval* (a,b) *at the point* $\left(t^{(p)}, \varphi_p\left(t^{(p)}\right)\right)$.

(*v*) *There exists exactly one solution* $p = p_0 \in (A,B)$ *of the equation*

$$\left|\varphi_p(a)\right| = \varphi_p\left(t^{(p)}\right) \qquad \left(i.e.\ \varphi_p(a) = \left|\varphi_p\left(t^{(p)}\right)\right|\right).$$

The function $\varphi_{p_0}(x)$ *is the minimax approximant of the family on the interval* (a,b). *The approximation error is*

$$d_0 = \left|\varphi_{p_0}(a)\right| = \varphi_{p_0}\left(t^{(p_0)}\right) \qquad \left(i.e.\ d_0 = \varphi_{p_0}(a) = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right|\right).$$

Proof. It is analogous to the proof of Theorem 2.1. □

3. Applications

In this section, we illustrate applications of the novel parametric method on the Cusa-Huygens inequality, Mitrinović-Adamović-type inequality from [17] and Jordan-type inequality from [20, 33].

3.1. Application 1 (Cusa-Huygens inequality)

The inequality

$$x > \frac{3\sin x}{2+\cos x}$$

for $x \in \left(0, \frac{\pi}{2}\right)$, is called the Cusa-Huygens inequality [1–3, 7, 27, 35, 39, 41, 48].

In [28, 31], the Cusa-Huygens inequality was considered by introducing the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, where

$$\varphi_p(x) = \begin{cases} x - \dfrac{(p+1)\sin x}{p+\cos x}, & x \in \left(0, \frac{\pi}{2}\right], \\ 0, & x = 0, \end{cases}$$

which is defined for $x \in \left[0, \frac{\pi}{2}\right]$ and $p \in \mathbb{P} = \mathbb{R}\backslash(-1,0)$.

LEMMA 3.1. (Lemma 1 [28]) *The family of functions* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ *is*

(*i*) *increasingly stratified on the interval* $\left(0, \frac{\pi}{2}\right]$ *with respect to the parameter* $p \in \mathbb{P}_1 = [0, +\infty)$,

(*ii*) *increasingly stratified on the interval* $\left(0, \frac{\pi}{2}\right]$ *with respect to the parameter* $p \in \mathbb{P}_2 = (-\infty, -1]$.

REMARK 3.1. The family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is not stratified on the interval $\left(0, \frac{\pi}{2}\right]$ with respect to the parameter $p \in \mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$. Moreover,

$$\left(\forall x \in \left(0, \tfrac{\pi}{2}\right]\right) (\forall p_1 \in \mathbb{P}_1)(\forall p_2 \in \mathbb{P}_2)\ \varphi_{p_1}(x) < \varphi_{p_2}(x).$$

By applying the novel parametric method (Theorem 2.1), we provide a simpler proof of the following statement from [28, 31].

THEOREM 3.1. *Let*

$$A = \frac{1+\sqrt{5}}{2} = 1.61803\ldots, \quad B = \frac{2}{\pi - 2} = 1.75193\ldots \quad \text{and} \quad C = 2.$$

Then, it holds:

(*i*) *If* $p \in (0, B)$, *it holds that*

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right)\ x < \frac{(B+1)\sin x}{B+\cos x} < \frac{(p+1)\sin x}{p+\cos x}$$

and the constant B *is the best possible.*

(*ii*) *If* $p \in (B,C)$*, the equation* $\varphi_p(x) = 0$ *has a unique solution* $x_0^{(p)}$ *and it holds that*

$$\left(\forall x \in \left(0, x_0^{(p)}\right)\right) \ x < \frac{(p+1)\sin x}{p+\cos x}$$

and

$$\left(\forall x \in \left(x_0^{(p)}, \frac{\pi}{2}\right)\right) \ x > \frac{(p+1)\sin x}{p+\cos x}.$$

(*iii*) *If* $p \in (-\infty, -1] \cup (C, +\infty)$*, it holds that*

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \ x > \frac{(C+1)\sin x}{C+\cos x} > \frac{(p+1)\sin x}{p+\cos x}$$

and the constant C *is the best possible.*

(*iv*) *Each function* $\varphi_p(x)$*, for* $p \in (A,C)$*, has exactly one stationary point on the interval* $\left(0, \frac{\pi}{2}\right)$*. Moreover, for* $p \in (B,C)$*, each function* $\varphi_p(x)$ *has exactly one local minimum on the interval* $\left(0, \frac{\pi}{2}\right)$ *at the point* $\left(t^{(p)}, \varphi_p\left(t^{(p)}\right)\right)$*.*

There exists exactly one solution of the equation $\left|\varphi_p\left(t^{(p)}\right)\right| = \varphi_p\left(\frac{\pi}{2}\right)$ *with respect to the parameter* $p \in (B,C)$*, which is numerically determined as*

$$p_0 = 1.78114\ldots.$$

The minimax approximant of the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ *on the interval* $\left(0, \frac{\pi}{2}\right)$ *is*

$$\varphi_{p_0}(x) = x - \frac{(p_0+1)\sin x}{p_0+\cos x},$$

which determines the corresponding minimax approximation

$$x \approx \frac{2.78114\ldots \sin x}{1.78114\ldots + \cos x}$$

with the approximation error

$$d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}\left(\frac{\pi}{2}\right) = 0.0093601\ldots.$$

Proof. Based on Lemma 3.1, we distinguish the following two cases:

Case 1. $p \in \mathbb{P}_1 = [0, +\infty)$:

We will show that for the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}_1}$, the conditions of Theorem 2.1 are satisfied.

Condition **(1)** of Theorem 2.1 holds based on Lemma 3.1 and the fact that the family $\{\varphi_p(x)\}_{p\in\mathbb{P}_1}$ is compressed at the point 0.

For $x \in \left(0, \frac{\pi}{2}\right)$, the following equivalence holds:

$$\frac{\partial \varphi_p(x)}{\partial x} = 1 - \frac{(p+1)(1+p\cos x)}{(p+\cos x)^2} = 0 \iff \begin{array}{l} p = g_1^-(x) = \frac{1}{2}\left(1-\sqrt{5+4\cos x}\right) \\ \vee \; p = g_1^+(x) = \frac{1}{2}\left(1+\sqrt{5+4\cos x}\right). \end{array}$$

The function $g_1^-(x)$ is monotonically increasing on the interval $\left(0, \frac{\pi}{2}\right)$ and $g_1^-\left(\left(0, \frac{\pi}{2}\right)\right) = \left(-1, \frac{1-\sqrt{5}}{2}\right) \not\subset \mathbb{P}_1$. Thus, we do not consider the function $g_1^-(x)$ in the following.

The function $g_1^+(x)$ is monotonically decreasing on the interval $\left(0, \frac{\pi}{2}\right)$ and $g_1^+\left(\left(0, \frac{\pi}{2}\right)\right) = (A, C) = \left(\frac{1+\sqrt{5}}{2}, 2\right) \subset \mathbb{P}_1$. Thus, the function $g_1^+(x)$ satisfies conditions **(2)** and **(3)** of Theorem 2.1.

For the value

$$p = B = \frac{2}{\pi - 2},$$

it holds that $\varphi_B\left(\frac{\pi}{2}\right) = 0$, which means that condition **(4)** of Theorem 2.1 is satisfied.

Condition **(5)** of Theorem 2.1 follows from the Taylor expansion of $\varphi_p(x)$ in a neighbourhood of the point 0, which is given by

$$\varphi_p(x) = \frac{p-2}{6(p+1)} x^3 + o\left(x^3\right).$$

Hence, all conditions for the application of Theorem 2.1 are satisfied, from which parts ***(i)***, ***(ii)*** and ***(iv)***, as well as part ***(iii)*** for $p \in (2, +\infty)$, follow.

Case 2. $p \in \mathbb{P}_2 = (-\infty, -1]$:

Part ***(iii)*** for $p \in (-\infty, -1]$ follows from Remark 3.1. □

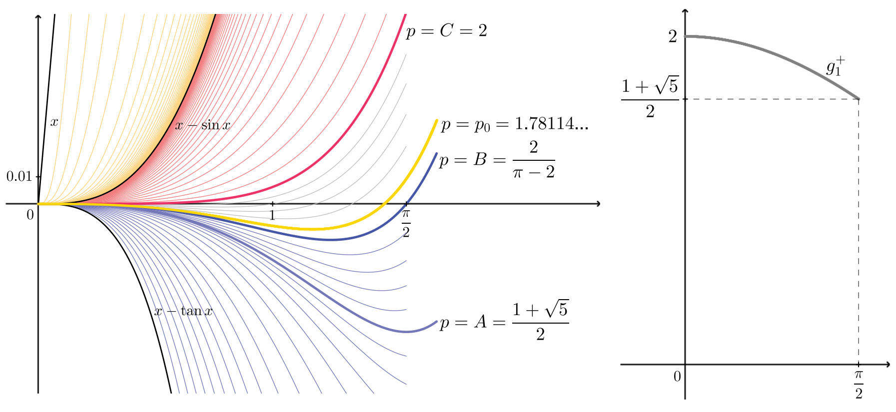

Figure 2: *Some functions from the family* $\{\varphi_p(x)\}_{p \in \mathbb{P}}$ *and the function* g_1^+.

3.2. Application 2 (Mitrinović-Adamović-type inequality)

The inequality

$$\left(\frac{\sin x}{x}\right)^3 > \cos x$$

for $x \in \left(0, \frac{\pi}{2}\right)$, is called the Mitrinović-Adamović inequality [4, 11, 17, 36, 41, 47, 49]. In [17], the following Mitrinović-Adamović-type inequality is given.

THEOREM 3.2. *For $x \in \left(0, \frac{\pi}{2}\right)$, it holds that*

$$1 - \left(\frac{1}{2} + \frac{\pi^3 - 16}{\pi^4} x \sin x\right) \sin^2 x < \left(\frac{\sin x}{x}\right)^3 < 1 - \left(\frac{1}{2} + \frac{7}{120} x \sin x\right) \sin^2 x$$

and the constants $\frac{\pi^3-16}{\pi^4}$ and $\frac{7}{120}$ are the best possible.

Based on the previous inequality, let us introduce the family of functions $\{\varphi_p(x)\}_{p \in \mathbb{P}}$, where

$$\varphi_p(x) = \begin{cases} \left(\dfrac{\sin x}{x}\right)^3 - 1 + \left(\dfrac{1}{2} + p x \sin x\right) \sin^2 x, & x \in \left(0, \frac{\pi}{2}\right], \\ 0, & x = 0, \end{cases}$$

which is defined for $x \in \left[0, \frac{\pi}{2}\right]$ and $p \in \mathbb{P} = \mathbb{R}$.

LEMMA 3.2. *The family of functions $\{\varphi_p(x)\}_{p \in \mathbb{P}}$ is increasingly stratified on the interval $\left(0, \frac{\pi}{2}\right]$ with respect to the parameter $p \in \mathbb{P} = \mathbb{R}$.*

Proof. It holds that $\dfrac{\partial \varphi_p(x)}{\partial p} = x \sin^3 x > 0$ for $x \in \left(0, \frac{\pi}{2}\right]$. □

By applying the novel parametric method (Theorem 2.2), we give the following improvement of Theorem 3.2.

THEOREM 3.3. *Let*

$$A = \frac{7}{120} = 0.058\overline{3}\,, \quad B = \frac{\pi^3 - 16}{\pi^4} = 0.15405\ldots \quad \textit{and} \quad C = \frac{48}{\pi^4} = 0.49276\ldots.$$

Then, it holds:

(i) *If $p \in (-\infty, A)$, it holds that*

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \left(\frac{\sin x}{x}\right)^3 < 1 - \left(\frac{1}{2} + A x \sin x\right) \sin^2 x < 1 - \left(\frac{1}{2} + p x \sin x\right) \sin^2 x$$

and the constant A is the best possible.

(*ii*) *If* $p\in(A,B)$, *the equation* $\varphi_p(x)=0$ *has a unique solution* $x_0^{(p)}$ *and it holds that*

$$\left(\forall x\in\left(0,x_0^{(p)}\right)\right)\ \left(\frac{\sin x}{x}\right)^3>1-\left(\frac{1}{2}+px\sin x\right)\sin^2 x$$

and

$$\left(\forall x\in\left(x_0^{(p)},\frac{\pi}{2}\right)\right)\ \left(\frac{\sin x}{x}\right)^3<1-\left(\frac{1}{2}+px\sin x\right)\sin^2 x.$$

(*iii*) *If* $p\in(B,+\infty)$, *it holds that*

$$\left(\forall x\in\left(0,\frac{\pi}{2}\right)\right)\ \left(\frac{\sin x}{x}\right)^3>1-\left(\frac{1}{2}+Bx\sin x\right)\sin^2 x>1-\left(\frac{1}{2}+px\sin x\right)\sin^2 x$$

and the constant B *is the best possible.*

(*iv*) *Each function* $\varphi_p(x)$, *for* $p\in(A,C)$, *has exactly one stationary point on the interval* $\left(0,\frac{\pi}{2}\right)$. *Moreover, for* $p\in(A,B)$, *each function* $\varphi_p(x)$ *has exactly one local maximum on the interval* $\left(0,\frac{\pi}{2}\right)$ *at the point* $\left(t^{(p)},\varphi_p\left(t^{(p)}\right)\right)$.

There exists exactly one solution of the equation $\varphi_p\left(t^{(p)}\right)=\left|\varphi_p\left(\frac{\pi}{2}\right)\right|$ *with respect to the parameter* $p\in(A,B)$, *which is numerically determined as*

$$p_0=0.13306\ldots.$$

The minimax approximant of the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ *on the interval* $\left(0,\frac{\pi}{2}\right)$ *is*

$$\varphi_{p_0}(x)=\left(\frac{\sin x}{x}\right)^3-1+\left(\frac{1}{2}+p_0x\sin x\right)\sin^2 x,$$

which determines the corresponding minimax approximation

$$\left(\frac{\sin x}{x}\right)^3\approx 1-\left(\frac{1}{2}+0.13306\ldots x\sin x\right)\sin^2 x$$

with the approximation error

$$d_0=\varphi_{p_0}\left(t^{(p_0)}\right)=\left|\varphi_{p_0}\left(\frac{\pi}{2}\right)\right|=0.032963\ldots.$$

Proof. We will show that for the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, the conditions of Theorem 2.2 are satisfied.

Condition **(1)** of Theorem 2.2 holds based on Lemma 3.2 and the fact that the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is compressed at the point 0.

We now show that condition **(2)** of Theorem 2.2 is satisfied. For $x \in \left(0, \frac{\pi}{2}\right)$, the following equivalence holds:

$$\frac{\partial \varphi_p(x)}{\partial x} = \frac{\sin x\,(3px^5 \sin x \cos x - px^4 \cos^2 x + x^4 \cos x + px^4 + 3x \sin x \cos x + 3\cos^2 x - 3)}{x^4} = 0$$

$$\iff \quad p = g_1(x) = \frac{-x^4 \cos x - 3x \sin x \cos x + 3\sin^2 x}{x^4 \sin x\,(3x\cos x + \sin x)}.$$

The first derivative of the function $g_1(x)$ is

$$\frac{dg_1(x)}{dx} = \frac{h_1(x)}{x^5 \sin^2 x\, h_2(x)},$$

where

$$\begin{aligned} h_1(x) \;=\;& 3x^6 \cos^3 x + 4x^5 \sin x \cos^2 x + x^5 \sin x - 36x^2 \cos^4 x + 24x^2 \cos^2 x + \\ &+ 36x \sin x \cos^3 x - 36x \sin x \cos x - 12\cos^4 x + 24\cos^2 x + 12x^2 - 12 \end{aligned}$$

and

$$h_2(x) = 9x^2 \cos^2 x + 6x \sin x \cos x - \cos^2 x + 1.$$

It is evident that $h_2(x) > 0$ on the interval $\left(0, \frac{\pi}{2}\right)$ since $\cos^2 x < 1$.

Let us prove that $h_1(x) > 0$ on the interval $\left(0, \frac{\pi}{2}\right)$ by applying the method for proving MTP inequalities. It holds that

$$\begin{aligned} h_1(x) \;=\;& \tfrac{9}{4}x^6 \cos x + \tfrac{3}{4}x^6 \cos 3x + 2x^5 \sin x + x^5 \sin 3x - 9x \sin(2x) + \\ &+ \tfrac{9}{2}x \sin 4x - 6\left(x^2 - 1\right)\cos 2x - \tfrac{3}{2}\left(3x^2 + 1\right)\cos 4x + \tfrac{21}{2}x^2 - \tfrac{9}{2}. \end{aligned}$$

Let $T_n^{\phi,a}$ denote the Taylor expansion of order n of a function ϕ in a neighbourhood of the point a. One downward polynomial approximation of the function $h_1(x)$ on the interval $\left(0, \frac{\pi}{2}\right)$ is

$$\begin{aligned} P(x) =& \tfrac{9}{4}x^6 T_{10}^{\cos,0}(x) + \tfrac{3}{4}x^6 T_{10}^{\cos,0}(3x) + 2x^5 T_{11}^{\sin,0}(x) + x^5 T_{11}^{\sin,0}(3x) - 9x T_{13}^{\sin,0}(2x) + \\ &+ \tfrac{9}{2}x T_{15}^{\sin,0}(4x) + 6T_{14}^{\cos,0}(2x) - 6x^2 T_{12}^{\cos,0}(2x) \underbrace{-\tfrac{3}{2}\left(3x^2+1\right)}_{(<0)} T_{12}^{\cos,0}(4x) + \tfrac{21}{2}x^2 - \tfrac{9}{2} \\ =& x^{10} Q(x), \end{aligned}$$

where

$$Q(x) = -\frac{46172263}{2270268000}x^6 + \frac{3049559}{45405360}x^4 - \frac{661}{1575}x^2 + \frac{62}{63}.$$

By applying Sturm's theorem to the polynomial $Q(x)$, we conclude that this polynomial does not have zeros on the segment $\left[0, \frac{\pi}{2}\right]$. Thus, the polynomial $P(x)$ does not have zeros on the interval $\left(0, \frac{\pi}{2}\right)$. Since $P\left(\frac{\pi}{4}\right) = 0.066627\ldots > 0$, we conclude that $P(x) > 0$ on the interval $\left(0, \frac{\pi}{2}\right)$. Therefore, $\frac{dg_1(x)}{dx} > 0$ and thus the function $g_1(x)$ is monotonically increasing on the interval $\left(0, \frac{\pi}{2}\right)$. Hence, condition **(2)** of Theorem 2.2 is satisfied.

There exist limit values

$$\lim_{x\to 0+} g_1(x) = A = \frac{7}{120} \quad \text{and} \quad \lim_{x\to\frac{\pi}{2}-} g_1(x) = C = \frac{48}{\pi^4}$$

and thus condition **(3)** of Theorem 2.2 is satisfied.

For the value

$$p = B = \frac{\pi^3 - 16}{\pi^4},$$

it holds that $\varphi_B\left(\frac{\pi}{2}\right) = 0$, which means that condition **(4)** of Theorem 2.2 is satisfied.

Condition **(5)** of Theorem 2.2 follows from the Taylor expansion of $\varphi_p(x)$ in a neighbourhood of the point 0, which is given by

$$\varphi_p(x) = \left(p - \frac{7}{120}\right)x^4 + o\left(x^4\right).$$

Hence, all conditions for the application of Theorem 2.2 are satisfied, which concludes the proof. □

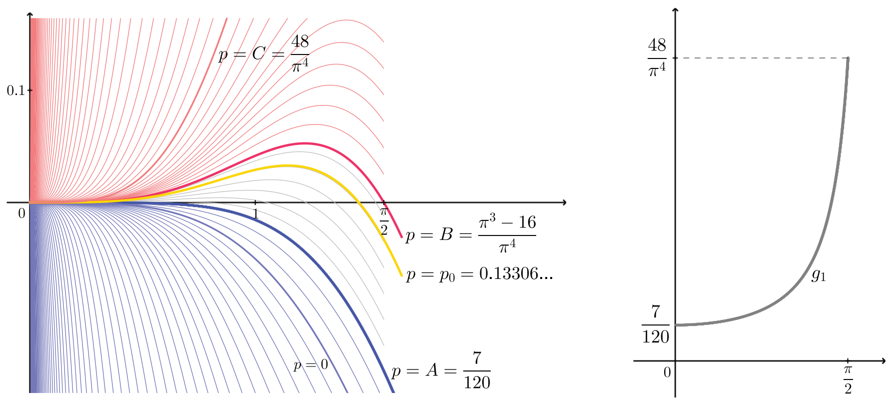

Figure 3: *Some functions from the family* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ *and the function* g_1.

3.3. Application 3 (Jordan-type inequality)

In this section, we consider one family of functions to which the parametric method from [28] could not be applied, but the novel parametric method could.

The inequality

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x}$$

for $x \in \left(0, \frac{\pi}{2}\right]$, is called the Jordan's inequality [20, 33, 35, 40, 43, 46]. In [20], the following Jordan-type inequality is given.

THEOREM 3.4. *For $x \in \left(0, \frac{\pi}{2}\right]$ and $n \in \mathbb{N}$, it holds that*

$$\frac{2}{\pi} + \frac{2}{\pi^2}(\pi - 2x) \geqslant \frac{\sin x}{x} \geqslant \frac{2}{\pi} + \frac{\pi-2}{\pi^2}(\pi - 2x)$$

and

$$\frac{2}{\pi} + \frac{2}{n\,\pi^{n+1}}(\pi^n - (2x)^n) \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi-2}{\pi^{n+1}}(\pi^n - (2x)^n) \quad (\textit{for } n \geqslant 2).$$

Based on the left sides of the inequalities from the previous theorem, according to [33], we introduce the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, where

$$\varphi_p(x) = \begin{cases} \dfrac{\sin x}{x} - \dfrac{2}{\pi} - \dfrac{2}{p\,\pi^{p+1}}(\pi^p - (2x)^p), & x \in \left(0, \frac{\pi}{2}\right], \\[2ex] 1 - \dfrac{2}{\pi}\left(1 + \dfrac{1}{p}\right), & x = 0, \end{cases}$$

which is defined for $x \in \left[0, \frac{\pi}{2}\right]$ and $p \in \mathbb{P} = \mathbb{R}\backslash\{0\}$. Let us notice that

$$\varphi_p\left(\frac{\pi}{2}\right) = 0$$

for each $p \in \mathbb{P}$, i.e. the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is compressed at the point $\frac{\pi}{2}$.

In [33], it was proved that the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is stratified on the interval $\left(0, \frac{\pi}{2}\right)$ with respect to the parameter $p \in \mathbb{R}^+$ (Lemma 3 [33]). Analogously to that proof, the following statement could be proved.

LEMMA 3.3. *The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is increasingly stratified on the interval $\left[0, \frac{\pi}{2}\right)$ with respect to the parameter $p \in \mathbb{P} = \mathbb{R}\backslash\{0\}$.*

Since the equation

$$\varphi_p(x) = \frac{\sin x}{x} - \frac{2}{\pi} + \frac{2}{p\,\pi^{p+1}}(\pi^p - (2x)^p) = 0$$

is not directly solvable with respect to p, in [33], the parametric method was not applied.

The minimax approximant of the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ and the corresponding inequalities for $p \in \mathbb{R}^+$ in [33] were obtained based on the analysis of the function g_1 that satisfies $(*)$. In this paper, by applying the novel parametric method (Theorem 2.3), we consider the corresponding inequalities for $p \in \mathbb{R}\backslash\{0\}$ and prove the following theorem.

THEOREM 3.5. *Let*

$$A = \frac{\pi^2}{4} - 1 = 1.46740\ldots, \quad B = \frac{2}{\pi - 2} = 1.75193\ldots \quad \textit{and} \quad C = 2.$$

Then, it holds:

(*i*) *If* $p \in (-\infty, 0) \cup (0, A)$*, it holds that*

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \ \frac{\sin x}{x} < \frac{2}{\pi} + \frac{2}{A\,\pi^{A+1}}\left(\pi^A - (2x)^A\right) < \frac{2}{\pi} + \frac{2}{p\,\pi^{p+1}}\left(\pi^p - (2x)^p\right)$$

and the constant A *is the best possible.*

(*ii*) *If* $p \in (A, B)$*, the equation* $\varphi_p(x) = 0$ *has a unique solution* $x_0^{(p)}$ *and it holds that*

$$\left(\forall x \in \left(0, x_0^{(p)}\right)\right) \ \frac{\sin x}{x} < \frac{2}{\pi} + \frac{2}{p\,\pi^{p+1}}\left(\pi^p - (2x)^p\right)$$

and

$$\left(\forall x \in \left(x_0^{(p)}, \frac{\pi}{2}\right)\right) \ \frac{\sin x}{x} > \frac{2}{\pi} + \frac{2}{p\,\pi^{p+1}}\left(\pi^p - (2x)^p\right).$$

(*iii*) *If* $p \in (B, +\infty)$*, it holds that*

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \ \frac{\sin x}{x} > \frac{2}{\pi} + \frac{2}{B\,\pi^{B+1}}\left(\pi^B - (2x)^B\right) > \frac{2}{\pi} + \frac{2}{p\,\pi^{p+1}}\left(\pi^p - (2x)^p\right)$$

and the constant B *is the best possible.*

(*iv*) *Each function* $\varphi_p(x)$*, for* $p \in (A, C)$*, has exactly one stationary point on the interval* $\left(0, \frac{\pi}{2}\right)$*. Moreover, for* $p \in (A, B)$*, each function* $\varphi_p(x)$ *has exactly one local maximum on the interval* $\left(0, \frac{\pi}{2}\right)$ *at the point* $\left(t^{(p)}, \varphi_p\left(t^{(p)}\right)\right)$.

There exists exactly one solution of the equation $\left|\varphi_p(0)\right| = \varphi_p\left(t^{(p)}\right)$ *with respect to the parameter* $p \in (A, B)$*, which is numerically determined as*

$$p_0 = 1.72287\ldots.$$

The minimax approximant of the family $\{\varphi_p(x)\}_{p \in \mathbb{P}}$ *on the interval* $\left(0, \frac{\pi}{2}\right)$ *is*

$$\varphi_{p_0}(x) = \frac{\sin x}{x} - \frac{2}{\pi} - \frac{2}{p_0\,\pi^{p_0+1}}\left(\pi^{p_0} - (2x)^{p_0}\right),$$

which determines the corresponding minimax approximation

$$\frac{\sin x}{x} \approx \frac{2}{\pi} + 0.051415\ldots\left(\pi^{1.72287\ldots} - (2x)^{1.72287\ldots}\right)$$

with the approximation error

$$d_0 = \left|\varphi_{p_0}(0)\right| = \varphi_{p_0}\left(t^{(p_0)}\right) = 0.0061296\ldots.$$

Proof. We will show that for the family of functions $\{\varphi_p(x)\}_{p \in \mathbb{P}}$, the conditions of Theorem 2.3 are satisfied.

Condition **(1)** of Theorem 2.3 holds based on Lemma 3.3 and the fact that the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is compressed at the point $\frac{\pi}{2}$.

For $x\in\left(0,\frac{\pi}{2}\right)$, the following equivalence holds:

$$\frac{\partial\varphi_p(x)}{\partial x}=\frac{x\cos x-\sin x+\left(\dfrac{2x}{\pi}\right)^{p+1}}{x^2}=0 \iff p=g_1(x)=\frac{\ln\dfrac{2x}{\pi(\sin x-x\cos x)}}{\ln\dfrac{\pi}{2x}}.$$

By applying L'Hôpital's rule for monotonicity [16, 42] and the method for proving MTP inequalities, in [33], it was proved that the function $g_1(x)$ is monotonically decreasing on the interval $\left(0,\frac{\pi}{2}\right)$.

There exist limit values

$$\lim_{x\to 0+} g_1(x)=C=2 \quad \text{and} \quad \lim_{x\to\frac{\pi}{2}-} g_1(x)=A=\frac{\pi^2}{4}-1$$

and thus conditions **(2)** and **(3)** of Theorem 2.3 are satisfied.

For the value

$$p=B=\frac{2}{\pi-2},$$

it holds that $\varphi_B(0)=0$, which means that condition **(4)** of Theorem 2.3 is satisfied.

Condition **(5)** of Theorem 2.3 follows from the Taylor expansion of $\varphi_p(x)$ in a neighbourhood of the point $\frac{\pi}{2}$, which is given by

$$\varphi_p(x)=\frac{4p-\pi^2+4}{\pi^3}\left(x-\frac{\pi}{2}\right)^2+o\left(\left(x-\frac{\pi}{2}\right)^2\right).$$

Hence, all conditions for the application of Theorem 2.3 are satisfied, which concludes the proof. □

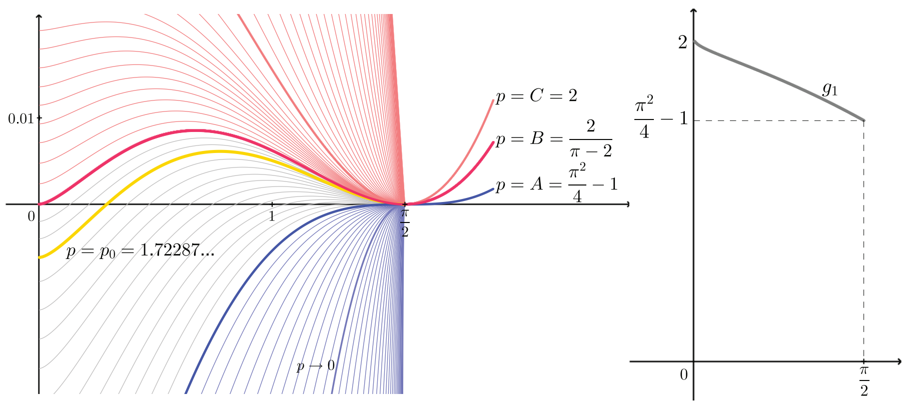

Figure 4: *Some functions from the family* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ *and the function* g_1.

4. Conclusion

The novel method for proving some analytic inequalities and determining minimax approximants, introduced in this paper, could be applied to numerous inequalities [1,2, 4–6,12,13,15,17,18,20,24,28,31–35,37,43,48,49]. By using this method, the best constants for the corresponding inequalities are obtained.

The minimax approximants are, in [5,24,31,32], determined using Nike theorem (Theorem 3 [31]) and Theorem 1, i.e. Theorem 1' from [31]. Determining the minimax approximant using the novel parametric method is often significantly simpler, as shown in Application 1. In Application 2, a generalisation and improvement of the Mitrinović-Adamović-type inequality was obtained. In Application 3, the novel parametric method is applied to a family of functions to which the parametric method from [28] could not be applied.

It is to be expected that the application of the novel parametric method will enable the improvement of the existing inequalities and discovering new ones, as well as determining the corresponding minimax approximations. The subject of future research will be the analysis of the cases when the function g_1 that satisfies $(*)$ has local extrema and/or an infinite limit on the observed interval.

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