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# NOVEL INEQUALITIES INVOLVING CONVEX FUNCTIONS AND THEIR APPLICATIONS

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*Abstract.* This paper presents a comprehensive generalization of the recent work by Yang and Zhang [J. Math. Inequal., **19** (2) (2025), 441–459]. We extend their piecewise interpolation approach originally developed for specific power-type functions to a broader setting involving general convex functions on the unit interval. By introducing new refinement and reverse inequalities, we establish sharper bounds for convex combinations and derive improved versions of classical results such as Jensen's inequality and Young-type inequalities. Our methods incorporate convex analysis, interpolation theory, and weak submajorization techniques, leading to new applications in real and matrix analysis. In particular, we obtain refined inequalities for various matrix means, unitarily invariant norms, and numerical radius bounds, offering enhanced tools for use in operator theory, functional analysis, and quantum information theory.

## 1. Introduction

Convexity is a fundamental concept in mathematics with broad applications in optimization, functional analysis, and matrix theory. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\eta \in [0,1]$, we have:

$$f((1-\eta)x+\eta y) \leqslant (1-\eta)f(x)+\eta f(y).$$

If the inequality is strict for $x \neq y$, then $f$ is strictly convex.

By analyzing the convexity of the function $g(t) = f((1-t)x+ty)$ for $t \in [0,1]$, where $f$ is a convex function defined on the interval $[x,y]$, the convexity condition is equivalent to the following inequality:

$$f(\eta) \leqslant (1-\eta)f(0)+\eta f(1). \tag{1.1}$$

Moreover, a positive function $f$ is said to be log-convex if $\log f$ is convex. This property can be expressed through the inequality:

$$f((1-\eta)x+\eta y) \leqslant f(x)^{1-\eta} f(y)^{\eta}. \tag{1.2}$$

Convex functions enjoy several useful properties, such as continuity on the interior of their domain, subdifferentiability, and the fact that any local minimum is also

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a global minimum. These properties make convexity a powerful tool in analysis and optimization.

In matrix analysis, convexity plays a critical role through the theory of operator convex functions and matrix inequalities. Let $\mathscr{T}, \mathscr{S}$ be Hermitian matrices of the same size. A function $f : I \to \mathbb{R}$ is said to be matrix convex if:

$$f((1-\eta)\mathscr{T}+\eta\mathscr{S}) \preceq (1-\eta)f(\mathscr{T})+\eta f(\mathscr{S}),$$

for all $\eta \in [0,1]$, where $\preceq$ denotes the Löwner partial order.

An important application arises in Jensen's inequality for matrices: if $f$ is operator convex and $\Psi$ is a unital positive linear map, then

$$f(\Psi(\mathscr{T})) \preceq \Psi(f(\mathscr{T})).$$

Moreover, convexity is central in defining entropy, divergence measures, and optimization problems in machine learning and economics. Thus, its role in both pure and applied mathematics is indispensable.

In mathematics, refining an inequality involves adding a strictly positive term to make the inequality sharper and more informative. This approach is particularly important in the study of convex functions, where the classical convexity inequality can sometimes be too general. By quantifying the deviation from linearity, refined inequalities provide deeper insight into the structure and behavior of functions. In certain contexts, inequalities can also be reversed especially for concave functions or under specific assumptions leading to meaningful lower bounds. Both refinement and reverse enrich the original inequality, offering more precise estimations, tighter bounds, and greater applicability. These techniques play a crucial role in optimization, functional analysis, and matrix theory, where they help assess convergence, stability, and error margins. Ultimately, they transform inequalities from simple comparative tools into powerful analytical instruments.

One particularly noteworthy contribution we would like to emphasize is the result established by D. Choi et al. [4], which introduces two refined inequalities that significantly enhance our understanding of convex functions. These refinements provide sharper bounds than the classical convexity inequality and represent a meaningful development in the field of inequality theory.

THEOREM 1.1. [4] *Let $f : [0,1] \to \mathbb{R}$ be a convex function and let $\eta \in [0,1]$. Then*

$$\begin{aligned} 2r\left(\frac{f(0)+f(1)}{2} - f\left(\frac{1}{2}\right)\right) &\leqslant (1-\eta)f(0)+\eta f(1) - f(\eta) \\ &\leqslant 2R\left(\frac{f(0)+f(1)}{2} - f\left(\frac{1}{2}\right)\right) \end{aligned}$$

*where* $r = \min\{\eta, 1-\eta\}$ *and* $R = \max\{\eta, 1-\eta\}$.

The inequality presented in Theorem 1.1 has important applications in refining operator and matrix inequalities. Specifically, it can be employed to sharpen classical results such as the Jensen operator inequality and the Heinz mean inequality. In matrix analysis, many inequalities involving convex (or operator convex) functions, such as those of the form $f(A) \leqslant f(B)$, benefit from tighter bounds obtained via scalar refinements. By applying this refined inequality entrywise or through trace functionals, more accurate estimations can be established. For instance, when $f(t) = t^p$ with $p \geqslant 1$, improved versions of matrix power means can be derived. Moreover, such inequalities play a role in quantum information theory, where convex trace functionals govern entropy bounds and quantum divergences. These refinements thus bridge scalar inequalities with matrix and operator-level precision.

For further results on convexity inequalities and their applications in means and matrix inequalities, the readers are encouraged to consult the following references: [8, 9, 10, 11, 18, 20, 21, 22, 23, 24, 25].

Additionally, if we consider the convex function $f(\eta) = x^{1-\eta}y^{\eta}$, the refined inequality in Theorem 1.1 allows us to recover an important refinement and reverse of the classical Young's inequality, as established by Kittaneh and Manasrah in [14, 15]. This result provides sharper bounds for the convex combination of positive numbers and has significant implications in matrix and operator theory:

$$r(\sqrt{x}-\sqrt{y})^2 \leqslant (1-\eta)x+\eta y - x^{1-\eta}y^{\eta} \leqslant R(\sqrt{x}-\sqrt{y})^2, \tag{1.3}$$

where $x, y \geqslant 0$, $0 \leqslant \eta \leqslant 1$, $r = \min\{\eta, 1-\eta\}$, and $R = \max\{\eta, 1-\eta\}$. This refined form captures the difference between the arithmetic and geometric means more precisely and has been widely applied in the study of norm inequalities and quantum entropy bounds.

Very recently, Yang et al.[28] introduced a significant refinement and generalization of inequality (1.3), offering a more precise estimation framework within the context of the Young-type inequalities. Their result establishes a piecewise linear interpolation approach depending on the value of the parameter $\eta \in [0,1]$, divided into $m$ uniform subintervals.

THEOREM 1.2. ([28]) *Let $x, y \geqslant 0$, $\eta \in [0,1]$, and let $m$ be a positive integer. If $\eta \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$ for some $k \in \{0, 1, \ldots, m-1\}$, then the following refined inequalities hold:*

$$\begin{aligned} x^{1-\eta}y^{\eta} &+ (m\eta - k)\left(\left(1-\frac{k+1}{m}\right)x + \frac{k+1}{m}y - x^{1-\frac{k+1}{m}}y^{\frac{k+1}{m}}\right) \\ &+ ((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)x + \frac{k}{m}y - x^{1-\frac{k}{m}}y^{\frac{k}{m}}\right) \leqslant (1-\eta)x+\eta y, \end{aligned}$$

*and the reverse bound as follows*

$$\begin{aligned} (1-\eta)x+\eta y \leqslant x^{1-\eta}y^{\eta} &+ (m\eta - k)\left(\left(1-\frac{k+1}{m}\right)x + \frac{k+1}{m}y + x^{\frac{k+1}{m}}y^{1-\frac{k+1}{m}}\right) \\ &+ ((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)x + \frac{k}{m}y + x^{\frac{k}{m}}y^{1-\frac{k}{m}}\right) - 2\sqrt{xy}. \end{aligned}$$

These inequalities offer a refined approximation of the convex combination $(1-\eta)x+\eta y$, incorporating weighted interpolations of arithmetic and geometric means. Their structure makes them applicable in further sharpening matrix inequalities and improving operator norm bounds in functional analysis and quantum information theory. For further results on Young's inequality, the reader is encouraged to consult the following interesting papers: [2, 7, 13, 16, 19, 26, 27].

The main objective of this paper is to extend Theorem 1.2, originally established for specific power-type functions, to the broader class of convex functions defined on the unit interval. This generalization not only preserves the structure of the original result but also provides a more flexible framework for deriving refined inequalities. By working within the general setting of convexity, we introduce a new refinement of Theorem 1.1 that leads to tighter bounds and improved estimates. Our approach builds upon the interpolation techniques introduced by Yang et al., while incorporating additional convexity-based arguments to ensure broader applicability. The refined inequalities presented in this work enhance the classical Jensen-type and Young-type inequalities by offering sharper control over the convex deviation. Moreover, these results open the door to further applications in matrix analysis, operator theory, and related fields, where convex functions play a central role. Overall, this contribution deepens the theoretical understanding of convex inequalities and enriches the available toolbox for future developments in mathematical analysis.

This paper is organized as follows. In Section 2, we introduce several new inequalities involving convex functions, including refined versions of the classical convexity inequality and their reverses. These results generalized earlier work and establish sharper bounds for convex combinations. Section 3 is devoted to inequalities derived from weak submajorization, which allow us to formulate powerful results for both scalars and vector valued convex functions. In Section 4, we present applications of our results to classical means, including arithmetic, geometric, harmonic, and power means, highlighting the strength of our approach in various settings. Section 5 explores applications to matrix inequalities, such as those involving unitarily invariant norms and trace norms. In Section 6, we focus on refinements of numerical radius inequalities using convexity and log-convexity, providing new insights relevant to operator theory and functional analysis.

## 2. New inequalities for convex functions

In this section, we investigate generalized refinements of classical convexity inequalities, including their reverses. We begin by sharpening Theorem 1.1, proposing a more precise version. This approach allows us to derive tighter bounds and extend the scope of the original inequality. The results obtained lay the groundwork for broader applications to convex and log-convex functions. Our analysis thus contributes to a deeper understanding of structured inequalities in real analysis.

THEOREM 2.1. *Let $f:[0,1]\to\mathbb{R}$, be a convex function, $0\leqslant\eta\leqslant 1$, and let $m$ be a positive integer. If $\eta\in\left[\frac{k}{m},\frac{k+1}{m}\right]$ for some $k=0,1,\ldots,m-1$, then*

$$
\begin{aligned}
&f(\eta)+(m\eta-k)\left(\left(1-\frac{k+1}{m}\right)f(0)+\frac{k+1}{m}f(1)-f\left(\frac{k+1}{m}\right)\right)\\
&\quad+((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)f(0)+\frac{k}{m}f(1)-f\left(\frac{k}{m}\right)\right)\\
&\leqslant(1-\eta)f(0)+\eta f(1).
\end{aligned}
$$

*In particular:*

*(i) If $\eta\in\left[0,\frac{1}{m}\right]$, then*

$$
\begin{aligned}
&f(\eta)+2\eta\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)\\
&\quad+m\eta\left(\left(1-\frac{2}{m}\right)f(0)+\frac{2}{m}f\left(\frac{1}{2}\right)-f\left(\frac{1}{m}\right)\right)\\
&\leqslant(1-\eta)f(0)+\eta f(1).
\end{aligned}
$$

*(ii) If $\eta\in\left[1-\frac{1}{m},1\right]$, then*

$$
\begin{aligned}
&f(\eta)+2(1-\eta)\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)\\
&\quad+(m-m\eta)\left(\left(1-\frac{2}{m}\right)f(1)+\frac{2}{m}f\left(\frac{1}{2}\right)-f\left(1-\frac{1}{m}\right)\right)\\
&\leqslant(1-\eta)f(0)+\eta f(1).
\end{aligned}
$$

*Proof.* The convexity of the function $f$ leads to

$$
\begin{aligned}
&(1-\eta)f(0)+\eta f(1)-(m\eta-k)\left(\left(1-\frac{k+1}{m}\right)f(0)+\frac{k+1}{m}f(1)-f\left(\frac{k+1}{m}\right)\right)\\
&-((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)f(0)+\frac{k}{m}f(1)-f\left(\frac{k}{m}\right)\right)\\
&=(m\eta-k)f\left(\frac{k+1}{m}\right)+((k+1)-m\eta)f\left(\frac{k}{m}\right)\\
&\geqslant f\left((m\eta-k)\frac{k+1}{m}+((k+1)-m\eta)\frac{k}{m}\right)\\
&=f(\eta).
\end{aligned}
$$

For inequality (i), we take $k=0$, and for the second inequality, we choose $k=m-1$. $\square$

REMARK 2.1. Note that the convexity of the function $f$ ensures that

$$\left(1-\frac{k+1}{m}\right)f(0)+\frac{k+1}{m}f(1)-f\left(\frac{k+1}{m}\right)\geqslant 0$$

and

$$\left(1-\frac{k}{m}\right)f(0)+\frac{k}{m}f(1)-f\left(\frac{k}{m}\right)\geqslant 0,$$

which implies Theorem 2.1 is a further refinement of (1.1).

REMARK 2.2. By considering the function $f(\eta)=x^{1-\eta}y^{\eta}$ for $x,y>0$ and $\eta\in[0,1]$, we are able to recover Theorem 1.2 as a special case of our general framework. This highlights not only the consistency of our approach with known results but also underscores the strength and broad applicability of the new inequalities presented in this work.

The following result provides a refined upper bound for convex functions defined on a closed interval $[x,y]$, incorporating a piecewise correction term that depends on the partition index $m\in\mathbb{N}$. By evaluating the function at affine combinations of the endpoints and interpolating values, this inequality improves the classical convexity inequality $f((1-\eta)x+\eta y)\leqslant(1-\eta)f(x)+\eta f(y)$. In particular, the refinement takes into account the deviation of $f$ from linearity over subintervals of $[x,y]$, and includes specific improved estimates for the boundary cases when $\eta\in\left[0,\frac{1}{m}\right]$ and $\eta\in\left[1-\frac{1}{m},1\right]$. This enhancement reflects the local behavior of convex functions and is useful in applications where sharper bounds are needed over discretized domains.

THEOREM 2.2. *Let $f:[x,y]\to\mathbb{R}$ be a convex function, where $x<y$, and let $m$ be a positive integer. For $\eta\in[0,1]$, define the affine point $z_\eta=(1-\eta)x+\eta y$. If $\eta\in\left[\frac{k}{m},\frac{k+1}{m}\right]$ for some $k=0,1,\dots,m-1$, then*

$$\begin{aligned}
&f(z_\eta)+(m\eta-k)\left(\left(1-\frac{k+1}{m}\right)f(x)+\frac{k+1}{m}f(y)-f\left(z_{\frac{k+1}{m}}\right)\right)\\
&\quad+((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)f(x)+\frac{k}{m}f(y)-f\left(z_{\frac{k}{m}}\right)\right)\\
&\leqslant(1-\eta)f(x)+\eta f(y).
\end{aligned}$$

*In particular:*

*(i) If $\eta\in\left[0,\frac{1}{m}\right]$, then*

$$\begin{aligned}
&f(z_\eta)+2\eta\left(\frac{f(x)+f(y)}{2}-f\left(z_{\frac{1}{2}}\right)\right)\\
&\quad+m\eta\left(\left(1-\frac{2}{m}\right)f(x)+\frac{2}{m}f\left(z_{\frac{1}{2}}\right)-f\left(z_{\frac{1}{m}}\right)\right)\\
&\leqslant(1-\eta)f(x)+\eta f(y),
\end{aligned}$$

*(ii) If* $\eta \in \left[1-\frac{1}{m},1\right]$*, then*

$$f(z_\eta)+2(1-\eta)\left(\frac{f(x)+f(y)}{2}-f\left(z_{\frac{1}{2}}\right)\right)$$
$$+(m-m\eta)\left(\left(1-\frac{2}{m}\right)f(y)+\frac{2}{m}f\left(z_{\frac{1}{2}}\right)-f\left(z_{1-\frac{1}{m}}\right)\right)$$
$$\leqslant (1-\eta)f(x)+\eta f(y).$$

*Proof.* The result follows by applying a change of variable. Define the function $g:[0,1]\to\mathbb{R}$ by

$$g(t)=f((1-t)x+ty),$$

where $f:[x,y]\to\mathbb{R}$ is convex. Since $g$ is the composition of a convex function $f$ with an affine map, it is convex on $[0,1]$. Moreover, the converse also holds: $f$ is convex on $[x,y]$ if and only if $g$ is convex on $[0,1]$. Now, the theorem is a direct consequence of the previous theorem. □

For the specific case $m=3$, we derive the following corollary. It serves as a direct consequence of Theorem 2.1.

COROLLARY 2.1. *Let* $f:[0,1]\to\mathbb{R}$*, be a convex function and* $0\leqslant\eta\leqslant 1$.

*(i) If* $\eta\in\left[0,\frac{1}{3}\right]$*, then*

$$f(\eta)+2\eta\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)+3\eta\left(\frac{1}{3}f(0)+\frac{2}{3}f\left(\frac{1}{2}\right)-f\left(\frac{1}{3}\right)\right)$$
$$\leqslant (1-\eta)f(0)+\eta f(1).$$

*(ii) If* $\eta\in\left[\frac{1}{3},\frac{2}{3}\right]$*, then*

$$f(\eta)+(2-3\eta)\left(\frac{2}{3}f(0)+\frac{1}{3}f(1)-f\left(\frac{1}{3}\right)\right)+(3\eta-1)\left(\frac{1}{3}f(0)+\frac{2}{3}f(1)-f\left(\frac{2}{3}\right)\right)$$
$$\leqslant (1-\eta)f(0)+\eta f(1).$$

*(iii) If* $\eta\in\left[\frac{2}{3},1\right]$*, then*

$$f(\eta)+2(1-\eta)\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)+(3-3\eta)\left(\frac{1}{3}f(1)+\frac{2}{3}f\left(\frac{1}{2}\right)-f\left(\frac{2}{3}\right)\right)$$
$$\leqslant (1-\eta)f(0)+\eta f(1).$$

REMARK 2.3. It is obvious that the first and third inequalities in Corollary 2.1 provide sharper bounds than the left-hand side of the inequalities presented in Theorem 1.1. However, the second inequality in Corollary 2.1 and the left-hand side of the inequalities in Theorem 1.1 are not comparable in general. To illustrate this, let us consider the convex function $f(t) = t^2$ on the interval $[0,1]$, and choose $\eta = \frac{1}{2} \in \left[\frac{1}{3}, \frac{2}{3}\right]$.

We first compute the refined bound given by Corollary 2.1:

$$
\begin{aligned}
&(2-3\eta)\left(\frac{2}{3}f(0)+\frac{1}{3}f(1)-f\left(\frac{1}{3}\right)\right)+(3\eta-1)\left(\frac{1}{3}f(0)+\frac{2}{3}f(1)-f\left(\frac{2}{3}\right)\right)\\
&=(0.5)\left(\frac{2}{3}(0)+\frac{1}{3}(1)-\left(\frac{1}{3}\right)^2\right)+(0.5)\left(\frac{1}{3}(0)+\frac{2}{3}(1)-\left(\frac{2}{3}\right)^2\right)\\
&=0.5\cdot\left(\frac{1}{3}-\frac{1}{9}\right)+0.5\cdot\left(\frac{2}{3}-\frac{4}{9}\right)\\
&=0.5\cdot\frac{2}{9}+0.5\cdot\frac{2}{9}=\frac{2}{9}\approx 0.2222.
\end{aligned}
$$

Now we compute the corresponding refinement from Theorem 1.1:

$$
2r\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)=1\cdot\left(\frac{0+1}{2}-\left(\frac{1}{2}\right)^2\right)=(0.5-0.25)=0.25,
$$

where $r = \min\{\eta, 1-\eta\} = \frac{1}{2}$.

Thus, in this case, the second inequality provides a tighter bound. However, for other values of $\eta$, such as $\eta = \frac{1}{3}$, the first inequality will be sharper. This confirms that the two inequalities are not universally comparable; their relative performance depends on the specific value of $\eta$ and the convex function under consideration.

Next, we focus on refined versions of the reverse convexity inequality. These refinements aim to sharpen the gap between the convex combination and the function's value. Such improvements allow for tighter bounds, particularly in applications involving operator and matrix inequalities.

THEOREM 2.3. *Let $f:[0,1]\to\mathbb{R}$, be a convex function, and let $m$ be a positive integer. If $\eta \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$ for some $k = 0,1,\ldots,m-1$, then*

$$
\begin{aligned}
&(1-\eta)f(0)+\eta f(1)\\
&\leqslant f(\eta)+(m\eta-k)\left(\left(1-\frac{k+1}{m}\right)f(0)+\frac{k+1}{m}f(1)+f\left(1-\frac{k+1}{m}\right)\right)\\
&\quad+((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)f(0)+\frac{k}{m}f(1)+f\left(1-\frac{k}{m}\right)\right)-2f\left(\frac{1}{2}\right).
\end{aligned}
$$

*In particular:*

*(i) If $\eta \in \left[0, \frac{1}{m}\right]$, then*

$$
\begin{aligned}
(1-\eta)f(0)+\eta f(1)\leqslant f(\eta)&+2(1-\eta)\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)\\
&-m\eta\left(\left(1-\frac{2}{m}\right)f(1)+\frac{2}{m}f\left(\frac{1}{2}\right)-f\left(1-\frac{1}{m}\right)\right).
\end{aligned}
\tag{2.1}
$$

*(ii) If* $\eta \in \left[1-\frac{1}{m}, 1\right]$*, then*

$$(1-\eta)f(0)+\eta f(1) \leqslant f(\eta)+2\eta\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right) \\ -(m-m\eta)\left(\left(1-\frac{2}{m}\right)f(0)+\frac{2}{m}f\left(\frac{1}{2}\right)-f\left(\frac{1}{m}\right)\right). \tag{2.2}$$

*Proof.* Utilizing the convexity of the function $f$, if $\eta \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$, $(k=0,1,\ldots, m-1)$, we have

$$\begin{aligned} & f(\eta)+(m\eta-k)\left(\left(1-\frac{k+1}{m}\right)f(0)+\frac{k+1}{m}f(1)+f\left(1-\frac{k+1}{m}\right)\right) \\ & +((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)f(0)+\frac{k}{m}f(1)+f\left(1-\frac{k}{m}\right)\right)-2f\left(\frac{1}{2}\right) \\ & -((1-\eta)f(0)+\eta f(1)) \\ = & f(\eta)+(m\eta-k)f\left(1-\frac{k+1}{m}\right)+((k+1)-m\eta)f\left(1-\frac{k}{m}\right)-2f\left(\frac{1}{2}\right) \\ \geqslant & f(\eta)+f(1-\eta)-2f\left(\frac{1}{2}\right) \geqslant 0. \end{aligned}$$

This completes the proof. □

REMARK 2.4.

(1) When $\eta \in \left[0, \frac{1}{m}\right] \cup \left[1-\frac{1}{m}, 1\right]$, where $m$ is a positive integer, inequalities (i) and (ii) are sharper than the second inequality in Theorem 1.1.

(2) For $m=2$, Theorem 2.3 coincides with the second inequality in Theorem 1.1.

Let $m=3$ in Theorem 2.3, we derive the following corollary.

COROLLARY 2.2. *Let* $f:[0,1]\to\mathbb{R}$*, be a convex function and* $0 \leqslant \eta \leqslant 1$.

*(i) If* $\eta \in \left[0, \frac{1}{3}\right]$*, then*

$$(1-\eta)f(0)+\eta f(1) \leqslant f(\eta)+2(1-\eta)\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right) \\ -3\eta\left(\frac{1}{3}f(1)+\frac{2}{3}f\left(\frac{1}{2}\right)-f\left(\frac{2}{3}\right)\right).$$

*(ii) If* $\eta \in \left[\frac{1}{3}, \frac{2}{3}\right]$*, then*

$$(1-\eta)f(0)+\eta f(1) \leqslant f(\eta)+(3\eta-1)\left(\frac{1}{3}f(0)+\frac{2}{3}f(1)+f\left(\frac{1}{3}\right)\right) \\ +(2-3\eta)\left(\frac{2}{3}f(0)+\frac{1}{3}f(1)+f\left(\frac{2}{3}\right)\right)-2f\left(\frac{1}{2}\right).$$

*(iii) If* $\eta \in \left[\frac{2}{3}, 1\right]$, *then*

$$(1-\eta)f(0)+\eta f(1) \leqslant f(\eta)+2\eta\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)$$
$$-(3-3\eta)\left(\frac{1}{3}f(0)+\frac{2}{3}f\left(\frac{1}{2}\right)-f\left(\frac{1}{3}\right)\right).$$

REMARK 2.5. As we explained in Remark 2.4: the first and third inequalities in Corollary 2.2 provide sharper bounds than the right-hand side of the inequalities presented in Theorem 1.1. However, we should reminder readers that the second inequality in Corollary 2.2 and the right-hand side of the inequalities in Theorem 1.1 are not comparable in general. To illustrate this, consider the convex function $f(t) = t^2$ on the interval $[0,1]$, and let $\eta = \frac{1}{2} \in \left[\frac{1}{3}, \frac{2}{3}\right]$.

We compute the following quantity:

$$\begin{aligned}
&(3\eta-1)\left(\frac{1}{3}f(0)+\frac{2}{3}f(1)+f\left(\frac{1}{3}\right)\right)\\
&\quad+(2-3\eta)\left(\frac{2}{3}f(0)+\frac{1}{3}f(1)+f\left(\frac{2}{3}\right)\right)-2f\left(\frac{1}{2}\right)\\
&=(0.5)\left(\frac{1}{3}(0)+\frac{2}{3}(1)+\left(\frac{1}{3}\right)^2\right)+(0.5)\left(\frac{2}{3}(0)+\frac{1}{3}(1)+\left(\frac{2}{3}\right)^2\right)-2\cdot\left(\frac{1}{2}\right)^2\\
&=0.5\cdot\left(\frac{2}{3}+\frac{1}{9}\right)+0.5\cdot\left(\frac{1}{3}+\frac{4}{9}\right)-0.5\\
&=0.5\cdot\left(\frac{7}{9}\right)+0.5\cdot\left(\frac{7}{9}\right)-0.5=\frac{7}{18}+\frac{7}{18}-\frac{1}{2}=\frac{5}{18}\approx 0.2778.
\end{aligned}$$

Now we compute the corresponding bound from Theorem 1.1:

$$2R\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)=2\cdot\frac{1}{2}\cdot\left(\frac{0+1}{2}-\left(\frac{1}{2}\right)^2\right)=0.25.$$

In this case, the quantity from Corollary 2.2 gives a slightly weaker bound compared to the corresponding refinement from Theorem 1.1.

However, if we take $\eta = \frac{5}{12}$ and $f(x) = 2^{1-x}3^x$, then we get the opposite conclusion, that is

$$\begin{aligned}
&(3\eta-1)\left(\frac{1}{3}f(0)+\frac{2}{3}f(1)+f\left(\frac{1}{3}\right)\right)+(2-3\eta)\left(\frac{2}{3}f(0)+\frac{1}{3}f(1)+f\left(\frac{2}{3}\right)\right)-2f\left(\frac{1}{2}\right)\\
&=\frac{1}{4}\left(\frac{1}{3}(2)+\frac{2}{3}(3)+2^{\frac{2}{3}}3^{\frac{1}{3}}\right)+\frac{3}{4}\left(\frac{2}{3}(2)+\frac{1}{3}(3)+2^{\frac{1}{3}}3^{\frac{2}{3}}\right)-2\sqrt{6}\\
&\approx 0.0556
\end{aligned}$$

and

$$2R\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)=2\cdot\frac{7}{12}\cdot\left(\frac{2+3}{2}-\sqrt{6}\right)\approx 0.0589,$$

which confirming that the bound of Corollary 2.2 is stronger than the corresponding conclusion in Theorem 1.1.

## 3. Convex and log-convex inequalities derived from weak submajorization

In many areas of applied mathematics and optimization theory, the concept of majorization plays a crucial role in comparing vectors in $\mathbb{R}^n$ based on the ordering of their components. In particular, weak submajorization is a partial ordering that captures the idea that one vector is, in a certain sense, more spread out or less concentrated than another.

Let $U=(U_1,U_2,\dots,U_n)$ and $V=(V_1,V_2,\dots,V_n)$ be vectors in $\mathbb{R}^n$. Denote by $U^{\downarrow}$ and $V^{\downarrow}$ the vectors obtained by rearranging the components of $U$ and $V$ in non-increasing order, respectively.

We say that $U$ is *weakly submajorized* by $V$, denoted $U\prec_w V$, if:

$$\sum_{i=1}^{k} U_i^{\downarrow}\leqslant \sum_{i=1}^{k} V_i^{\downarrow},\quad \text{for all } k=1,2,\dots,n.$$

This concept is particularly useful when analyzing inequalities involving convex functions and vector arguments.

LEMMA 3.1. (Fundamental Inequality for Weak Submajorization [17, pp. 13]) *Let $U=(U_i)_{i=1}^n, V=(V_i)_{i=1}^n\in\mathbb{R}^n$, and let $J\subset\mathbb{R}$ be an interval containing all components of $U$ and $V$. If $U\prec_w V$ and $\Psi:J\to\mathbb{R}$ is a continuous, increasing, and convex function, then the following inequality holds:*

$$\sum_{i=1}^{n}\Psi(U_i)\leqslant\sum_{i=1}^{n}\Psi(V_i).$$

LEMMA 3.2. *Let $f:[0,1]\to\mathbb{R}^+$ be a convex function and $U=(U_1,U_2)$ and $V=(V_1,V_2)\in\mathbb{R}^2$ be two vectors with components defined as follows:*

$$U_1=f(\eta),\ V_1=(1-\eta)f(0)+\eta f(1),\ U_2=\eta((m-1)f(0)+f(1)) \text{ and } V_2=m\eta f\left(\tfrac{1}{m}\right),$$

*where $m$ is a positive integer, $\eta\in\left[0,\frac{1}{m}\right]$. Let $K_1=\max\{U_1,U_2\}$ and $K_2=\min\{U_1,U_2\}$. If $K=(K_1,K_2)$, then we have*

$$K\prec_w V.$$

*that is,*

$$K_2\leqslant K_1\leqslant V_1,\ \ V_2\leqslant V_1;\ \text{ and }\ K_1+K_2\leqslant V_1+V_2.$$

*Proof.* Observe that

$$V_1-U_2=(1-\eta)f(0)+\eta f(1)-\eta((m-1)f(0)+f(1))=(1-m\eta)f(0)\geqslant 0,$$

that is $V_1\geqslant U_2$. Based on convexity of the function $f$, we deduce that $U_1\leqslant V_1$ and $U_2\geqslant V_2$, so we have $K_2\leqslant K_1\leqslant V_1$ and $V_2\leqslant V_1$.

On the other hand, $U_1+U_2=K_1+K_2\leqslant V_1+V_2$ follows from the second inequality in Theorem 2.1 directly. $\square$

LEMMA 3.3. *Let $f:[0,1]\to\mathbb{R}^+$ be a convex function and $U=(U_1,U_2)$ and $V=(V_1,V_2)\in\mathbb{R}^2$ be two vectors with components defined as follows: We consider the following expressions:*

$$\begin{aligned}
&U_1=(1-\eta)f(0)+\eta f(1),\\
&U_2=2f\left(\tfrac{1}{2}\right),\\
&V_2=f(\eta),\\
&V_1=(1-\eta)f(0)+(1-(m-1)\eta)f(1)+m\eta f\left(1-\tfrac{1}{m}\right),
\end{aligned}$$

*where $\eta\in\left[0,\frac{1}{m}\right]$. Let $K_1=\max\{U_1,U_2\}$ and $K_2=\min\{U_1,U_2\}$. If $K=(K_1,K_2)$, then we have*

$$K\prec_w V.$$

*that is,*

$$K_2\leqslant K_1\leqslant V_1,\ \ V_2\leqslant V_1;\ \text{ and }\ K_1+K_2\leqslant V_1+V_2.$$

*Proof.* First, by applying Jensen's inequality for three variables, we conclude

$$V_1=2\left(\frac{(1-\eta)}{2}f(0)+\frac{(1-(m-1)\eta)}{2}f(1)+\frac{m\eta}{2}f\left(1-\frac{1}{m}\right)\right)\geqslant 2f\left(\frac{1}{2}\right)=U_2.$$

Moreover, by the convexity of the function $f$, we have

$$V_1-U_1=(1-m\eta)f(1)+m\eta f\left(1-\frac{1}{m}\right)\geqslant 0\quad\text{and}\quad U_1\geqslant V_2,$$

so we have $K_2\leqslant K_1\leqslant V_1$ and $V_2\leqslant V_1$.

On the other hand, $U_1+U_2=K_1+K_2\leqslant V_1+V_2$ follows from the second part of Theorem 2.3 directly. □

Next, we discuss the situations when $\eta\in\left[0,\frac{1}{m}\right]\cup\left[1-\frac{1}{m},1\right]$ and $m$ is a positive integer, based on Theorems 2.1 and 2.3.

THEOREM 3.1. *Let $f:[0,1]\to\mathbb{R}^+$, be a convex function, $\Psi:J\to\mathbb{R}$ is a continuous, increasing and convex function, and $m$ a positive integer, and $\eta\in\left[0,\frac{1}{m}\right]\cup\left[1-\frac{1}{m},1\right]$.*

*(i) If $\eta\in\left[0,\frac{1}{m}\right]$, then*

$$\begin{aligned}
&\Psi\left(\eta\left((m-1)f(0)+f(1)\right)\right)-\Psi\left(m\eta f\left(\tfrac{1}{m}\right)\right)\\
&\leqslant\Psi\left((1-\eta)f(0)+\eta f(1)\right)-\Psi\left(f(\eta)\right)\\
&\leqslant\Psi\left((1-\eta)f(0)+(1-(m-1)\eta)f(1)+m\eta f\left(1-\tfrac{1}{m}\right)\right)-\Psi\left(2f\left(\tfrac{1}{2}\right)\right).
\end{aligned}\tag{3.1}$$

*(ii) If* $\eta \in \left[1-\frac{1}{m}, 1\right]$*, then*

$$\begin{aligned}&\Psi\left((1-\eta)\left(f(0)+(m-1)f(1)\right)\right)-\Psi\left((m-m\eta)f\left(1-\tfrac{1}{m}\right)\right)\\ \leqslant\,&\Psi\left((1-\eta)f(0)+\eta f(1)\right)-\Psi\left(f(\eta)\right)\\ \leqslant\,&\Psi\left((1-(m-1)(1-\eta))f(0)+\eta f(1)+(m-m\eta)f\left(\tfrac{1}{m}\right)\right)-\Psi\left(2f\left(\tfrac{1}{2}\right)\right).\end{aligned} \tag{3.2}$$

*Proof.* Let us consider the vectors $V=(V_1,V_2)$ and $K=(K_1,K_2)$ as defined in Lemma 3.2, we have $K \prec_w V$.

Then, by applying Lemma 3.1, it follows that

$$\Psi(V_1)+\Psi(V_2) \geqslant \Psi(K_1)+\Psi(K_2),$$

which implies

$$\Psi(V_1)-\Psi(U_1) \geqslant \Psi(U_2)-\Psi(V_2).$$

A similar argument applies to the reverse inequality.

The second inequality follows by observing that if the function $x \mapsto f(x)$ is convex, then so is $x \mapsto f(1-x)$. Moreover, if $\eta \in \left[1-\frac{1}{m}, 1\right]$, then $1-\eta \in \left[0, \frac{1}{m}\right]$. Therefore, by replacing $f(x)$ with $f(1-x)$ and $\eta$ with $1-\eta$, the desired inequality follows directly. □

By selecting the function $\Psi(x)=x^\lambda$ with $\lambda \geqslant 1$, we derive an intriguing refinement as well as a reversed form of the convexity inequality (1.1). This approach reveals deeper insights and extends the classical results.

THEOREM 3.2. *Let* $f:[0,1]\to\mathbb{R}^+$ *be a convex function, and let* $\lambda \geqslant 1$*,* $m$ *be a positive integer, and* $\eta \in \left[0, \frac{1}{m}\right] \cup \left[1-\frac{1}{m}, 1\right]$.

*(i) If* $\eta \in \left[0, \frac{1}{m}\right]$*, then*

$$\begin{aligned}&\left[\eta\left((m-1)f(0)+f(1)\right)\right]^\lambda-\left[m\eta f\left(\tfrac{1}{m}\right)\right]^\lambda\\ \leqslant\,&\left[(1-\eta)f(0)+\eta f(1)\right]^\lambda-\left[f(\eta)\right]^\lambda\\ \leqslant\,&\left[(1-\eta)f(0)+(1-(m-1)\eta)f(1)+m\eta f\left(1-\tfrac{1}{m}\right)\right]^\lambda-\left[2f\left(\tfrac{1}{2}\right)\right]^\lambda.\end{aligned}$$

*(ii) If* $\eta \in \left[1-\frac{1}{m}, 1\right]$*, then*

$$\begin{aligned}&\left[(1-\eta)\left(f(0)+(m-1)f(1)\right)\right]^\lambda-\left[(m-m\eta)f\left(1-\tfrac{1}{m}\right)\right]^\lambda\\ \leqslant\,&\left[(1-\eta)f(0)+\eta f(1)\right]^\lambda-\left[f(\eta)\right]^\lambda\\ \leqslant\,&\left[(1-(m-1)(1-\eta))f(0)+\eta f(1)+(m-m\eta)f\left(\tfrac{1}{m}\right)\right]^\lambda-\left[2f\left(\tfrac{1}{2}\right)\right]^\lambda.\end{aligned}$$

By choosing the function $\Psi(x)=\exp(x)$, we obtain the following interesting refinement and reverse of the log-convexity inequality (1.2).

THEOREM 3.3. *Let* $f : [0,1] \to (0,+\infty)$ *be a log-convex function,* $m \in \mathbb{N}$*, and let* $\eta \in \left[0, \frac{1}{m}\right] \cup \left[1-\frac{1}{m}, 1\right]$.

*(i) If* $\eta \in \left[0, \frac{1}{m}\right]$*, then*

$$\begin{aligned} &\left[f(0)^{m-1} f(1)\right]^{\eta} - f\left(\tfrac{1}{m}\right)^{m\eta} \\ \leqslant &\left[f(0)^{1-\eta} f(1)^{\eta}\right] - f(\eta) \\ \leqslant &\left[f(0)^{1-\eta} f(1)^{1-(m-1)\eta} f\left(1-\tfrac{1}{m}\right)^{m\eta}\right] - f\left(\tfrac{1}{2}\right)^{2}. \end{aligned}$$

*(ii) If* $\eta \in \left[1-\frac{1}{m}, 1\right]$*, then*

$$\begin{aligned} &\left[f(0) f(1)^{m-1}\right]^{1-\eta} - f\left(1-\tfrac{1}{m}\right)^{m-m\eta} \\ \leqslant &\, f(0)^{1-\eta} f(1)^{\eta} - f(\eta) \\ \leqslant &\, f(0)^{1-(m-1)(1-\eta)} f(1)^{\eta} f\left(\tfrac{1}{m}\right)^{m-m\eta} - f\left(\tfrac{1}{2}\right)^{2}. \end{aligned}$$

## 4. Application to some classical means

Means are fundamental mathematical tools used to represent central tendencies or averages of numbers. Among the most common are the arithmetic, geometric, and harmonic means, which have extensive applications in analysis, statistics, optimization, and operator theory.

### (1). Arithmetic mean

The *arithmetic mean* (AM) of two positive numbers $x$ and $y$ with weight $\eta \in [0,1]$ is defined as:

$$x \nabla_{\eta} y := (1-\eta)x + \eta y.$$

### (2). Geometric mean

The *geometric mean* (GM) captures multiplicative relationships and is defined (for $x, y > 0$) by:

$$x \sharp_{\eta} y := x^{1-\eta} y^{\eta}.$$

### (3). Harmonic mean

The harmonic mean is defined as the reciprocal of the arithmetic mean of the reciprocals:

$$x !_{\eta} y := \left((1-\eta)x^{-1} + \eta y^{-1}\right)^{-1}, \quad x, y > 0.$$

**(4). Power means**

More generally, the *power mean* (or Hölder mean) of order $p \in \mathbb{R}$ is defined by:

$$x\sharp_{\eta,p}y := ((1-\eta)x^p + \eta y^p)^{\frac{1}{p}}, \quad p \neq 0,$$

and

$$x\sharp_{\eta,0}y := x^{1-\eta}y^{\eta}.$$

Special cases of power mean include:

- $p = 1$: Arithmetic mean
- $p = 0$: Geometric mean
- $p = -1$: Harmonic mean
- $p \to \infty$, $x\sharp_{\eta,p}y \to \max\{x,y\}$; and $p \to -\infty$, $x\sharp_{\eta,p}y \to \min\{x,y\}$.

It is well known that the function

$$f(\eta) = x\sharp_{\eta,p}y := ((1-\eta)x^p + \eta y^p)^{\frac{1}{p}} \quad \text{for } \ \mathrm{p} \leqslant 1$$

is convex. As an application of Theorem 2.1, we obtain the following inequality, which provides a refinement of the classical inequality between the power means and the arithmetic mean.

THEOREM 4.1. *Let $x,y > 0$, $0 \leqslant \eta \leqslant 1$, and let $m$ be a positive integer. If $\eta \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$ for some $k = 0,1,\ldots,m-1$, then*

$$\begin{aligned}
&x\sharp_{\eta,p}y + (m\eta - k)\left(\left(1 - \frac{k+1}{m}\right)x + \frac{k+1}{m}y - x\sharp_{\frac{k+1}{m},p}y\right) \\
&\quad + ((k+1) - m\eta)\left(\left(1 - \frac{k}{m}\right)x + \frac{k}{m}y - x\sharp_{\frac{k}{m},p}y\right) \\
&\leqslant (1-\eta)x + \eta y.
\end{aligned}$$

*In particular:*

*(i) If $\eta \in \left[0, \frac{1}{m}\right]$, then*

$$\begin{aligned}
&x\sharp_{\eta,p}y + 2\eta\left(\frac{x+y}{2} - x\sharp_{1/2,p}y\right) \\
&\quad + m\eta\left(\left(1 - \frac{2}{m}\right)x + \frac{2}{m}x\sharp_{1/2,p}y - x\sharp_{1/m,p}y\right) \\
&\leqslant (1-\eta)x + \eta y.
\end{aligned}$$

*(ii) If* $\eta \in \left[1-\frac{1}{m},1\right]$*, then*

$$\begin{aligned} & x\sharp_{\eta,p}y + 2(1-\eta)\left(\frac{x+y}{2} - x\sharp_{1/2,p}y\right) \\ & \quad + (m-m\eta)\left(\left(1-\frac{2}{m}\right)y + \frac{2}{m}x\sharp_{1/2,p}y - x\sharp_{1-1/m,p}y\right) \\ & \leqslant (1-\eta)x + \eta y. \end{aligned}$$

If we take $p=-1$, we obtain the following inequality, which provides a refinement of the classical inequality relating the harmonic and arithmetic means.

THEOREM 4.2. *Let* $x,y>0$, $0 \leqslant \eta \leqslant 1$*, and let* $m$ *be a positive integer. If* $\eta \in \left[\frac{k}{m},\frac{k+1}{m}\right]$ *for some* $k=0,1,\ldots,m-1$*, then*

$$\begin{aligned} & x!_{\eta}y + (m\eta-k)\left(\left(1-\frac{k+1}{m}\right)x + \frac{k+1}{m}y - x!_{\frac{k+1}{m}}y\right) \\ & \quad + ((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)x + \frac{k}{m}y - x!_{\frac{k}{m}}y\right) \\ & \leqslant (1-\eta)x + \eta y. \end{aligned}$$

*In particular:*

*(i) If* $\eta \in \left[0,\frac{1}{m}\right]$*, then*

$$\begin{aligned} & x!_{\eta}y + 2\eta\left(\frac{x+y}{2} - x!_{1/2}y\right) \\ & \quad + m\eta\left(\left(1-\frac{2}{m}\right)x + \frac{2}{m}x!_{1/2}y - x!_{1/m}y\right) \\ & \leqslant (1-\eta)x + \eta y. \end{aligned}$$

*(ii) If* $\eta \in \left[1-\frac{1}{m},1\right]$*, then*

$$\begin{aligned} & x!_{\eta}y + 2(1-\eta)\left(\frac{x+y}{2} - x!_{1/2}y\right) \\ & \quad + (m-m\eta)\left(\left(1-\frac{2}{m}\right)y + \frac{2}{m}x!_{1/2}y - x!_{1-1/m}y\right) \\ & \leqslant (1-\eta)x + \eta y. \end{aligned}$$

As $p$ approaches zero, we obtain the following inequality, which provides a refinement of the classical Young's inequality. This refinement is precisely stated in Theorem 1.2 and serves as a key illustration of the strength and scope of our results.

THEOREM 4.3. *Let $x,y>0$, $0\leqslant\eta\leqslant 1$, and let $m$ be a positive integer. If $\eta\in\left[\frac{k}{m},\frac{k+1}{m}\right]$ for some $k=0,1,\ldots,m-1$, then*

$$\begin{aligned}
&x\sharp_\eta y+(m\eta-k)\left(\left(1-\frac{k+1}{m}\right)x+\frac{k+1}{m}y-x\sharp_{\frac{k+1}{m}}y\right)\\
&\quad+((k+1)-m\eta)\left(\left(1-\frac{k}{m}\right)x+\frac{k}{m}y-x\sharp_{\frac{k}{m}}y\right)\\
&\leqslant(1-\eta)x+\eta y.
\end{aligned}$$

*In particular:*

*(i) If $\eta\in\left[0,\frac{1}{m}\right]$, then*

$$\begin{aligned}
&x\sharp_\eta y+2\eta\left(\frac{x+y}{2}-x\sharp_{1/2}y\right)\\
&\quad+m\eta\left(\left(1-\frac{2}{m}\right)x+\frac{2}{m}x\sharp_{1/2}y-x\sharp_{1/m}y\right)\\
&\leqslant(1-\eta)x+\eta y.
\end{aligned}$$

*(ii) If $\eta\in\left[1-\frac{1}{m},1\right]$, then*

$$\begin{aligned}
&x\sharp_\eta y+2(1-\eta)\left(\frac{x+y}{2}-x\sharp_{1/2}y\right)\\
&\quad+(m-m\eta)\left(\left(1-\frac{2}{m}\right)y+\frac{2}{m}x\sharp_{1/2}y-x\sharp_{1-1/m}y\right)\\
&\leqslant(1-\eta)x+\eta y.
\end{aligned}$$

## 5. Application to matrix inequalities

Matrix inequalities offer valuable tools for analyzing linear transformations, particularly in understanding the interplay between norms, spectra, and matrix structures. A central theme in this field is enhancing classical results such as norm comparisons and inequalities involving matrix means and extending them within broader mathematical contexts. Investigations into convexity and log-convexity are especially influential, facilitating the development of sharper inequalities and generalized frameworks. These advancements not only deepen our theoretical understanding but also find applications in disciplines such as quantum theory, numerical analysis, and optimization. As a result, matrix inequalities remain a vibrant area of mathematical research with significant analytical and practical relevance.

Before delving into our main results, we introduce some notational conventions. Let $\mathbf{M}_n$ denote the algebra of all complex $n\times n$ matrices. A matrix $\mathscr{T}\in\mathbf{M}_n$ is said to be *Hermitian* if $\mathscr{T}=\mathscr{T}^*$, where $\mathscr{T}^*$ represents the conjugate transpose (adjoint) of $\mathscr{T}$. We write $\mathscr{T}\geqslant 0$ (or $\mathscr{T}>0$) to signify that $\mathscr{T}$ is positive semi-definite (or positive definite). For Hermitian matrices $\mathscr{T},\mathscr{S} \in \mathbf{M}_n$, the inequality $\mathscr{T} \geqslant \mathscr{S}$ means that $\mathscr{T}-\mathscr{S}$ is positive semi-definite.

We define the set of positive semi-definite matrices as

$$\mathbf{M}_n^+ := \{\mathscr{T} \in \mathbf{M}_n : \langle \mathscr{T}u,u\rangle \geqslant 0 \text{ for all } u \in \mathbb{C}^n\},$$

and the set of positive definite matrices by

$$\mathbf{M}_n^{++} := \{\mathscr{T} \in \mathbf{M}_n : \langle \mathscr{T}u,u\rangle > 0 \text{ for all nonzero } u \in \mathbb{C}^n\}.$$

The singular values of a matrix $\mathscr{T} \in \mathbf{M}_n$ are defined as the eigenvalues of the positive semi-definite matrix $|\mathscr{T}| := (\mathscr{T}^*\mathscr{T})^{1/2}$. These values are denoted by $s_j(\mathscr{T})$ for $j = 1,\dots,n$, and are ordered from largest to smallest.

A norm $\|\cdot\|$ on $\mathbf{M}_n$ is called *unitarily invariant* if it satisfies

$$\|U\mathscr{T}V\| = \|\mathscr{T}\|$$

for all unitary matrices $U,V \in \mathbf{M}_n$ and any $\mathscr{T} \in \mathbf{M}_n$. Two common examples are:

- **Trace norm:** $\|\mathscr{T}\|_1 := \mathrm{tr}|\mathscr{T}| = \sum_{j=1}^n s_j(\mathscr{T})$,
- **Hilbert-Schmidt norm:**

$$\|\mathscr{T}\|_2 := (\mathrm{tr}(\mathscr{T}\mathscr{T}^*))^{1/2} = \left(\sum_{i,j} |t_{ij}|^2\right)^{1/2},$$

  where $\mathscr{T} = (t_{ij})$.

For all $\mathscr{T},\mathscr{S} \in \mathbf{M}_n$, any positive real number $r$, and every unitarily invariant norm, Horn and Mathias [5, 6] established the following matrix version of the Cauchy-Schwarz inequality:

$$\||\mathscr{T}^*\mathscr{S}|^r\|^2 \leqslant \|(\mathscr{T}^*\mathscr{T})^r\|\|(\mathscr{S}^*\mathscr{S})^r\|.$$

Bhatia and Davis extended this result in [3] to a more generalized setting. Specifically, for all $\mathscr{T},\mathscr{S} \in \mathbf{M}_n^+$, any $X \in \mathbf{M}_n$, and $r > 0$, they established:

$$\||\mathscr{T}^*X\mathscr{S}|^r\| \leqslant \||\mathscr{T}\mathscr{T}^*X|^r\|\||X\mathscr{S}\mathscr{S}^*|^r\|.$$

This is equivalent to the following inequality:

$$\||\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}|^r\|^2 \leqslant \||\mathscr{T}X|^r\|\||X\mathscr{S}|^r\|. \tag{5.1}$$

For all $\mathscr{T},\mathscr{S} \in \mathbf{M}_n^+$ and $t \in [0,1]$, a Hölder-type inequality has been derived in [12]:

$$\||\mathscr{T}^{1-t}X\mathscr{S}^t|^r\| \leqslant \||\mathscr{T}X|^r\|^{1-t} \cdot \||X\mathscr{S}|^r\|^t. \tag{5.2}$$

Given $\mathscr{T},\mathscr{S} \in \mathbf{M}_n^{++}$ and $X \in \mathbf{M}_n$, the function $f(t) = \||\mathscr{T}^{1-t}X\mathscr{S}^t|^r\|$ with $r > 0$ is log-convex on the interval $[0,1]$ for every unitarily invariant norm $\|\cdot\|$ on $\mathbf{M}_n$ (see [12]). Utilizing Theorem 3.3, we obtain a novel refinement and reverse of the Hölder-type inequality (5.2).

THEOREM 5.1. *Let* $\mathscr{T},\mathscr{S}\in\mathbf{M}_n^{++}$ *and* $0\neq X\in\mathbf{M}_n$, $r>0$, $m\in\mathbb{N}$, *and let* $\eta\in\left[0,\frac{1}{m}\right]\cup\left[1-\frac{1}{m},1\right]$. *Then:*

*(i) If* $\eta\in\left[0,\frac{1}{m}\right]$, *then*

$$\begin{aligned}
&\left[\||\mathscr{T}X|^r\|^{m-1}\,\||X\mathscr{S}|^r\|\right]^{\eta}-\left\|\left|\mathscr{T}^{1-\frac{1}{m}}X\mathscr{S}^{\frac{1}{m}}\right|^r\right\|^{m\eta}\\
&\leqslant\left[\||\mathscr{T}X|^r\|^{1-\eta}\,\||X\mathscr{S}|^r\|^{\eta}\right]-\left\|\left|\mathscr{T}^{1-\eta}X\mathscr{S}^{\eta}\right|^r\right\|\\
&\leqslant\left[\||\mathscr{T}X|^r\|^{1-\eta}\,\||X\mathscr{S}|^r\|^{1-(m-1)\eta}\left\|\left|\mathscr{T}^{\frac{1}{m}}X\mathscr{S}^{1-\frac{1}{m}}\right|^r\right\|^{m\eta}\right]-\left\|\left|\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}\right|^r\right\|^2.
\end{aligned}$$

*(ii) If* $\eta\in\left[1-\frac{1}{m},1\right]$, *then*

$$\begin{aligned}
&\left[\||\mathscr{T}X|^r\|\,\||X\mathscr{S}|^r\|^{m-1}\right]^{1-\eta}-\left\|\left|\mathscr{T}^{\frac{1}{m}}X\mathscr{S}^{1-\frac{1}{m}}\right|^r\right\|^{m-m\eta}\\
&\leqslant\||\mathscr{T}X|^r\|^{1-\eta}\,\||X\mathscr{S}|^r\|^{\eta}-\left\|\left|\mathscr{T}^{1-\eta}X\mathscr{S}^{\eta}\right|^r\right\|\\
&\leqslant\||\mathscr{T}X|^r\|^{1-(m-1)(1-\eta)}\,\||X\mathscr{S}|^r\|^{\eta}\left\|\left|\mathscr{T}^{1-\frac{1}{m}}X\mathscr{S}^{\frac{1}{m}}\right|^r\right\|^{m-m\eta}-\left\|\left|\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}\right|^r\right\|^2.
\end{aligned}$$

For every $\mathscr{T},\mathscr{S}\in\mathbf{M}_n^{++}$ and $X\in\mathbf{M}_n$, the function

$$f(t)=\||\mathscr{T}^tX\mathscr{S}^t|^r\|$$

with $r>0$ is log-convex on $[0,1]$, for any unitarily invariant norm $\|\cdot\|$ on $\mathbf{M}_n$ (see [12]).

THEOREM 5.2. *Let* $\mathscr{T},\mathscr{S}\in\mathbf{M}_n^{++}$ *and* $0\neq X\in\mathbf{M}_n$, $r>0$, $m\in\mathbb{N}$, *and let* $\eta\in\left[0,\frac{1}{m}\right]\cup\left[1-\frac{1}{m},1\right]$. *Then:*

*(i) If* $\eta\in\left[0,\frac{1}{m}\right]$, *then*

$$\begin{aligned}
&\left[\||X|^r\|^{m-1}\,\||\mathscr{T}X\mathscr{S}|^r\|\right]^{\eta}-\left\|\left|\mathscr{T}^{\frac{1}{m}}X\mathscr{S}^{\frac{1}{m}}\right|^r\right\|^{m\eta}\\
&\leqslant\left[\||X|^r\|^{1-\eta}\,\||\mathscr{T}X\mathscr{S}|^r\|^{\eta}\right]-\||\mathscr{T}^{\eta}X\mathscr{S}^{\eta}|^r\|\\
&\leqslant\left[\||X|^r\|^{1-\eta}\,\||\mathscr{T}X\mathscr{S}|^r\|^{1-(m-1)\eta}\left\|\left|\mathscr{T}^{1-\frac{1}{m}}X\mathscr{S}^{1-\frac{1}{m}}\right|^r\right\|^{m\eta}\right]-\left\|\left|\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}\right|^r\right\|^2.
\end{aligned}$$

*(ii) If* $\eta\in\left[1-\frac{1}{m},1\right]$, *then*

$$\begin{aligned}
&\left[\||X|^r\|\,\||\mathscr{T}X\mathscr{S}|^r\|^{m-1}\right]^{1-\eta}-\left\|\left|\mathscr{T}^{1-\frac{1}{m}}X\mathscr{S}^{1-\frac{1}{m}}\right|^r\right\|^{m-m\eta}\\
&\leqslant\||X|^r\|^{1-\eta}\,\||\mathscr{T}X\mathscr{S}|^r\|^{\eta}-\||\mathscr{T}^{\eta}X\mathscr{S}^{\eta}|^r\|\\
&\leqslant\||X|^r\|^{1-(m-1)(1-\eta)}\,\||\mathscr{T}X\mathscr{S}|^r\|^{\eta}\left\|\left|\mathscr{T}^{\frac{1}{m}}X\mathscr{S}^{\frac{1}{m}}\right|^r\right\|^{m-m\eta}-\left\|\left|\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}\right|^r\right\|^2.
\end{aligned}$$

REMARK 5.1. Notice that Theorem 5.2 provides both a refinement and a reverse form of the Hölder-type inequalities for unitarily invariant norms. In particular, by setting $X = I$, we obtain refinement and reverse of the classical inequality.

$$|||\mathscr{T}\mathscr{S}|^r\|^t \geqslant |||\mathscr{T}^t\mathscr{S}^t|^r\|,$$

for $0 < t \leqslant 1$.

It is established in [23] that for two matrices $\mathscr{T}, \mathscr{S} \in \mathbf{M}_n^+$, the function $f(t) = \mathrm{tr}(\mathscr{T}^{1-t}\mathscr{S}^t)$ is log-convex on the interval $[0,1]$. Utilizing Theorem 3.3, we derive the following theorem, which provides a refinement and reverse of the classical Hölder's type inequality for the trace of matrices.

THEOREM 5.3. *Let* $\mathscr{T}, \mathscr{S} \in \mathbf{M}_n^+$, $m \in \mathbb{N}$, *and let* $\eta \in \left[0, \frac{1}{m}\right] \cup \left[1 - \frac{1}{m}, 1\right]$. *Then:*

*(i) If* $\eta \in \left[0, \frac{1}{m}\right]$, *then*

$$\begin{aligned}
&\left[\mathrm{tr}(\mathscr{T})^{m-1}\mathrm{tr}(\mathscr{S})\right]^{\eta} - \mathrm{tr}\left(\mathscr{T}^{1-\frac{1}{m}}\mathscr{S}^{\frac{1}{m}}\right)^{m\eta} \\
&\leqslant \left[\mathrm{tr}(\mathscr{T})^{1-\eta}\mathrm{tr}(\mathscr{S})^{\eta}\right] - \mathrm{tr}\left(\mathscr{T}^{1-\eta}\mathscr{S}^{\eta}\right) \\
&\leqslant \left[\mathrm{tr}(\mathscr{T})^{1-\eta}\mathrm{tr}(\mathscr{S})^{1-(m-1)\eta}\mathrm{tr}\left(\mathscr{T}^{\frac{1}{m}}\mathscr{S}^{1-\frac{1}{m}}\right)^{m\eta}\right] - \mathrm{tr}\left(\mathscr{T}^{\frac{1}{2}}\mathscr{S}^{\frac{1}{2}}\right)^{2}.
\end{aligned}$$

*(ii) If* $\eta \in \left[1 - \frac{1}{m}, 1\right]$, *then*

$$\begin{aligned}
&\left[\mathrm{tr}(\mathscr{T})\mathrm{tr}(\mathscr{S})^{m-1}\right]^{1-\eta} - \mathrm{tr}\left(\mathscr{T}^{\frac{1}{m}}\mathscr{S}^{1-\frac{1}{m}}\right)^{m-m\eta} \\
&\leqslant \mathrm{tr}(\mathscr{T})^{1-\eta}\mathrm{tr}(\mathscr{S})^{\eta} - \mathrm{tr}\left(\mathscr{T}^{1-\eta}\mathscr{S}^{\eta}\right) \\
&\leqslant \mathrm{tr}(\mathscr{T})^{1-(m-1)(1-\eta)}\mathrm{tr}(\mathscr{S})^{\eta}\mathrm{tr}\left(\mathscr{T}^{1-\frac{1}{m}}\mathscr{S}^{\frac{1}{m}}\right)^{m-m\eta} - \mathrm{tr}\left(\mathscr{T}^{\frac{1}{2}}\mathscr{S}^{\frac{1}{2}}\right)^{2}.
\end{aligned}$$

## 6. Application to refined numerical radius inequalites

The numerical radius of an operator $\mathscr{T} \in \mathbf{M}_n$, denoted by $w(\mathscr{T})$, is defined as

$$w(\mathscr{T}) = \sup_{\|u\|=1} |\langle \mathscr{T}u, u\rangle|,$$

and serves as a fundamental tool for measuring the size or "magnitude" of an operator. This quantity not only provides valuable information about the spectral behavior of $\mathscr{T}$, but also plays a central role in many areas of operator theory and functional analysis. Over the years, numerous inequalities have been established that connect the numerical radius with classical operator norms, spectral properties, and various matrix means, thus highlighting its wide applicability.

To broaden this framework, Abbas *et al.* [1] introduced a generalization based on an arbitrary norm $N(\cdot)$ on $\mathbf{M}_n$. The resulting generalized numerical radius of $\mathscr{T}$, denoted by $w_N(\mathscr{T})$, is defined as

$$w_N(\mathscr{T}) = \sup_{\theta\in\mathbb{R}} N\Big(\mathrm{Re}\big(e^{i\theta}\mathscr{T}\big)\Big),$$

where the real part of an operator $Z$ is given by $\mathrm{Re}(Z) = \frac{Z+Z^*}{2}$. This generalization provides a flexible framework that accommodates different choices of norms, thereby extending the scope of classical numerical radius inequalities.

Moreover, for a matrix $X \in \mathbf{M}_n$ with Cartesian decomposition

$$X = \mathrm{Re}(X) + i\,\mathrm{Im}(X),$$

one has

$$\mathrm{Re}(X) = \frac{X+X^*}{2}, \quad \mathrm{Im}(X) = \frac{X-X^*}{2i}.$$

These decompositions play a crucial role in analyzing the structure of matrices and in deriving inequalities involving the generalized numerical radius.

It follows that when $N$ is the usual operator norm derived from the Hilbert space inner product, $w_N(\cdot)$ coincides with the classical numerical radius $w(\cdot)$.

Building on the results in [1], the function

$$f(t) = w_N\left(\mathscr{T}^{1-t}X\mathscr{T}^{t} + \mathscr{T}^{t}X\mathscr{T}^{1-t}\right)$$

is convex on the interval $[0,1]$ for any unitarily invariant norm $N(\cdot)$ on $\mathbf{M}_n$, where $\mathscr{T} \in \mathbf{M}_n^+$ and $X \in \mathbf{M}_n$. This convexity property allows us to invoke Theorem 3.2 to establish Heinz-type inequalities involving the generalized numerical radius.

THEOREM 6.1. *Let $\mathscr{T} \in \mathbf{M}_n^+$ and $X \in \mathbf{M}_n$. Then, for $\lambda \geqslant 1$, $m \in \mathbb{N}$, and $\eta \in \left[0,\frac{1}{m}\right] \cup \left[1-\frac{1}{m},1\right]$, the following hold:*

*(i) If $\eta \in \left[0,\frac{1}{m}\right]$, then*

$$\begin{aligned}
&\left[m\eta\, w_N(\mathscr{T}X+X\mathscr{T})\right]^{\lambda} - \left[m\eta\, w_N(\mathscr{T}^{1-\frac{1}{m}}X\mathscr{T}^{\frac{1}{m}} + \mathscr{T}^{\frac{1}{m}}X\mathscr{T}^{1-\frac{1}{m}})\right]^{\lambda}\\
\leqslant &\left[w_N(\mathscr{T}X+X\mathscr{T})\right]^{\lambda} - \left[w_N(\mathscr{T}^{1-\eta}X\mathscr{T}^{\eta} + \mathscr{T}^{\eta}X\mathscr{T}^{1-\eta})\right]^{\lambda}\\
\leqslant &\left[(2-m\eta)\, w_N(\mathscr{T}X+X\mathscr{T}) + m\eta\, w_N(\mathscr{T}^{1-\frac{1}{m}}X\mathscr{T}^{\frac{1}{m}} + \mathscr{T}^{\frac{1}{m}}X\mathscr{T}^{1-\frac{1}{m}})\right]^{\lambda}\\
&- \left[2\, w_N(\mathscr{T}^{1/2}X\mathscr{T}^{1/2})\right]^{\lambda}.
\end{aligned}$$

*(ii) If* $\eta \in \left[1-\frac{1}{m},1\right]$*, then*

$$\left[m(1-\eta)\,w_N(\mathscr{T}X+X\mathscr{T})\right]^{\lambda}-\left[(m-m\eta)\,w_N(\mathscr{T}^{1-\frac{1}{m}}X\mathscr{T}^{\frac{1}{m}}+\mathscr{T}^{\frac{1}{m}}X\mathscr{T}^{1-\frac{1}{m}})\right]^{\lambda}$$
$$\leqslant \left[w_N(\mathscr{T}X+X\mathscr{T})\right]^{\lambda}-\left[w_N(\mathscr{T}^{1-\eta}X\mathscr{T}^{\eta}+\mathscr{T}^{\eta}X\mathscr{T}^{1-\eta})\right]^{\lambda}$$
$$\leqslant \left[(2-m(1-\eta))\,w_N(\mathscr{T}X+X\mathscr{T})+(m-m\eta)\,w_N(\mathscr{T}^{1-\frac{1}{m}}X\mathscr{T}^{\frac{1}{m}}+\mathscr{T}^{\frac{1}{m}}X\mathscr{T}^{1-\frac{1}{m}})\right]^{\lambda}$$
$$-\left[2\,w_N(\mathscr{T}^{1/2}X\mathscr{T}^{1/2})\right]^{\lambda}.$$

As stated in [1], if $\mathscr{T},\mathscr{S}\in\mathbf{M}_n^+$ and $X\in\mathbf{M}_n$, the function $f(t)=w_N(\mathscr{T}^{1-t}X\mathscr{S}^{t})$ is convex on the interval $[0,1]$ for any unitarily invariant norm $N(\cdot)$ on $\mathbf{M}_n$. Hence, by applying Theorem 3.2 to this function, we obtain the following generalized refinement with multiple terms of the Young-type inequality for the numerical radius.

THEOREM 6.2. *Let* $\mathscr{T},\mathscr{S}\in\mathbf{M}_n^+$ *and* $X\in\mathbf{M}_n$*, and let* $\lambda\geqslant 1$*,* $m\in\mathbb{N}$*, and* $\eta\in\left[0,\frac{1}{m}\right]\cup\left[1-\frac{1}{m},1\right]$*. Then:*

*(i) If* $\eta\in\left[0,\frac{1}{m}\right]$*, then*

$$\left[\eta\left((m-1)w_N(\mathscr{T}X)+w_N(X\mathscr{S})\right)\right]^{\lambda}-\left[m\eta\cdot w_N\left(\mathscr{T}^{1-\frac{1}{m}}X\mathscr{S}^{\frac{1}{m}}\right)\right]^{\lambda}$$
$$\leqslant \left[(1-\eta)w_N(\mathscr{T}X)+\eta w_N(X\mathscr{S})\right]^{\lambda}-\left[w_N\left(\mathscr{T}^{1-\eta}X\mathscr{S}^{\eta}\right)\right]^{\lambda}$$
$$\leqslant \left[(1-\eta)w_N(\mathscr{T}X)+(1-(m-1)\eta)w_N(X\mathscr{S})+m\eta w_N\left(\mathscr{T}^{\frac{1}{m}}X\mathscr{S}^{1-\frac{1}{m}}\right)\right]^{\lambda}$$
$$-\left[2w_N\left(\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}\right)\right]^{\lambda}.$$

*(ii) If* $\eta\in\left[1-\frac{1}{m},1\right]$*, then*

$$\left[(1-\eta)\left(w_N(\mathscr{T}X)+(m-1)w_N(X\mathscr{S})\right)\right]^{\lambda}-\left[(m-m\eta)w_N\left(\mathscr{T}^{\frac{1}{m}}X\mathscr{S}^{1-\frac{1}{m}}\right)\right]^{\lambda}$$
$$\leqslant \left[(1-\eta)w_N(\mathscr{T}X)+\eta w_N(X\mathscr{S})\right]^{\lambda}-\left[w_N\left(\mathscr{T}^{1-\eta}X\mathscr{S}^{\eta}\right)\right]^{\lambda}$$
$$\leqslant \left[(1-(m-1)(1-\eta))w_N(\mathscr{T}X)+\eta w_N(X\mathscr{S})+(m-m\eta)w_N\left(\mathscr{T}^{1-\frac{1}{m}}X\mathscr{S}^{\frac{1}{m}}\right)\right]^{\lambda}$$
$$-\left[2w_N\left(\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}\right)\right]^{\lambda}.$$

It has been shown (see [1] and [22, Proposition 2.5]) that for any matrices $\mathscr{T},\mathscr{S}\in\mathbf{M}_n^+$ and $X\in\mathbf{M}_n$, the mapping

$$f(t)=w_N(\mathscr{T}^{t}X\mathscr{S}^{t})$$

is log-convex over the interval $[0,1]$, where $N(\cdot)$ denotes any unitarily invariant norm on $\mathbf{M}_n$. Consequently, by utilizing Theorem 3.3 on this function, one can derive

Hölder-type inequalities involving the numerical radius within the framework of unitarily invariant norms.

THEOREM 6.3. *Let* $\mathscr{T},\mathscr{S}\in\mathbf{M}_n^+$, $X\in\mathbf{M}_n$ *and let* $\eta\in\left[0,\frac{1}{m}\right]\cup\left[1-\frac{1}{m},1\right]$. *Then:*

*(i) If* $\eta\in\left[0,\frac{1}{m}\right]$, *we have*

$$\begin{aligned}&\left[w_N(X)^{m-1}w_N(\mathscr{T}X\mathscr{S})\right]^{\eta}-w_N\left(\mathscr{T}^{\frac{1}{m}}X\mathscr{S}^{\frac{1}{m}}\right)^{m\eta}\\ \leqslant\ &\left[w_N(X)^{1-\eta}w_N(\mathscr{T}X\mathscr{S})^{\eta}\right]-w_N\left(\mathscr{T}^{\eta}X\mathscr{S}^{\eta}\right)\\ \leqslant\ &\left[w_N(X)^{1-\eta}w_N(\mathscr{T}X\mathscr{S})^{1-(m-1)\eta}w_N\left(\mathscr{T}^{1-\frac{1}{m}}X\mathscr{S}^{1-\frac{1}{m}}\right)^{m\eta}\right]-w_N\left(\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}\right)^{2}.\end{aligned}$$

*(ii) If* $\eta\in\left[1-\frac{1}{m},1\right]$, *we have*

$$\begin{aligned}&\left[w_N(X)w_N(\mathscr{T}X\mathscr{S})^{m-1}\right]^{1-\eta}-w_N\left(\mathscr{T}^{1-\frac{1}{m}}X\mathscr{S}^{1-\frac{1}{m}}\right)^{m-m\eta}\\ \leqslant\ &w_N(X)^{1-\eta}w_N(\mathscr{T}X\mathscr{S})^{\eta}-w_N\left(\mathscr{T}^{\eta}X\mathscr{S}^{\eta}\right)\\ \leqslant\ &w_N(X)^{1-(m-1)(1-\eta)}w_N(\mathscr{T}X\mathscr{S})^{\eta}w_N\left(\mathscr{T}^{\frac{1}{m}}X\mathscr{S}^{\frac{1}{m}}\right)^{m-m\eta}-w_N\left(\mathscr{T}^{\frac{1}{2}}X\mathscr{S}^{\frac{1}{2}}\right)^{2}.\end{aligned}$$

In particular, for $X=I$, we obtain the following theorem, which presents a refinement and reverse of the classical inequality:

$$w_N^t(\mathscr{T}\mathscr{S})\geqslant w_N(\mathscr{T}^t\mathscr{S}^t),$$

for $t\in[0,1]$.

## 7. Conclusions

In this paper, we have presented a comprehensive generalization of classical convexity-based inequalities, extending recent developments such as those by Yang and Zhang to a wider class of convex and log-convex functions. By employing interpolation techniques and weak submajorization, we established novel refinements and reverses of Jensen-type and Young-type inequalities, yielding sharper bounds and more flexible analytical tools.

Our results demonstrate the versatility of convexity in both scalar and matrix settings, with applications ranging from inequalities for classical means to refined matrix norm and numerical radius inequalities. In particular, we have shown how these inequalities enhance the precision of operator bounds and open up new avenues in functional analysis and quantum information theory.

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