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ANALYSIS OF MACLAURIN'S INEQUALITY WITH APPLICATIONS IN NUMERICAL ANALYSIS

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Abstract. Maclaurin's inequality estimates the error bounds of a three-point open method named as Maclaurin's procedure. The current study aims to explore the error boundaries of Maclaurin's rule by utilizing the convexity of the mappings. We derive a new twice-differentiable Maclaurin's identity. Based on newly developed identity, the convexity of mappings, and the elementary properties of inequalities, we derive some new Maclaurin's type inequalities. Also, we apply the obtained bounds to formulate the relation between means, composite quadrature bounds, and a novel two-step iterative method with a cubic order of convergence. Lastly, we explore our major findings and the iterative method through illustrative examples and visuals.

1. Introduction and preliminaries

The theory of inequalities is the core of mathematical analysis, and it has experienced exponential progression over the last few decades due to the dependency of several other domains of physical sciences. It has studied from multiple aspects, for example, to conclude novel refinements to evacuate the limitations, unify, and increase the applicable domain of existing results. We have deployed several approaches, which include different generalizations of calculus, operator theory, and convex function theory. Convex functions, such as Jensen's inequality, the converse Jensen's inequality, Jensen-Mercer's inequality, Hermite-Hadamard's inequality, Hölder's type inequalities, and several error inequalities of quadrature and cubature procedures, can be used to prove several classical inequalities. Both Jensen's and the trapezium inequalities are cornerstones to investigate the convexity of mappings. Let us revisit the notion of a convex function.

DEFINITION 1. Let $f:[a,b]\to\mathbb{R}$ be a convex mapping, then

$$f((1-t)x+ty)\leqslant(1-t)f(x)+tf(y),\quad \forall x,y\in[a,b] \tag{1}$$

where $t\in[0,1]$.

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Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping, then

$$(b-a)f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)\mathrm{d}x \leqslant (b-a)\frac{f(a)+f(b)}{2}. \tag{2}$$

This inequality has gained valuable attention from mathematicians due to its geometrical, analytical properties, and applicable aspects. It has been transformed and proved in different frameworks through various approaches. One of the key strategies for exploring this inequality is to calculate both the left and right estimations of it. In [12], Dragomir and Agarwal presented the right estimations of it in association with first-order differentiable convex mappings, which have an immense number of applications. In fact, they computed the error upper boundaries of the trapezoidal procedure and set a new venue of thinking to explore the bounds of the remaining terms of quadrature and cubature rules. For more detail, consult [10, 14, 18–20, 28, 29].

Observing that midpoint rules meet the constraints of closed Newton-Cotes schemes, we cannot apply closed procedures to functions that exhibit discontinuities at their end points. Now, we recollect the error inequalities of Simpson's and Maclaurin's procedures, respectively.

THEOREM 1. [13] *If $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable on (a,b), and $||f^{(4)}||_\infty = \sup_{x\in(a,b)} |f^{(4)}| < \infty$, then*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leqslant \frac{1}{2880}||f^{(4)}||_\infty(b-a)^5.$$

THEOREM 2. *Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable on (a,b), and $||f^{(4)}||_\infty = \sup_{x\in(a,b)} |f^{(4)}| < \infty$, then*

$$\left|\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leqslant \frac{7(b-a)^4}{51840}||f^{(4)}||_\infty.$$

Dragomir and Agarwal [13] investigated the Simpson's inequality through several classes of functions with applications and is still a gateway to conducting research on these kinds of inequalities. Dragomir et al. [15] used the unified approach to look into the error approximations of Ostrowski's, Hermite-Hadamard's, and Simpson's-like inequalities by using monotonic functions and finding interesting uses for them. Yang and his coauthors [30] concluded some new refinements of Simpson's inequality through (s,m)-convexity. Ujevac [27] analyzed the sharpness of both Simpson's and Ostrowski's like inequalities. Alomari et al. [3] discussed the error boundaries of Simpson's rule, employing Breckner convexity. Awan et al. [5] came up with some new weighted error inequalities for the three-point closed rule using generalised convexity and showed how they could be used. In [22], Noor et al. studied the harmonic convexity over a rectangular domain and developed trapezoid-like inequalities. Chu et al. [9] defined a new class of functions by bridging the idea of n-polynomial and higher strongly convexity. Additionally, they demonstrated the strong bounds of Simpson's inequality using this class of functions.

Alomari and Dragomir [2,4] looked at the error estimates of a number of closed and open methods, such as the Euler procedure, using different types of differentiable mappings and talked about Milne's-like inequalities. Bin-Mohsin et al. [6] presented the refinements of Milne's-like inequalities, integrating both quantum calculus and the Mercer approach. In [7], authors computed the fractional analogues of Milne's-like inequalities via bounded variation, Lipschitz, and convex functions. Pecaric et al. [23] delivered novel generalizations of Maclaurin's-like inequalities with potential applications. Meftah et al. [21] used Yang local fractional to look at the error limits of Maclaurin's method using generalised local convexity. Hezenci and his fellows [17] explored fractional analogues of Maclaurin's inequality via convex mappings. For more details, see [24,26].

The primary focus of this study is to investigate the tight bounds of Maclaurin's inequalities involving convex mappings. First, we will establish a new Maclaurin's equation, which will act as an auxiliary result to produce our major findings leveraging the convexity. We also validate our results using numerical and visual techniques. To enhance the impact of the current study, we present some novel applications, especially an iterative scheme. Our result will provide tight bounds as compared to first-order differentiable convex functions.

2. Main results

This section presents significant advancements in Maclaurin's inequality through the use of twice differentiable convex functions. First, we construct an identity based on two differentiable continuous mappings.

LEMMA 1. *Let $f: I \to \mathbb{R}$ be a differential mapping on $I^o, a, b \in I^o$ with $a < b$ and $f \in L[a,b]$, then the following equality hold*

$$\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \tag{3}$$

$$\begin{aligned}
&=\frac{(b-a)^2}{16}\left[\int_0^{\frac{1}{6}}(-8t^2)f''((1-t)a+tb)\mathrm{d}t\right.\\
&\quad+\int_{\frac{1}{6}}^{\frac{1}{2}}(2t-1)(1-4t)f''((1-t)a+tb)\mathrm{d}t\\
&\quad+\int_{\frac{1}{2}}^{\frac{5}{6}}(2t-1)(3-4t)f''((1-t)a+tb)\mathrm{d}t\\
&\quad\left.+\int_{\frac{5}{6}}^{1}(t-1)(8-8t)f''((1-t)a+tb)\mathrm{d}t\right].
\end{aligned}$$

Proof. Let

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{6}} (-8t^2) f''((1-t)a+tb)\mathrm{d}t \\
I_2 &= \int_{\frac{1}{6}}^{\frac{1}{2}} (2t-1)(1-4t) f''((1-t)a+tb)\mathrm{d}t \\
I_3 &= \int_{\frac{1}{2}}^{\frac{5}{6}} (2t-1)(3-4t) f''((1-t)a+tb)\mathrm{d}t \\
I_4 &= \int_{\frac{5}{6}}^{1} (t-1)(8-8t) f''((1-t)a+tb)\mathrm{d}t.
\end{aligned}
$$

Implementing the integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{6}} (-8t^2) f''((1-t)a+tb)\mathrm{d}t \\
&= \left| \frac{-8t^2 f'((1-t)a+tb)}{b-a} \right|_0^1 + \frac{16}{b-a}\int_0^{\frac{1}{6}} t f'((1-t)a+tb)\mathrm{d}t \\
&= \frac{-2}{9(b-a)} f'\left(\frac{5a+b}{6}\right) + \frac{8}{3(b-a)^2} f\left(\frac{5a+b}{6}\right) - \frac{16}{(b-a)^3}\int_a^{\frac{5a+b}{6}} f(x)\mathrm{d}x. \qquad (4)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{6}}^{\frac{1}{2}} (2t-1)(1-4t) f''((1-t)a+tb)\mathrm{d}t \\
&= \frac{2}{9(b-a)} f'\left(\frac{5a+b}{6}\right) + \frac{2}{(b-a)^2} f\left(\frac{a+b}{2}\right) + \frac{10}{3(b-a)^2} f\left(\frac{5a+b}{6}\right) \\
&\quad - \frac{16}{(b-a)^3}\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} f(x)\mathrm{d}x, \qquad (5) \\
I_3 &= \int_{\frac{1}{2}}^{\frac{5}{6}} (2t-1)(3-4t) f''((1-t)a+tb)\mathrm{d}t \\
&= -\frac{2}{9(b-a)} f'\left(\frac{a+5b}{6}\right) + \frac{2}{(b-a)^2} f\left(\frac{a+b}{2}\right) + \frac{10}{3(b-a)^2} f\left(\frac{a+5b}{6}\right) \\
&\quad - \frac{16}{(b-a)^3}\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} f(x)\mathrm{d}x, \qquad (6) \\
I_4 &= \int_{\frac{5}{6}}^{1} (t-1)(8-8t) f''((1-t)a+tb)\mathrm{d}t \\
&= \frac{2}{9(b-a)} f'\left(\frac{a+5b}{6}\right) + \frac{8}{3(b-a)^2} f\left(\frac{a+5b}{6}\right) - \frac{16}{(b-a)^3}\int_{\frac{a+5b}{6}}^{b} f(x)\mathrm{d}x. \qquad (7)
\end{aligned}
$$

Summing (4–7) and then taking the product of obtained result by $\frac{(b-a)^2}{16}$, we get (3). □

THEOREM 3. *Assume that all of the requirements of Lemma 1 are fulfilled. If $|f''|$ is a convex mapping on $[a,b]$, then*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leqslant \frac{(b-a)^2}{384}(|f''(a)|+|f''(b)|). \tag{8}$$

Proof. Considering Lemma 1 and implementing the modulus characteristic and the convexity of $|f''|$, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\leqslant \frac{(b-a)^2}{16}\left[\int_0^{\frac{1}{6}} |(-8t^2)||f''((1-t)a+tb)|\mathrm{d}t\right. \\
&\quad +\int_{\frac{1}{6}}^{\frac{1}{2}} |(2t-1)(1-4t)||f''((1-t)a+tb)|\mathrm{d}t \\
&\quad +\int_{\frac{1}{2}}^{\frac{5}{6}} |(2t-1)(3-4t)||f''((1-t)a+tb)|\mathrm{d}t \\
&\quad \left.+\int_{\frac{5}{6}}^{1} |(t-1)(8-8t)||f''((1-t)a+tb)|\mathrm{d}t\right] \\
&\leqslant \frac{(b-a)^2}{16}\left[\int_0^{\frac{1}{6}} 8t^2((1-t)|f''(a)|+t|f''(b)|)\mathrm{d}t\right. \\
&\quad +\int_{\frac{1}{6}}^{\frac{1}{2}} |(2t-1)(1-4t)|((1-t)|f''(a)|+t|f''(b)|)\mathrm{d}t \\
&\quad +\int_{\frac{1}{2}}^{\frac{5}{6}} |(2t-1)(3-4t)|((1-t)|f''(a)|+t|f''(b)|)\mathrm{d}t \\
&\quad \left.+\int_{\frac{5}{6}}^{1} |(t-1)(8-8t)|((1-t)|f''(a)|+t|f''(b)|)\mathrm{d}t\right] \\
&= \frac{(b-a)^2}{16}\left[\int_0^{\frac{1}{6}} 8t^2((1-t)|f''(a)|+t|f''(b)|)\mathrm{d}t\right. \\
&\quad +\left(\int_{\frac{1}{6}}^{\frac{1}{4}} (1-2t)(1-4t)+\int_{\frac{1}{4}}^{\frac{1}{2}} (1-2t)(4t-1)\right)((1-t)|f''(a)|+t|f''(b)|)\mathrm{d}t \\
&\quad +\left(\int_{\frac{1}{2}}^{\frac{3}{4}} (2t-1)(3-4t)+\int_{\frac{3}{4}}^{\frac{1}{2}} (2t-1)(4t-3)\right)((1-t)|f''(a)|+t|f''(b)|)\mathrm{d}t \\
&\quad \left.+\int_{\frac{5}{6}}^{1} (1-t)(8-8t)((1-t)|f''(a)|+t|f''(b)|)\mathrm{d}t\right] \\
&= \frac{(b-a)^2}{384}(|f''(a)|+|f''(b)|).
\end{aligned}$$

The proof is completed. □

COROLLARY 1. *Assuming* $|f|'' \leqslant M$ *in Theorem* 3*, we obtain*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leqslant \frac{M(b-a)^2}{192}.$$

THEOREM 4. *Assume that all of the requirements of Lemma* 1 *are fulfilled. If* $|f''|^q$ *is a convex mapping on* $[a,b]$, $q>1$, *then*

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\leqslant \frac{(b-a)^2}{16}\left[\left(\frac{8}{6^{2p+1}(2p+1)}\right)^{\frac{1}{p}}\left(\frac{11}{72}|f''(a)|^q+\frac{1}{72}|f''(b)|^q\right)^{\frac{1}{q}}\right.\\ &\quad +(A)^{\frac{1}{p}}\left(\frac{2}{9}|f''(a)|^q+\frac{1}{9}|f''(b)|^q\right)^{\frac{1}{q}}+(B)^{\frac{1}{p}}\left(\frac{1}{9}|f''(a)|^q+\frac{2}{9}|f''(b)|^q\right)^{\frac{1}{q}}\\ &\quad \left.+\left(\frac{8}{6^{2p+1}(2p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{72}|f''(a)|^q+\frac{11}{72}|f''(b)|^q\right)^{\frac{1}{q}}\right],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

$$\int_0^{\frac{1}{6}} 8t^{2p}\mathrm{d}t=\int_{\frac{5}{6}}^1((1-t)(8-8t))^p\mathrm{d}t=\frac{8}{6^{2p+1}(2p+1)},$$

$$A=\int_{\frac{1}{6}}^{\frac{1}{2}}|(2t-1)(1-4t)|^p\,\mathrm{d}t,\quad B=\int_{\frac{1}{2}}^{\frac{5}{6}}|(2t-1)(3-4t)|^p\,\mathrm{d}t.$$

Proof. Taking $p>1$. Considering Lemma 1, and making the utility of notable Hölder's integral inequality and the convexity of $|f''|^q$, we achieve

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\leqslant \frac{(b-a)^2}{16}\left[\int_0^{\frac{1}{6}}|-8t^2||f''((1-t)a+tb)|\mathrm{d}t\right.\\ &\quad +\int_{\frac{1}{6}}^{\frac{1}{2}}|(2t-1)(1-4t)||f''((1-t)a+tb)|\mathrm{d}t\\ &\quad +\int_{\frac{1}{2}}^{\frac{5}{6}}|(2t-1)(3-4t)||f''((1-t)a+tb)|\mathrm{d}t\\ &\quad \left.+\int_{\frac{5}{6}}^{1}|(t-1)(8-8t)||f''((1-t)a+tb)|\mathrm{d}t\right]\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{(b-a)^2}{16}\left[\left(\int_0^{\frac{1}{6}} 8t^{2p}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{6}} |f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} |(2t-1)(1-4t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}} |f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} |(2t-1)(3-4t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}} |f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{5}{6}}^{1} |(t-1)(8-8t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{5}{6}}^{1} |f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
&\leqslant \frac{(b-a)^2}{16}\left[\left(\int_0^{\frac{1}{6}} 8t^{2p}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{6}} ((1-t)|f''(a)|^q+t|f''(b)^q)\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} |(2t-1)(1-4t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}} ((1-t)|f''(a)|^q+t|f''(b)^q)\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} |(2t-1)(3-4t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}} ((1-t)|f''(a)|^q+t|f''(b)^q)\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{5}{6}}^{1} |(1-t)(8-8t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{5}{6}}^{1} ((1-t)|f''(a)|^q+t|f''(b)^q)\mathrm{d}t\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

which ends the proof. □

THEOREM 5. *Assume that all of the requirements of Lemma 1 are fulfilled. If $|f''|^q$ is a convex mapping on $[a,b]$, $q \geqslant 1$, then*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\
&\leqslant \frac{(b-a)^2}{16}\left\{\left(\frac{1}{81}\right)^{1-\frac{1}{q}}\left[\left(\frac{7}{648}|f''(a)|^q+\frac{1}{648}|f''(b)|^q\right)^{\frac{1}{q}}\right.\right.\\
&\quad\left.+\left(\frac{1}{648}|f''(a)|^q+\frac{7}{648}|f''(b)^q\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\frac{19}{648}\right)^{1-\frac{1}{q}}\left[\left(\frac{103}{5184}|f''(a)|^q+\frac{49}{5184}|f''(b)^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\left(\frac{49}{5184}|f''(a)|^q+\frac{103}{5184}|f''(b)^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}
$$

Proof. Considering Lemma 1, implementing the power-mean's inequality and the convexity of $|f''|^q$, we get

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\leqslant \frac{(b-a)^2}{16}\left[\int_0^{\frac{1}{6}}|-8t^2||f''((1-t)a+tb)|\mathrm{d}t\right. \\
&\quad+\int_{\frac{1}{6}}^{\frac{1}{2}}|(2t-1)(1-4t)||f''((1-t)a+tb)|\mathrm{d}t \\
&\quad+\int_{\frac{1}{2}}^{\frac{5}{6}}|(2t-1)(3-4t)||f''((1-t)a+tb)|\mathrm{d}t \\
&\quad\left.+\int_{\frac{5}{6}}^{1}|(t-1)(8-8t)||f''((1-t)a+tb)|\mathrm{d}t\right] \\
&\leqslant \frac{(b-a)^2}{16}\left[\left(\int_0^{\frac{1}{6}}8t^2\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{6}}8t^2((1-t)|f''(a)|^q+t|f''(b)^q)\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}|(2t-1)(1-4t)|\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad\left(\int_{\frac{1}{6}}^{\frac{1}{2}}|(2t-1)(1-4t)|((1-t)|f''(a)|^q+t|f''(b)^q)\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}|(2t-1)(3-4t)|\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad\left(\int_{\frac{1}{2}}^{\frac{5}{6}}|(2t-1)(3-4t)|((1-t)|f''(a)|^q+t|f''(b)^q)\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{5}{6}}^{1}|(t-1)(8-8t)|\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad\left.\left(\int_{\frac{5}{6}}^{1}|(t-1)(8-8t)|((1-t)|f''(a)|^q+t|f''(b)^q)\mathrm{d}t\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Hence, we acquire our required result. □

THEOREM 6. *Assume that all of the requirements of Lemma 1 are fulfilled. If* $|f''|^q$ *is a convex mapping on* $[a,b]$, $q>1$, *then*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\leqslant \frac{(b-a)^2}{16}\left\{6\left[\left(\frac{8^p}{6^{2p+2}(2p+1)(2p+2)}\right)^{\frac{1}{p}}\left(\frac{17}{1296}|f''(a)|^q+\frac{1}{1296}|f''(b)|^q\right)^{\frac{1}{q}}\right.\right. \\
&\quad \left. + \left(\frac{8^p}{6^{2p+2}(2p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{81}|f''(a)|^q+\frac{1}{648}|f''(b)|^q\right)^{\frac{1}{q}}\right] \\
&\quad +3\left[(C)^{\frac{1}{p}}\left(\frac{13}{324}|f''(a)|^q+\frac{5}{324}|f''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. +(D)^{\frac{1}{p}}\left(\frac{11}{324}|f''(a)|^q+\frac{7}{324}|f''(b)|^q\right)^{\frac{1}{q}}\right] \\
&\quad +3\left[(E)^{\frac{1}{p}}\left(\frac{7}{324}|f''(a)|^q+\frac{11}{324}|f''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. +(F)^{\frac{1}{p}}\left(\frac{5}{324}|f''(a)|^q+\frac{13}{324}|f''(b)|^q\right)^{\frac{1}{q}}\right] \\
&\quad +6\left[(G)^{\frac{1}{p}}\left(\frac{1}{648}|f''(a)|^q+\frac{1}{81}|f''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.\left. +(H)^{\frac{1}{p}}\left(\frac{1}{1296}|f''(a)|^q+\frac{17}{1296}|f''(b)|^q\right)^{\frac{1}{q}}\right]\right\},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

$$
\begin{aligned}
C&=\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\frac{1}{2}-t\right)|(2t-1)(1-4t)|^p\,\mathrm{d}t, \quad D=\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)|(2t-1)(1-4t)|^p\,\mathrm{d}t, \\
E&=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)|(2t-1)(3-4t)|^p\,\mathrm{d}t, \quad F=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(t-\frac{1}{2}\right)|(2t-1)(3-4t)|^p\,\mathrm{d}t, \\
G&=\int_{\frac{5}{6}}^{1}(1-t)|(t-1)(8-8t)|^p\,\mathrm{d}t, \quad H=\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)|(t-1)(8-8t)|^p\,\mathrm{d}t.
\end{aligned}
$$

Proof. Considering Lemma 1, by implementing the improved Hölder's inequality and the convexity of $|f''|^q$, we get

$$
\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\leqslant \frac{(b-a)^2}{16}\left[\int_0^{\frac{1}{6}}|-8t^2||f''((1-t)a+tb)|\mathrm{d}t\right.
\end{aligned}
$$

$$+\int_{\frac{1}{6}}^{\frac{1}{2}} |(2t-1)(1-4t)||f''((1-t)a+tb)|\mathrm{d}t$$

$$+\int_{\frac{1}{2}}^{\frac{5}{6}} |(2t-1)(3-4t)||f''((1-t)a+tb)|\mathrm{d}t$$

$$\left.+\int_{\frac{5}{6}}^{1} (|t-1)(8-8t)||f''((1-t)a+tb)|\mathrm{d}t\right]$$

$$\leqslant \frac{(b-a)^2}{16}\left\{6\left[\left(\int_0^{\frac{1}{6}} (8t^2)^p \left(\frac{1}{6}-t\right)\mathrm{d}t\right)^{\frac{1}{p}}\right.\right.$$

$$\left(\int_0^{\frac{1}{6}} \left(\frac{1}{6}-t\right)|f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^{\frac{1}{6}} 8^p t^{2p+1}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{6}} t|f''((1-t)a+tb)|\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$+3\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left(\frac{1}{2}-t\right)|(2t-1)(1-4t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right.$$

$$\left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left(\frac{1}{2}-t\right)|f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left(t-\frac{1}{6}\right)|(2t-1)(1-4t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

$$\left.\left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left(t-\frac{1}{6}\right)|f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$+3\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6}-t\right)|(2t-1)(3-4t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right.$$

$$\left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6}-t\right)|f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left(t-\frac{1}{2}\right)|(2t-1)(3-4t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

$$\left.\left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left(t-\frac{1}{2}\right)|f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$+6\left[\left(\int_{\frac{5}{6}}^{1}(1-t)\,|(t-1)(8-8t)|^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}\right.$$
$$\left(\int_{\frac{5}{6}}^{1}(1-t)\,|f''((1-t)a+tb)|^{q}\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)|(t-1)(8-8t)|^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}$$
$$\left.\left(\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)|f''((1-t)a+tb)|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\right]\Bigg\}.$$

Calculating the integrals presented in aforementioned inequality generate the required outcome. □

THEOREM 7. *Assume that all of the requirements of Lemma 1 are fulfilled. If* $|f''|^{q}$ *is a convex mapping on* $[a,b]$, $q \geqslant 1$, *then*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\mathrm{d}x\right|$$
$$\leqslant\frac{(b-a)^{2}}{16}\left\{6\left[\left(\frac{1}{1944}\right)^{1-\frac{1}{q}}\left(\frac{1}{2160}|f''(a)|^{q}+\frac{1}{19440}|f''(b)|^{q}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{13}{9720}|f''(a)|^{q}+\frac{1}{4860}|f''(b)|^{q}\right)^{\frac{1}{q}}\right]$$
$$+3\left[\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left(\frac{1181}{311040}|f''(a)|^{q}+\frac{439}{311040}|f''(b)|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{71}{15552}\right)^{1-\frac{1}{q}}\left(\frac{293}{103680}|f''(a)|^{q}+\frac{541}{311040}|f''(b)|^{q}\right)^{\frac{1}{q}}\right]$$
$$+3\left[\left(\frac{71}{15552}\right)^{1-\frac{1}{q}}\left(\frac{541}{311040}|f''(a)|^{q}+\frac{293}{103680}|f''(b)|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left(\frac{439}{311040}|f''(a)|^{q}+\frac{1181}{311040}|f''(b)|^{q}\right)^{\frac{1}{q}}\right]$$
$$+6\left[\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{1}{4860}|f''(a)|^{q}+\frac{13}{9720}|f''(b)|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{1}{1944}\right)^{1-\frac{1}{q}}\left(\frac{1}{19440}|f''(a)|^{q}+\frac{1}{2160}|f''(b)|^{q}\right)^{\frac{1}{q}}\right]\right\}.$$

Proof. Through Lemma 1, implementing the improved power-mean inequality and the convexity of $|f''|^q$, we get

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leqslant \frac{(b-a)^2}{16}\left[\int_0^{\frac{1}{6}}|-8t^2||f''((1-t)a+tb)|\mathrm{d}t\right.$$
$$+\int_{\frac{1}{6}}^{\frac{1}{2}}|(2t-1)(1-4t)||f''((1-t)a+tb)|\mathrm{d}t$$
$$+\int_{\frac{1}{2}}^{\frac{5}{6}}|(2t-1)(3-4t)||f''((1-t)a+tb)|\mathrm{d}t$$
$$\left.+\int_{\frac{5}{6}}^{1}|(t-1)(8-8t)||f''((1-t)a+tb)|\mathrm{d}t\right]$$
$$\leqslant \frac{(b-a)^2}{16}\left\{6\left[\left(\int_0^{\frac{1}{6}}8t^2\left(\frac{1}{6}-t\right)\mathrm{d}t\right)^{1-\frac{1}{q}}\right.\right.$$
$$\left(\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)8t^2|f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^{\frac{1}{6}}8t^3\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{6}}8t^3|f''((1-t)a+tb)|\mathrm{d}t\right)^{\frac{1}{q}}\right]$$
$$+3\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\frac{1}{2}-t\right)|(2t-1)(1-4t)|\mathrm{d}t\right)^{1-\frac{1}{q}}\right.$$
$$\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\frac{1}{2}-t\right)|(2t-1)(1-4t)||f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)|(2t-1)(1-4t)|\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\left.\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)|(2t-1)(1-4t)||f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]$$
$$+3\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)|(2t-1)(3-4t)|\mathrm{d}t\right)^{1-\frac{1}{q}}\right.$$
$$\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)|(2t-1)(3-4t)||f''((1-t)a+tb)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(t-\frac{1}{2}\right)|(2t-1)(3-4t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(t-\frac{1}{2}\right)|(2t-1)(3-4t)||f''((1-t)a+tb)|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\Bigg]\\
&+6\left[\left(\int_{\frac{5}{6}}^{1}(1-t)|(t-1)(8-8t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\right.\\
&\left(\int_{\frac{5}{6}}^{1}(1-t)|(t-1)(8-8t)||f''((1-t)a+tb)|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)|(t-1)(8-8t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left.\left.\left(\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)|(t-1)(8-8t)||f''((1-t)a+tb)|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\right]\right\}\\
\leqslant\;&\frac{(b-a)^2}{16}\left\{6\left[\left(\frac{1}{1944}\right)^{1-\frac{1}{q}}\left(\frac{1}{2160}|f''(a)|^{q}+\frac{1}{19440}|f''(b)|^{q}\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{13}{9720}|f''(a)|^{q}+\frac{1}{4860}|f''(b)|^{q}\right)^{\frac{1}{q}}\right]\\
&+3\left[\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left(\frac{1181}{311040}|f''(a)|^{q}+\frac{439}{311040}|f''(b)|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{71}{15552}\right)^{1-\frac{1}{q}}\left(\frac{293}{103680}|f''(a)|^{q}+\frac{541}{311040}|f''(b)|^{q}\right)^{\frac{1}{q}}\right]\\
&+3\left[\left(\frac{71}{15552}\right)^{1-\frac{1}{q}}\left(\frac{541}{311040}|f''(a)|^{q}+\frac{293}{103680}|f''(b)|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left(\frac{439}{311040}|f''(a)|^{q}+\frac{1181}{311040}|f''(b)|^{q}\right)^{\frac{1}{q}}\right]\\
&+6\left[\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{1}{4860}|f''(a)|^{q}+\frac{13}{9720}|f''(b)|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\frac{1}{1944}\right)^{1-\frac{1}{q}}\left(\frac{1}{19440}|f''(a)|^{q}+\frac{1}{2160}|f''(b)|^{q}\right)^{\frac{1}{q}}\right]\right\},
\end{aligned}
$$

which ends the proof. □

3. Applications

In the subsequent sections, we will explore numerous potential uses of our significant developments. Initially, we set up a connection between means of nonnegative real numbers by looking at specific outcomes acquired in the previous section. This section also covers applications to composite numerical integration approaches.

3.1. The quadrature formula

Suppose a partition $\mathscr{P}: a = x_0 < x_1 < \ldots < x_i < x_{i+1} < x_{n-1} < x_n = b$ of $[a,b]$, with $i = 0, 1, \ldots, n-1$, then

$$\int_a^b f(x)dx = T(x) + R(x).$$

Here

$$T(x) = \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right],$$

and $R(x)$ denotes the error term.

PROPOSITION 3.1. From Theorem 3, we have

$$|R(x)| \leqslant \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^3}{324}\left(|f''(x_i)| + |f''(x_{i+1})|\right).$$

Proof. Applying Theorem 3 over subinterval $[x_i, x_{i+1}]$ and taking sum, we get desired bound. □

PROPOSITION 3.2. From Theorem 6, we have

$$|R(x)| \leqslant \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^2}{16}\left\{6\left[\left(\frac{8^p}{6^{2p+2}(2p+1)(2p+2)}\right)^{\frac{1}{p}}\right.\right.$$
$$\left(\frac{17}{1296}|f''(x_i)|^q + \frac{1}{1296}|f''(x_{i+1})|^q\right)^{\frac{1}{q}}$$
$$\left. + \left(\frac{8^p}{6^{2p+2}(2p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{81}|f''(x_i)|^q + \frac{1}{648}|f''(x_{i+1})|^q\right)^{\frac{1}{q}}\right]$$
$$+3\left[(C)^{\frac{1}{p}}\left(\frac{13}{324}|f''(x_i)|^q + \frac{5}{324}|f''(x_{i+1})|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + (D)^{\frac{1}{p}}\left(\frac{11}{324}|f''(a)|^q + \frac{7}{324}|f''(b)|^q\right)^{\frac{1}{q}}\right]$$

$$+3\left[(E)^{\frac{1}{p}}\left(\frac{7}{324}|f''(x_i)|^q+\frac{11}{324}|f''(x_{i+1})|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(F)^{\frac{1}{p}}\left(\frac{5}{324}|f''(a)|^q+\frac{13}{324}|f''(b)|^q\right)^{\frac{1}{q}}\right]$$
$$+6\left[(G)^{\frac{1}{p}}\left(\frac{1}{648}|f''(x_i)|^q+\frac{1}{81}|f''(x_{i+1})|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+(H)^{\frac{1}{p}}\left(\frac{1}{1296}|f''(a)|^q+\frac{17}{1296}|f''(b)|^q\right)^{\frac{1}{q}}\right]\right\}.$$

Proof. Applying Theorem 6 over subinterval $[x_i,x_{i+1}]$ and taking sum, we get desired bound. □

3.2. Applications to means

We recall some notable means for non-negative real numbers.

1. The arithmetic mean:
$$A(a,b)=\frac{a+b}{2}.$$

2. The Weighted arithmetic mean:
$$_wA(w_1,w_2;a,b)=\frac{aw_1+bw_2}{w_1+w_2}.$$

3. The log-mean:
$$L_r(a,b)=\left[\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{\frac{1}{r}};\ r\in\Re\setminus\{-1,0\}.$$

PROPOSITION 3.3. From Theorem 4, we get

$$\left|\frac{\theta}{8(r+2\theta)}\left[3A^{\frac{r}{\theta}+2}\left(a,b,\frac{5}{6},\frac{1}{6}\right)+2A^{\frac{r}{\theta}+2}(a,b)+3A^{\frac{r}{\theta}+2}\left(a,b,\frac{1}{6},\frac{5}{6}\right)\right]-3L_{\frac{r}{\theta}+2}^{\frac{r}{\theta}+2}(a,b)\right|$$
$$\leqslant\frac{(b-a)^2}{16}\left[\left(\frac{8}{6^{2p+1}(2p+1)}\right)^{\frac{1}{p}}\left(\frac{r+\theta}{36\theta}A\left(11|a^{\frac{r}{\theta}}|^q,|b^{\frac{r}{\theta}}|^q\right)\right)^{\frac{1}{q}}\right.$$
$$+(A)^{\frac{1}{p}}\left(\frac{2(r+\theta)}{9\theta}A\left(|a^{\frac{r}{\theta}}|^q,|b^{\frac{r}{\theta}}|^q\right)\right)^{\frac{1}{q}}+(B)^{\frac{1}{p}}\left(\frac{2(r+\theta)}{9\theta}A\left(|a^{\frac{r}{\theta}}|^q,2|b^{\frac{r}{\theta}}|^q\right)\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{8}{6^{2p+1}(2p+1)}\right)^{\frac{1}{p}}\left(\frac{r+\theta}{36\theta}A\left(|a^{\frac{r}{\theta}}|^q,11|b^{\frac{r}{\theta}}|^q\right)\right)^{\frac{1}{q}}\right].$$

Proof. The statement of claim is directly followed by substitution $f(t)=\frac{\theta}{r+2\theta}t^{\frac{r}{\theta}+2}$ in Theorem 4. □

PROPOSITION 3.4. From Theorem 5, we have

$$\left|\frac{\theta}{8(r+2\theta)}\left[3A^{\frac{r}{\theta}+2}\left(a,b,\frac{5}{6},\frac{1}{6}\right)+2A^{\frac{r}{\theta}+2}(a,b)+3A^{\frac{r}{\theta}+2}\left(a,b,\frac{1}{6},\frac{5}{6}\right)\right]-3L_{\frac{r}{\theta}+2}^{\frac{r}{\theta}+2}(a,b)\right|$$
$$\leqslant\frac{(b-a)^2}{16}\left\{\left(\frac{1}{81}\right)^{1-\frac{1}{q}}\left[\left(\frac{r+\theta}{324\theta}A\left(7|a^{\frac{r}{\theta}}|^q,|b^{\frac{r}{\theta}}|^q\right)\right)^{\frac{1}{q}}+\left(\frac{r+\theta}{324\theta}A\left(|a^{\frac{r}{\theta}}|^q,7|b^{\frac{r}{\theta}}|^q\right)\right)^{\frac{1}{q}}\right]\right.$$
$$+\left(\frac{19}{648}\right)^{1-\frac{1}{q}}\left[\left(\frac{r+\theta}{2592\theta}A\left(103|a^{\frac{r}{\theta}}|^q,49|b^{\frac{r}{\theta}}|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{r+\theta}{2592\theta}A\left(49|a^{\frac{r}{\theta}}|^q,103|b^{\frac{r}{\theta}}|^q\right)\right)^{\frac{1}{q}}\right]\right\}.$$

Proof. By substituting $f(t)=\frac{\theta}{r+2\theta}t^{\frac{r}{\theta}+2}$ in Theorem 5, we attain our desired result. □

3.3. Applications to probability theory

Let $p:[a,b]\to[0,1]$ be a probability density mapping over convex set X. Then cumulative distribution is demonstrated as:

$$Pr(X\leqslant b)=F(b)=\int_a^b p(t)\mathrm{d}t.$$

Utilizing the fact that

$$E(X)=\int_a^b t\,dF(t)$$
$$E(X)=b-\int_a^b F(t)\mathrm{d}t.$$

PROPOSITION 3.5. Considering Theorem 3, we have

$$\left|\frac{1}{8}\left[3Pr\left(X\leqslant\frac{5a+b}{6}\right)+2Pr\left(X\leqslant\frac{a+b}{2}\right)+3Pr\left(X\leqslant\frac{a+5b}{6}\right)\right]-\frac{b-E(X)}{b-a}\right|$$
$$\leqslant\frac{b-a}{324}(|p'(a)|+|p'(b)|),$$

and

$$\left|\frac{1}{8}\left[3Pr\left(X\leqslant\frac{5a+b}{6}\right)+2Pr\left(X\leqslant\frac{a+b}{2}\right)+3Pr\left(X\leqslant\frac{a+5b}{6}\right)\right]-\frac{b-E(X)}{b-a}\right|$$
$$\leqslant\frac{M(b-a)}{192}.$$

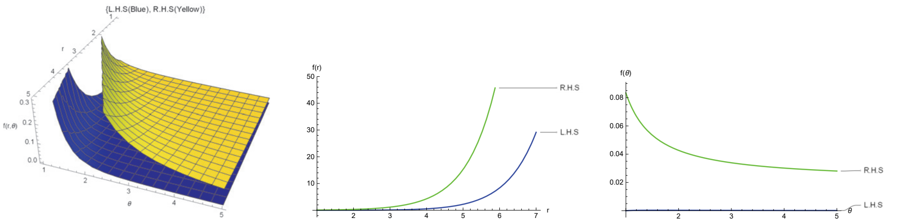
Proof. We conclude this result employing probability density function on Theorem 3. □

3.4. Graphical discussion of outcomes

We now give a graphical illustration of inequality presented in Theorem 3.

EXAMPLE 1. Since $f(t) = \frac{\theta}{r+2\theta} t^{\frac{r}{\theta}+2}$ such that $r \geqslant 1$, $\theta > 1$ defined on $\mathbb{R}^+$ and its both first and second derivative are convex functions. They satisfy the hypothesis of Theorem 3 and for $a = 1$ and $b = 3$, we have

$$\left| \frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}} + 2(2^{\frac{r+2\theta}{\theta}}) + 3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(2\theta + r)} - \frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}} - 1\right)}{2((r+2\theta)(r+3\theta))} \right| \leqslant \frac{r+\theta}{96\theta}\left(1 + 3^{\frac{r}{\theta}}\right).$$



(a) (b) (c)

Figure 1: *Graphical visuals of left and right sides of Theorem 3.*

We adjust $r \in [1,5]$ and $\theta \in [1,5]$ to interpret the Fig 5(a), Fig 5(b) and Fig 5(c). Clearly, these figures affirming the reliability of Theorem 3.

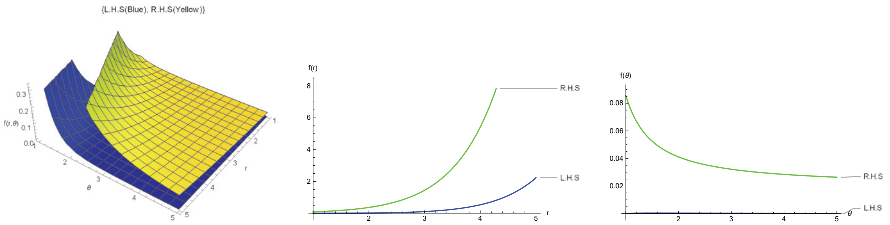
Below is the graphical illustration of Theorem 4.

EXAMPLE 2. Since $f(t) = \frac{\theta}{r+2\theta} t^{\frac{r}{\theta}+2}$ such that $r \geqslant 1$, $\theta > 1$ defined on $\mathbb{R}^+$ and its both first and second derivative are convex functions. They satisfy the hypothesis of Theorem 4 and for $a = 1$ and $b = 3$, we have

$$\left| \frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}} + 2(2^{\frac{r+2\theta}{\theta}}) + 3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)} - \frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}} - 1\right)}{2((r+2\theta)(r+3\theta))} \right|$$

$$\leqslant \frac{1}{4}\left[\sqrt{\frac{1}{4860}}\left(\sqrt{\frac{11}{72}\left(\frac{r+\theta}{\theta}\right)^2 + \frac{1}{72}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right) + \sqrt{\frac{4}{1215}}\left(\sqrt{\frac{2}{9}\left(\frac{r+\theta}{\theta}\right)^2 + \frac{1}{9}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.$$

$$+\sqrt{\frac{4}{1215}}\left(\sqrt{\frac{1}{9}\left(\frac{r+\theta}{\theta}\right)^2+\frac{2}{9}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)$$
$$+\sqrt{\frac{1}{4860}}\left(\sqrt{\frac{1}{72}\left(\frac{r+\theta}{\theta}\right)^2+\frac{11}{72}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\Bigg].$$



(a) (b) (c)

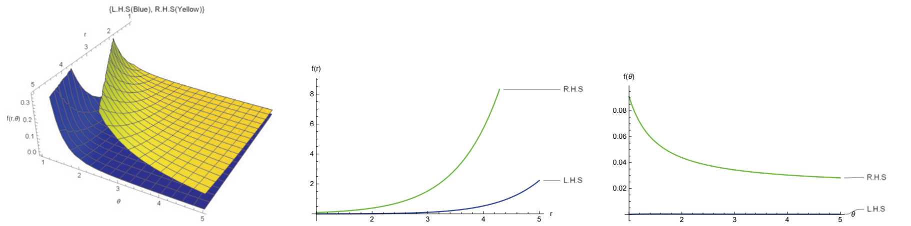
Figure 2: *Graphical visuals of left and right sides of Theorem 4.*

We adjust $r \in [1,5]$ and $\theta \in [1,5]$ to interpret the Fig 2(a), Fig 2(b) and Fig 2(c). Clearly, these figures affirming the reliability of Theorem 4.

Here is the visual analysis of Theorem 5.

EXAMPLE 3. Since $f(t)=\frac{\theta}{r+2\theta}t^{\frac{r}{\theta}+2}$ such that $r \geqslant 1$, $\theta > 1$ defined on $\mathbb{R}^+$ and its both first and second derivative are convex functions. They satisfy the hypothesis of Theorem 5 and for $a=1$ and $b=3$, we have

$$\left|\frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}}+2(2^{\frac{r+2\theta}{\theta}})+3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)}-\frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}}-1\right)}{2((r+2\theta)(r+3\theta))}\right|$$
$$\leqslant \frac{1}{4}\left[\sqrt{\frac{1}{81}}\left(\sqrt{\frac{7}{648}\left(\frac{r+\theta}{\theta}\right)^2+\frac{1}{648}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right.\right.$$
$$\left.+\sqrt{\frac{1}{648}\left(\frac{r+\theta}{\theta}\right)^2+\frac{7}{648}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)$$
$$+\sqrt{\frac{19}{648}}\left(\sqrt{\frac{103}{5184}\left(\frac{r+\theta}{\theta}\right)^2+\frac{49}{5184}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right.$$
$$\left.\left.+\sqrt{\frac{49}{5184}\left(\frac{r+\theta}{\theta}\right)^2+\frac{103}{5184}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right].$$



(a) (b) (c)

Figure 3: *Graphical visuals of left and right sides of Theorem 5.*

We adjust $r \in [1,5]$ and $\theta \in [1,5]$ to interpret the Fig 3(a), Fig 3(b) and Fig 3(c). Clearly, these figures affirming the reliability of Theorem 5.

EXAMPLE 4. Since $f(t) = \frac{\theta}{r+2\theta} t^{\frac{r}{\theta}+2}$ such that $r \geqslant 1$, $\theta > 1$ defined on $\mathbb{R}^+$ and its both first and second derivative are convex functions. They satisfy the hypothesis of Theorem 6 and for $a = 1$ and $b = 3$, we have

$$\left| \frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}} + 2(2^{\frac{r+2\theta}{\theta}}) + 3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)} - \frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}} - 1\right)}{2((r+2\theta)(r+3\theta))} \right|$$

$$\leqslant \frac{1}{4}\left\{ 6\left[\sqrt{\frac{1}{21870}}\left(\sqrt{\frac{17}{1296}\left(\frac{r+\theta}{\theta}\right)^2 + \frac{1}{1296}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.\right.$$

$$\left. + \sqrt{\frac{1}{4374}}\left(\sqrt{\frac{1}{81}\left(\frac{r+\theta}{\theta}\right)^2 + \frac{1}{648}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right]$$

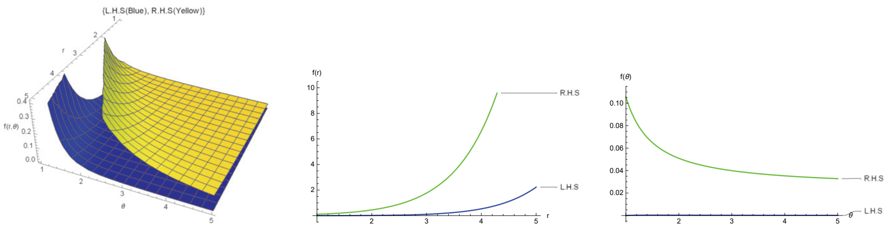
$$+3\left[\sqrt{\frac{7}{10935}}\left(\sqrt{\frac{13}{324}\left(\frac{r+\theta}{\theta}\right)^2 + \frac{5}{324}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.$$

$$\left. + \sqrt{\frac{1}{2187}}\left(\sqrt{\frac{11}{324}\left(\frac{r+\theta}{\theta}\right)^2 + \frac{7}{324}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right]$$

$$+3\left[\sqrt{\frac{1}{2187}}\left(\sqrt{\frac{7}{324}\left(\frac{r+\theta}{\theta}\right)^2 + \frac{11}{324}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.$$

$$\left. + \sqrt{\frac{7}{10935}}\left(\sqrt{\frac{5}{324}\left(\frac{r+\theta}{\theta}\right)^2 + \frac{13}{324}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right]$$

$$+6\left[\sqrt{\frac{1}{4374}}\left(\sqrt{\frac{1}{648}\left(\frac{r+\theta}{\theta}\right)^2+\frac{1}{81}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.$$
$$\left.+\sqrt{\frac{1}{21870}}\left(\sqrt{\frac{1}{1296}\left(\frac{r+\theta}{\theta}\right)^2+\frac{17}{1296}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right]\Bigg\}.$$



(a) (b) (c)

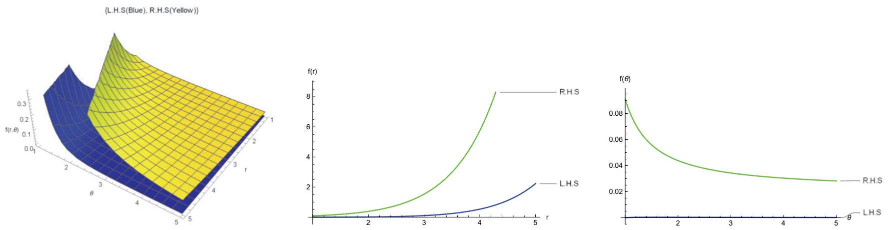
Figure 4: *Graphical visuals of left and right sides of Theorem 6.*

We adjust $r \in [1,5]$ and $\theta \in [1,5]$ to interpret the Fig 4(a), Fig 4(b) and Fig 4(c). Clearly, these figures affirming the reliability of Theorem 6.

EXAMPLE 5. Since $f(t)=\frac{\theta}{r+2\theta}t^{\frac{r}{\theta}+2}$ such that $r \geqslant 1$, $\theta > 1$ defined on $\mathbb{R}^+$ and its both first and second derivative are convex functions. They satisfy the hypothesis of Theorem 7 and for $a=1$ and $b=3$, we have

$$\left|\frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}}+2(2^{\frac{r+2\theta}{\theta}})+3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)}-\frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}}-1\right)}{2((r+2\theta)(r+3\theta))}\right|$$
$$\leqslant \frac{1}{4}\left\{6\left[\sqrt{\frac{1}{1944}}\left(\sqrt{\frac{1}{2160}\left(\frac{r+\theta}{\theta}\right)^2+\frac{1}{19440}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.\right.$$
$$\left.+\sqrt{\frac{1}{648}}\left(\sqrt{\frac{13}{9720}\left(\frac{r+\theta}{\theta}\right)^2+\frac{1}{4860}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right]$$
$$+3\left[\sqrt{\frac{1}{192}}\left(\sqrt{\frac{1181}{311040}\left(\frac{r+\theta}{\theta}\right)^2+\frac{439}{311040}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.$$
$$\left.+\sqrt{\frac{71}{15552}}\left(\sqrt{\frac{293}{103680}\left(\frac{r+\theta}{\theta}\right)^2+\frac{541}{311040}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right]$$
$$+3\left[\sqrt{\frac{71}{15552}}\left(\sqrt{\frac{541}{311040}\left(\frac{r+\theta}{\theta}\right)^2+\frac{293}{103680}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.$$

$$+\sqrt{\frac{1}{192}}\left(\sqrt{\frac{439}{311040}\left(\frac{r+\theta}{\theta}\right)^2+\frac{1181}{311040}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\Bigg]$$
$$+6\left[\sqrt{\frac{1}{648}}\left(\sqrt{\frac{1}{4860}\left(\frac{r+\theta}{\theta}\right)^2+\frac{13}{9720}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right.$$
$$\left.+\sqrt{\frac{1}{1944}}\left(\sqrt{\frac{1}{19440}\left(\frac{r+\theta}{\theta}\right)^2+\frac{1}{2160}\left(\frac{r+\theta}{\theta}(3)^{\frac{r}{\theta}}\right)^2}\right)\right]\Bigg\}.$$



(a) (b) (c)

Figure 5: *Graphical visuals of left and right sides of Theorem 7.*

We adjust $r \in [1,5]$ and $\theta \in [1,5]$ to interpret the Fig 5(a), Fig 5(b) and Fig 5(c). Clearly, these figures affirming the reliability of Theorem 7.

4. Iterative scheme

Now, we provide a new cubic order iterative scheme based on Maclaurin's method.

ALGORITHM 4.1. Suppose we have a non-linear equation $f(x)=0$, then

$$x_{n+1}=x_n-\frac{8f(x_n)}{3f'\left(\frac{5x_n+y_n}{6}\right)+2f'\left(\frac{x_n+y_n}{2}\right)+3f'\left(\frac{x_n+5y_n}{6}\right)}, \tag{9}$$

where

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}.$$

Proof. The proof directly follows from Theorem 3. □

4.1. Numerical analysis

The following part has some physical examples and their computational analysis of Algorithm 4.1.

1. The blood rheology and fractional non-linear equations model is used to examine the plug flow of Casson fluids, taking into account a non-linear fractional equation to assess a flow rate decline and which is given as:

$$\mathcal{F}(x) = 1 - \frac{16}{7}\sqrt{x} + \frac{4}{3}x - \frac{1}{21}x^4 - G,$$

where the fall rate is approximated based on $G = 0.4$. With an initial guess of $x_0 = 0.1$, then proposed Algorithm 4.1 approximate the desired solution $x = 0.104698651536548228\,12$ in four iterations.

2. The second issue discussed is fluid permeability in biogels ([16]), which corresponds to both pressure gradient and fluid velocity in media with pores such as agarose gel or extracellular microfibre matrix:

$$f(x) = \Re_e x^3 - 20\kappa(1-x)^2,$$

where $\Re_e = 10 \times 10^{-9}$ and $\kappa = 0.3655$. Algorithm 4.1 computes the approximate solution $x = 1.0000369883881891758$ in 20 iterations with initial guess of $x_0 = 2$.

3. The 3rd model is discussed in ([25]) as:

$$\mathcal{F}(x) = \frac{x}{1-x} - 5\log\left[\frac{0.4(1-x)}{0.4-0.5x}\right] + 4.45977, \tag{10}$$

Conversion of specie A in reactor is described by x and $\omega \in [0,1]$. Algorithm 4.1 predicts the approximate solution with initial guess 0.75 in 4 iterations.

Next, we discuss the comparative study of algorithm 4.1 with classic methods such as Newton method (NM) [8], Abbasbandy's method (AM) [1], Halley's method (HM) [8], Chun's method (CM) [11] through non-linear equations.

1. $\mathcal{F}(x) = x^3 + 4x^2 - 15$,
2. $\mathcal{F}(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$,
3. $\mathcal{F}(x) = 10xe^{-x^2} - 1$,
4. $\mathcal{F}(x) = e^{-x} + \cos x$.

We select tolerance $\varepsilon = 10^{-15}$ and

1. $|x_{n+1} - x_n| < \varepsilon$,
2. $|\mathcal{F}(x_{n+1})| < \varepsilon$.

Numerical evaluations were carried out on an Intel(R) Core(TM) i5 processor running at 1.60 GHz and 16GB of RAM. Maple 2020 served as the tool for software development, whereas visual analysis was carried out via Matlab 2021. After executing numerical evaluation on the program, we supply both tabular and graphic representations of Algorithm 4.1 for the scenarios above.

Table 1: *Comparison of Different Methods for Various Examples*

Methods	x_0	**IT**	x_n	$f(x_n)$	δ
NM	2	6	1.6319808055660635175	0	0
AM	2	4	1.6319808055660635175	0	0
HM	2	4	1.6319808055660635175	0	0
CM	2	4	1.6319808055660635175	0	0
ALG	2	4	1.6319808055660635175	0	0
NM	-1	6	-1.2076478271309189270	4.0×10^{-19}	7.58×10^{-17}
AM	-1	5	-1.2076478271309189270	4.0×10^{-19}	0
HM	-1	4	-1.2076478271309189270	4.0×10^{-19}	0
CM	-1	5	-1.2076478271309189270	4.0×10^{-19}	0
ALG	-1	5	-1.2076478271309189270	4.0×10^{-19}	0
NM	1.8	5	1.6796306104284499407	-9×10^{-20}	4.7395×10^{-15}
AM	1.8	4	1.6796306104284499407	-9×10^{-20}	1.0×10^{-19}
HM	1.8	4	1.6796306104284499407	-9×10^{-20}	0
CM	1.8	4	1.6796306104284499407	2.0×10^{-19}	0
ALG	1.8	4	1.6796306104284499407	-9×10^{-20}	0
NM	2	5	1.7461395304080124176	6.0×10^{-20}	1.0×10^{-19}
AM	2	4	1.7461395304080124176	-6×10^{-20}	1.0×10^{-19}
HM	2	4	1.7461395304080124176	6.0×10^{-20}	1.0×10^{-19}
CM	2	3	1.7461395304080124176	-6×10^{-20}	4.63×10^{-17}
ALG	2	4	1.7461395304080124176	6×10^{-20}	1.0×10^{-19}

Here, we give the visual comparison of Algorithm 4.1 with classical methods based on number of iterations.

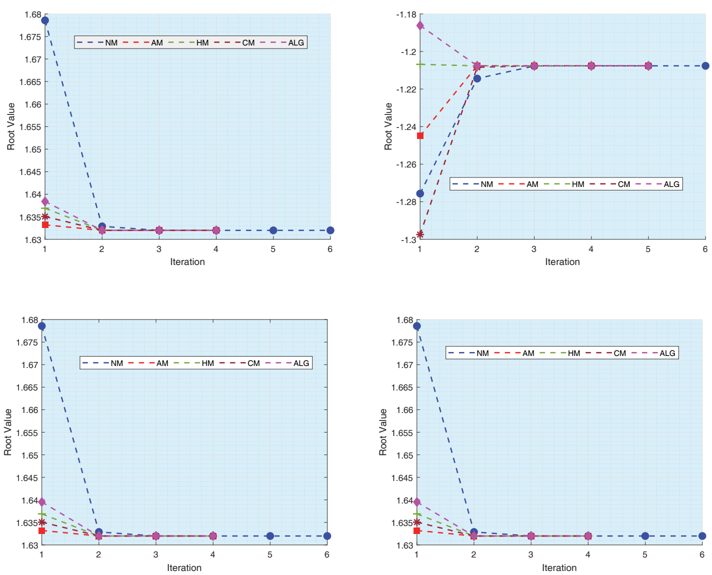


Figure 6: *Graphical visuals*

5. Conclusion

Recently, researchers have concentrated on analyzing errors in quadrature and cubature rules using a variety of analytical strategies. The main concern is to calculate the novel rectified forms of existing results. This work assessed some sharp inequalities of Maclaurin's type via twice differentiable convex functions. One of the most important parts of this study is that it figures out lower bounds for Maclaurin's inequalities for first-order differentiable functions and uses them to look at means, the error inequalities of Maclaurin's composite rule, and iterative schemes in non-linear analysis. The identity found in this study can also be used to look into strong limits for strong convexity, superquadraticity, uniform convexity, and non-convex mappings. To check this inequality in the future, we plan to change it into quantum, symmetric quantum calculus, fractional calculus, and interval-valued frameworks, using a variety of convex and non-convex mappings. We hope the strategy and findings of this paper will create new insights for carrying out further research.

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