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SHARP MULTIDIMENSIONAL MULTIPLICATIVE INEQUALITIES FOR WEIGHTED L_p SPACES WITH HOMOGENEOUS WEIGHTS

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Abstract. Let Ω be an arbitrary cone in $\mathbb{R}^n$ with the origin as a vertex. A multidimensional multiplicative inequality for weighted $L_p(\Omega)$-spaces with homogeneous weights is proved. The inequality is sharp and all cases of equality are pointed out. In particular, this inequality may be regarded as a weighted multidimensional extension of previous inequalities of Carlson, Beurling and Levin.

1. Introduction

Let f be a nonnegative function on $[0,\infty)$ such that $f^2(x)$ and $x^2f^2(x)$ are integrable on $[0,\infty)$. Then the inequality

$$\int_0^\infty f(x)dx \leqslant \sqrt{\pi}\left(\int_0^\infty f^2(x)dx\right)^{1/4}\left(\int_0^\infty x^2f^2(x)dx\right)^{1/4} \tag{1}$$

holds and $\sqrt{\pi}$ is the best possible constant. This is the well-known Carlson inequality [2]. It was generalized in many directions. We formulate next a general result obtained by V. I. Levin [3]. For other generalizations see the books [4, 5] and the references given there.

THEOREM 1. ([3]) *Let $p>1$, $q>1$, $\lambda>0$, $\mu>0$ and let f be a nonnegative function such that $x^{p-1-\lambda}f^p(x)$ and $x^{q-1+\mu}f^q(x)$ are integrable on $[0,\infty)$. Then*

$$\int_0^\infty f(x)dx \leqslant C\left(\int_0^\infty x^{p-1-\lambda}f^p(x)dx\right)^s\left(\int_0^\infty x^{q-1+\mu}f^q(x)dx\right)^t \tag{2}$$

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where[1]

$$C=\left(\frac{1}{ps}\right)^{s}\left(\frac{1}{qt}\right)^{t}\left(\frac{1}{\lambda+\mu}B\left(\frac{s}{1-s-t},\frac{t}{1-s-t}\right)\right)^{1-s-t}$$

and

$$s=\frac{\mu}{p\mu+q\lambda},\quad t=\frac{\lambda}{p\mu+q\lambda}.$$

The constant C is best possible.

We note that the inequality (2) is equivalent to the inequality

$$\int_{-\infty}^{\infty}f(x)dx\leqslant 2^{1-s-t}C\left(\int_{-\infty}^{\infty}|x|^{p-1-\lambda}f^{p}(x)dx\right)^{s}\left(\int_{-\infty}^{\infty}|x|^{q-1+\mu}f^{q}(x)dx\right)^{t}. \quad (3)$$

In particular, the inequality (1) is equivalent to the inequality

$$\int_{-\infty}^{\infty}f(x)dx\leqslant\sqrt{2\pi}\left(\int_{-\infty}^{\infty}f^{2}(x)dx\right)^{1/4}\left(\int_{-\infty}^{\infty}x^{2}f^{2}(x)dx\right)^{1/4}$$

first established by A. Beurling $[1]$. Indeed, in order to deduce (2) from (3) it is enough to apply (3) to the even extension of f from $[0,\infty)$ to $(-\infty,\infty)$. On the other hand from (2) and Hölder's inequality for sums it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}f(x)dx&=\int_{-\infty}^{0}f(x)dx+\int_{0}^{\infty}f(x)dx\\&\leqslant C\Bigg(\left(\int_{-\infty}^{0}|x|^{p-1-\lambda}f^{p}(x)dx\right)^{s}\left(\int_{-\infty}^{0}|x|^{q-1+\mu}f^{q}(x)dx\right)^{t}\\&\qquad+\left(\int_{0}^{\infty}|x|^{p-1-\lambda}f^{p}(x)dx\right)^{s}\left(\int_{0}^{\infty}|x|^{q-1+\mu}f^{q}(x)dx\right)^{t}\Bigg)\\&\leqslant 2^{1-s-t}C\left(\int_{-\infty}^{\infty}|x|^{p-1-\lambda}f^{p}(x)dx\right)^{s}\left(\int_{-\infty}^{\infty}|x|^{q-1+\mu}f^{q}(x)dx\right)^{t}\end{aligned}$$

since $a_1b_1+a_2b_2\leqslant 2^{1-s-t}\left(a_1^{1/s}+b_1^{1/s}\right)^{s}\left(a_2^{1/t}+b_2^{1/t}\right)^{t}$, $a_i,\ b_i\geqslant 0$.

Our aim is to give generalizations of (2) and (3) to the multidimensional case and general homogeneous weights.

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[1]Here $B(\sigma,\tau)$, $\sigma,\tau>0$, is the beta-function: $B(\sigma,\tau)=\frac{\Gamma(\sigma)\Gamma(\tau)}{\Gamma(\sigma+\tau)}$, where $\Gamma(\sigma)$, $\sigma>0$, is the gamma-function.

2. The Main Result

Let Ω be an arbitrary infinite cone in $\mathbb{R}^n$ with the origin as a vertex, i.e.,

$$\Omega = \{x \in \mathbb{R}^n : x = \rho\sigma,\, 0 < \rho < \infty,\, \sigma \in S\} \tag{4}$$

where $S \subseteq S^n = \{x \in \mathbb{R}^n : |x| = 1\}$ is a measurable set. We note that for arbitrary positive ε we have $\varepsilon\Omega = \Omega$.

THEOREM 2. *Let Ω be defined by* (4) *and w_i, $i = 0, 1, 2$, be functions defined on Ω. Suppose that w_i are positive, measurable and homogeneous*[2] *of orders $\alpha_i \in \mathbb{R}$, $i = 0, 1, 2$. Moreover, suppose that*

$$0 < p_0 < p_1, p_2 < \infty, \quad 0 < \theta < 1, \tag{5}$$

and put

$$d_i = \alpha_i + \frac{n}{p_i}, \;\; i = 0, 1, 2.$$

1. *In order that for some $A > 0$ and for all functions f measurable on Ω and satisfying $\|f w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1, 2$,*

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant A \|f w_1\|^{\theta}_{L_{p_1}(\Omega)} \|f w_2\|^{1-\theta}_{L_{p_2}(\Omega)}, \tag{6}$$

it is necessary and sufficient that

$$d_0 = \theta d_1 + (1-\theta) d_2, \;\; d_1 \neq d_2, \tag{7}$$

and

$$\left\| \frac{w_0}{w_1^{\theta} w_2^{1-\theta}} \right\|_{L_q(S)} < \infty, \tag{8}$$

where

$$\frac{1}{q} = \frac{1}{p_0} - \frac{\theta}{p_1} - \frac{1-\theta}{p_2}. \tag{9}$$

2. *If* (7) *and* (8) *are satisfied, then the minimal possible value A_* of A in* (6) *has the form*

$$A_* = \left(\frac{\theta}{p_1}\right)^{-\frac{\theta}{p_1}} \left(\frac{1-\theta}{p_2}\right)^{-\frac{1-\theta}{p_2}} (p_1 p_2)^{-\frac{1}{p_0}} \left(\frac{B(\theta\frac{q}{p_1}, (1-\theta)\frac{q}{p_2})}{|\, d_1 - d_2 \,|}\right)^{1/q} \times$$

$$\times \left(\frac{1}{p_0} - \frac{1}{q}\right)^{-\frac{1}{q}} \left\| \frac{w_0}{w_1^{\theta} w_2^{1-\theta}} \right\|_{L_q(S)}. \tag{10}$$

[2] I.e., for each $x \in \Omega$ and $\varepsilon > 0$, $w_i(\varepsilon x) = \varepsilon^{\alpha_i} w_i(x)$.

3. *Equality in* (6) *with* $A = A_*$ *holds if and only if, for some* $B \geqslant 0$, $\gamma > 0$,

$$|f(x)| = Bf_*(\gamma x) \tag{11}$$

for almost all $x \in \Omega$*, where*

$$f_* = \left(k\frac{w_0^{p_0}}{w_1^{p_1}}\right)^{\frac{1}{p_1 - p_0}}. \tag{12}$$

and the positive function k *is uniquely defined by*

$$\left(k^{1/p_1}\frac{w_0}{w_1}\right)^{r_1} = \left((1-k)^{1/p_2}\frac{w_0}{w_2}\right)^{r_2}, \tag{13}$$

where

$$\frac{1}{r_i} = \frac{1}{p_0} - \frac{1}{p_i}, \quad i = 1, 2. \tag{14}$$

REMARK 1. It is enough to prove Theorem 2 for the case in which $p_0 = 1$, $w_0 = 1$. The general case follows from this case if we replace w_i by $\left(\frac{w_i}{w_0}\right)^{p_0}$, p_i by $\frac{p_i}{p_0}$, $i = 1, 2$ and f by $(f w_0)^{p_0}$.

3. Proof of The Main Result

First we state the following lemma of independent intrest:

LEMMA 1. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a measurable set and let* w_0, w_1, w_2 *be positive and measurable functions on* Ω*. Moreover, suppose that* $0 < p_0 < p_1, p_2 < \infty$*, and* $\delta > 0$*. If* f *is measurable on* Ω *and* $\|f w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1, 2$*, then*

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant \left(M_1^{p_0}(\delta)\|f w_1\|^{p_0}_{L_{p_1}(\Omega)} + M_2^{p_0}(\delta)\|f w_1\|^{p_0}_{L_{p_1}(\Omega)}\right)^{1/p_0}, \tag{15}$$

where

$$M_i(\delta) = \left\|a_{i,\delta}^{1/p_0}\frac{w_0}{w_i}\right\|_{L_{r_i}(\Omega)}, \ i = 1, 2,$$

and r_i *are defined by* (14).

Here $0 < a_{1,\delta}(x) < 1$, $x \in \Omega$*, is uniquely defined by*

$$\left(a_{1,\delta}^{1/p_1}(x)\frac{w_0(x)}{w_1(x)}\right)^{r_1} = \delta\left((1 - a_{1,\delta}(x))^{1/p_2}\frac{w_0(x)}{w_2(x)}\right)^{r_2} \tag{16}$$

and $a_{2,\delta}(x) = 1 - a_{1,\delta}(x)$.

Inequality (15) *is sharp. Moreover equality holds if, and only if, for some nonnegative* λ,

$$|f| = \lambda \left(a_{1,\delta} \frac{w_0^{p_0}}{w_1^{p_1}} \right)^{\frac{1}{p_1 - p_0}} \tag{17}$$

almost everywhere on Ω.

Proof. Let the functions a_1, a_2 be positive and measurable on Ω and $a_1 + a_2 = 1$. Then

$$\int_\Omega |f|^{p_0} w_0^{p_0} dx = \int_\Omega a_1 \left(\frac{w_0}{w_1}\right)^{p_0} (|f| w_1)^{p_0} dx + \int_\Omega a_2 \left(\frac{w_0}{w_2}\right)^{p_0} (|f| w_2)^{p_0} dx.$$

For $p > 1$ we use as usual the notation $p' = \frac{p}{p-1}$. Since $p_i > p_0$ we have, by Hölder's inequality,

$$\begin{aligned} \int_\Omega |f|^{p_0} w_0^{p_0} dx \;&\leqslant\; \left\| a_1 \left(\frac{w_0}{w_1}\right)^{p_0} \right\|_{L_{\left(\frac{p_1}{p_0}\right)'}(\Omega)} \left\| (|f| w_1)^{p_0} \right\|_{L_{\frac{p_1}{p_0}}(\Omega)} \\ &\quad + \left\| a_2 \left(\frac{w_0}{w_2}\right)^{p_0} \right\|_{L_{\left(\frac{p_2}{p_0}\right)'}(\Omega)} \left\| (|f| w_2)^{p_0} \right\|_{L_{\frac{p_2}{p_0}}(\Omega)} \\ &= \left(\left\| a_1^{1/p_0} \frac{w_0}{w_1} \right\|_{L_{r_1}(\Omega)} \|f w_1\|_{L_{p_1}(\Omega)} \right)^{p_0} \\ &\quad + \left(\left\| a_2^{1/p_0} \frac{w_0}{w_2} \right\|_{L_{r_2}(\Omega)} \|f w_2\|_{L_{p_2}(\Omega)} \right)^{p_0}. \end{aligned} \tag{18}$$

Next we choose a_1 and a_2 in the optimal way keeping in mind that in (18) equality holds if and only if, for some λ_1, $\lambda_2 \geqslant 0$

$$(|f| w_1)^{p_1} = \lambda_1 \left(a_1 \left(\frac{w_0}{w_1}\right)^{p_0} \right)^{\left(\frac{p_1}{p_0}\right)'},$$

$$(|f| w_2)^{p_2} = \lambda_2 \left(a_2 \left(\frac{w_0}{w_2}\right)^{p_0} \right)^{\left(\frac{p_2}{p_0}\right)'}$$

almost everywhere on Ω, or

$$|f| = \lambda \left(a_1 \frac{w_0^{p_0}}{w_1^{p_1}} \right)^{\frac{1}{p_1 - p_0}}$$

where $\lambda = {\lambda_1}^{1/p_1}$ and

$$\left(a_1^{1/p_1}\frac{w_0}{w_1}\right)^{r_1} = \delta\left(a_2^{1/p_2}\frac{w_0}{w_2}\right)^{r_2}$$

where $\delta = \left({\lambda_2}^{1/p_2}{\lambda_1}^{-1/p_1}\right)^{p_0}$.

Hence the statement of the lemma follows.

Next we state a corollary of Lemma 1, which is useful for our later purposes.

LEMMA 2. *Let* Ω *be defined by* (4) *and* w_i *be functions defined on* Ω *which are positive, measurable and homogeneous of orders* α_i, $i = 1,2$. *Moreover, suppose that* $0 < p_0 < p_1, p_2 < \infty$ *and* $\varepsilon > 0$.

If f *is measurable on* Ω *and* $\|f w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1,2$, *then, for every* $\varepsilon > 0$,

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant \left(\left(\varepsilon^{(d_1-d_0)}A_1\,\|f w_1\|_{L_{p_1}(\Omega)}\right)^{p_0} + \left(\varepsilon^{(d_2-d_0)}A_2\,\|f w_2\|_{L_{p_2}(\Omega)}\right)^{p_0}\right)^{1/p_0}, \quad (19)$$

where

$$A_i = \left\|k_i^{1/p_0}\frac{w_0}{w_i}\right\|_{L_{r_i}(\Omega)}, \quad i = 1,2, \quad (20)$$

$k_1 \equiv k$, $k_2 \equiv 1-k$ *and* d_i, r_i *and* k *are defined in Theorem 2.*

Inequality (19) *is sharp. Moreover, equality holds if, and only if, for some* $\mu \geqslant 0$

$$|f(x)| = \mu f_*(\varepsilon x), \quad (21)$$

where f_* *is defined by* (12).

Proof: Replacing x by εx in (13) we get

$$\begin{aligned}\left(k^{1/p_1}(\varepsilon x)\frac{w_0(x)}{w_1(x)}\right)^{r_1} &= \varepsilon^{(\alpha_0-\alpha_2)r_2-(\alpha_0-\alpha_1)r_1}\left((1-k(\varepsilon x))^{1/p_2}\frac{w_0(x)}{w_2(x)}\right)^{r_2}\\ &= \varepsilon^{(d_0-d_2)r_2-(d_0-d_1)r_1}\left((1-k(\varepsilon x))^{1/p_2}\frac{w_0(x)}{w_2(x)}\right)^{r_2}.\end{aligned}$$

Consequently, if $\delta = \varepsilon^{(d_0-d_2)r_2-(d_0-d_1)r_1}$ and $a_{i,\delta}(x) = k_i(\varepsilon x)$, then the equality (16) is satisfied.

Hence, by Lemma 1, the inequality (15) is valid, where

$$\begin{aligned}M_i(\delta) &= \left\|k_i^{1/p_0}(\varepsilon x)\frac{w_0(x)}{w_i(x)}\right\|_{L_{r_i}(\Omega)}\\ &= \varepsilon^{-\frac{n}{r_i}}\left\|k_i^{1/p_0}(y)\frac{w_0(\frac{y}{\varepsilon})}{w_i(\frac{y}{\varepsilon})}\right\|_{L_{r_i}(\varepsilon\Omega)}\\ &= \varepsilon^{-\frac{n}{r_i}-\alpha_0+\alpha_i}\left\|k_i^{1/p_0}\frac{w_0}{w_i}\right\|_{L_{r_i}(\Omega)} = \varepsilon^{d_i-d_0}A_i\end{aligned}$$

and (19) follows.

Furthermore, by (17), equality in (19) holds if, and only if,

$$\begin{aligned} |f(x)| &= \lambda \left(k(\varepsilon x) \frac{w_0^{p_0}(x)}{w_1^{p_1}(x)} \right)^{\frac{1}{p_1-p_0}} \\ &= \lambda \varepsilon^{\frac{\alpha_1 p_1 - \alpha_0 p_0}{p_1 - p_0}} f_*(\varepsilon x) = \mu f_*(\varepsilon x). \end{aligned}$$

The proof is complete.

Proof of Theorem 2. Our proof is organized in six steps:

1. Let (7) and (8) be satisfied. According to (7) it yields that $d_1 - d_0 = (1-\theta)(d_1 - d_2)$ and $d_2 - d_0 = -\theta(d_1 - d_2)$. We set $\varepsilon^{(d_1-d_2)p_0} = \delta$ in Lemma 2. Since $d_1 \neq d_2$, by (19) we have, for all $\delta > 0$,

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant \left(\delta^{1-\theta} \left(A_1 \|f w_1\|_{L_{p_1}(\Omega)} \right)^{p_0} + \delta^{-\theta} \left(A_2 \|f w_2\|_{L_{p_2}(\Omega)} \right)^{p_0} \right)^{1/p_0}.$$

Since for $a,\, b > 0$

$$\min_{\delta>0}(a\delta^{1-\theta} + b\delta^{-\theta}) = (a\delta^{1-\theta} + b\delta^{-\theta}) \Big|_{\delta = \frac{\theta}{1-\theta}\frac{b}{a}} = (\theta^\theta (1-\theta)^{1-\theta})^{-1} a^\theta b^{1-\theta}$$

we obtain, by setting in (19) $\varepsilon = \varepsilon_0$, where

$$\varepsilon_0 = \varepsilon_0(f) = \left(\left(\frac{\theta}{1-\theta} \right)^{1/p_0} \frac{A_2 \|f w_2\|_{L_{p_2}(\Omega)}}{A_1 \|f w_1\|_{L_{p_1}(\Omega)}} \right)^{\frac{1}{d_1-d_2}}, \tag{22}$$

that inequality (6) is valid with $A = \tilde{A}$, where

$$\tilde{A} = \left(\theta^\theta (1-\theta)^{1-\theta} \right)^{-1/p_0} A_1^\theta A_2^{1-\theta}. \tag{23}$$

2. Let there be equality in (6) with $A = \tilde{A}$. Then, for $\varepsilon = \varepsilon_0(f)$ defined by (22) there is equality in (19). Hence, by (21), for some $\mu \geqslant 0$,

$$|f(x)| = \mu f_*\left(\varepsilon_0(f) x\right). \tag{24}$$

Thus an extremal function in (6) is necessarily of the form (11).

Let us verify that each function f defined by (11) is extremal. By Lemma 2 with $\varepsilon = 1$ and the elementary arithmetic-geometric mean inequality

$$x^\theta y^{1-\theta} \leqslant \theta^\theta (1-\theta)^{1-\theta}(x+y), \ x,\, y \geqslant 0,\, 0 < \theta < 1, \tag{25}$$

it follows that

$$\begin{aligned} \|f_* w_0\|_{L_{p_0}(\Omega)} &= \left(\left(A_1 \|f_* w_1\|_{L_{p_1}(\Omega)} \right)^{p_0} + \left(A_2 \|f_* w_2\|_{L_{p_2}(\Omega)} \right)^{p_0} \right)^{1/p_0} \\ &\geqslant \tilde{A} \|f_* w_1\|_{L_{p_1}(\Omega)}^\theta \|f_* w_2\|_{L_{p_2}(\Omega)}^{1-\theta}. \end{aligned} \tag{26}$$

Consequently, keeping in mind inequality (6) for $f = f_*$ with $A = \tilde{A}$, we see that there is equality in (26).

Thus $\tilde{A}$ is the minimal possible value of A in (6) and f_* is an extremal function, if $\|f_* w_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1,2$. This will be proved in step 4 of the proof. Moreover for every $B \geqslant 0$ and $\gamma > 0$ the function f defined by (11) is also an extremal function. This easily follows if we replace γx by y and apply the relation (7).

3. Next we note that

$$\|f_* w_i\|_{L_{p_i}(\Omega)} = \left\| k_i^{1/p_0} \frac{w_0}{w_i} \right\|_{L_{r_i}(\Omega)}^{\frac{r_i}{p_i}} = A_i^{\frac{r_i}{p_i}}, \ i = 1,2. \tag{27}$$

Since equality in (25) holds if, and only if, $\frac{x}{\theta} = \frac{y}{1-\theta} (= x+y)$ and there is equality in (26), we have, applying in addition (27), that

$$\frac{A_1^{r_1}}{\theta} = \frac{A_2^{r_2}}{1-\theta} = \|f_* w_0\|_{L_{p_0}(\Omega)}^{p_0}. \tag{28}$$

In particular, we note that due to (28) equation (22) is satisfied for the function f defined by (24). Moreover, we use (23) and (27) and find that

$$\begin{aligned} \tilde{A} &= \left(\theta^{\theta}(1-\theta)^{1-\theta}\right)^{-1/p_0} \theta^{\frac{\theta}{r_1}} (1-\theta)^{\frac{1-\theta}{r_2}} \|f_* w_0\|_{L_{p_0}(\Omega)}^{p_0\left(\frac{\theta}{r_1}+\frac{1-\theta}{r_2}\right)} \\ &= \theta^{-\frac{\theta}{p_1}} (1-\theta)^{-\frac{1-\theta}{p_2}} \left\| k^{\frac{1}{p_1}} \frac{w_0}{w_1} \right\|_{L_{r_1}(\Omega)}^{\frac{r_1}{q}}. \end{aligned} \tag{29}$$

4. Since the functions $w_i(x)$ are homogeneous of orders α_i we have $w_i(x) = \rho^{\alpha_i} w_i(\sigma)$ for $x \in \Omega$, where $\rho = |x|$ and $\sigma = \frac{x}{|x|} \in S$. Substituting this into (13) we get

$$k^{\frac{r_1}{p_1}}(x) = \left(\frac{w_1(\sigma)}{w_0(\sigma)}\right)^{r_1} \left(\frac{w_0(\sigma)}{w_2(\sigma)}\right)^{r_2} \rho^{(\alpha_0-\alpha_2)r_2-(\alpha_0-\alpha_1)r_1} (1-k(x))^{\frac{r_2}{p_2}}$$

or

$$k^{\frac{1}{p_1 r_2}}(x) = \left(\frac{w_1(\sigma)}{w_0(\sigma)}\right)^{\frac{1}{r_2}} \left(\frac{w_0(\sigma)}{w_2(\sigma)}\right)^{\frac{1}{r_1}} \rho^{(\frac{\theta}{r_1}+\frac{1-\theta}{r_2})(d_1-d_2)} (1-k(x))^{\frac{1}{p_2 r_1}}.$$

Thus

$$k^{\frac{1}{p_1 r_q}}(x) = (w(\sigma)\rho)^{\frac{d_1-d_2}{q}} (1-k(x))^{\frac{1}{p_2 r_1}},$$

where

$$w(\sigma) = \left(\left(\frac{w_1(\sigma)}{w_0(\sigma)}\right)^{\frac{1}{r_2}} \left(\frac{w_0(\sigma)}{w_2(\sigma)}\right)^{\frac{1}{r_1}}\right)^{\frac{q}{d_1-d_2}}.$$

Consequently

$$k(x) = \varphi\left(w(\sigma)\rho\right)$$

where the function φ of one variable is defined by

$$\varphi^{1/p_1 r_2}(z) = z^{\frac{d_1-d_2}{q}}\left(1-\varphi(z)\right)^{1/p_2 r_1}. \tag{30}$$

Furthermore,

$$\begin{aligned}
\|f_* w_0\|^{p_0}_{L_{p_0}(\Omega)} &= \int_\Omega \left(k^{1/p_1}(x)\frac{w_0(x)}{w_1(x)}\right)^{r_1} dx \\
&= \int_S \left(\int_0^\infty \varphi^{r_1/p_1}\left(w(\sigma)\rho\right)\left(\frac{w_0(\sigma)}{w_1(\sigma)}\right)^{r_1}\rho^{(\alpha_0-\alpha_1)r_1+n-1}d\rho\right) d\sigma \\
&= \int_S w(\sigma)^{-(d_0-d_1)r_1}\left(\frac{w_0(\sigma)}{w_1(\sigma)}\right)^{r_1} d\sigma \int_0^\infty \varphi^{r_1/p_1}(z)z^{(d_0-d_1)r_1-1}dz \\
&= \int_S \left(\frac{w_0}{w_1^\theta w_2^{1-\theta}}\right)^q d\sigma \int_0^\infty \varphi^{r_1/p_1}(z)z^{-(1-\theta)(d_1-d_2)r_1-1}dz. \qquad (31)
\end{aligned}$$

The latter integral can be evaluated in the following way: Substitute $\varphi(z) = y$ so that, by (30), $z = \left(y^{\frac{1}{p_1 r_2}}(1-y)^{-\frac{1}{r_1 p_2}}\right)^{\frac{q}{d_1-d_2}}$. Then

$$\begin{aligned}
&\int_0^\infty \varphi^{r_1/p_1}(z)z^{-(1-\theta)(d_1-d_2)r_1-1}dz \\
&\quad= \int_0^1 y^{\frac{r_1}{p_1}}\left(y^{\frac{1}{p_1 r_2}}(1-y)^{-\frac{1}{r_1 p_2}}\right)^{\frac{-[q(1-\theta)(d_1-d_2)r_1+1]}{|d_1-d_2|}} |z'(y)|dy \\
&\quad= \frac{1}{p_1 r_2}\frac{q}{|d_1-d_2|}B\left(\theta\frac{q}{p_1},(1-\theta)\frac{q}{p_2}+1\right) \\
&\qquad\quad + \frac{1}{r_1 p_2}\frac{q}{|d_1-d_2|}B\left(\theta\frac{q}{p_1}+1,(1-\theta)\frac{q}{p_2}\right).
\end{aligned}$$

We now only use the facts that $B(a,b+1) = \frac{a}{a+b}B(a,b)$ and $B(a,b) = B(b,a)$ and find that

$$\int_0^\infty \varphi^{r_1/p_1}(z)z^{-(1-\theta)(d_1-d_2)r_1-1}dz = \frac{B\left(\theta\frac{q}{p_1},(1-\theta)\frac{q}{p_2}\right)}{|d_1-d_2|(\theta p_2+(1-\theta p_1))}. \tag{32}$$

Hence $\|f_* w_i\|_{L_{p_i}(\Omega)} < \infty,\ i = 1,2$, and $\tilde{A} = A_*$ where A_* is defined by (10).

5. Let us prove that the first condition (7) and condition (8) are necessary. Suppose that inequality (6) is valid for all functions measurable on Ω satisfying $\|fw_i\|_{L_{p_i}(\Omega)} < \infty$, $i = 1,2$. Consider a fixed function f_0 which is not equivalent to zero, satisfying these conditions and apply (6) to $f_0\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$. Replacing $\frac{x}{\varepsilon}$ by y, we obtain

$$\left\|f_0\left(\frac{x}{\varepsilon}\right) w_i(x)\right\|_{L_{p_i}(\Omega)} = \varepsilon^{n/p_i}\|f_0 w_i(\varepsilon y)\|_{L_{p_i}(\frac{1}{\varepsilon}\Omega)} = \varepsilon^{d_i}\|f_0 w_i\|_{L_{p_i}(\Omega)}.$$

Hence $\varepsilon^{d_0-\theta d_1-(1-\theta)d_2} \leqslant A_1$ for all $\varepsilon > 0$, where A_1 is independent of ε, and it follows that $d_0 = \theta d_1 + (1-\theta)d_2$.

Next we set $w_{i,N}(x) = w_i(x)\chi_N\left(\frac{x}{|x|}\right)$, $i = 0,1,2,$ and $f_{*,N}(x) = f_*(x)\chi_N\left(\frac{x}{|x|}\right)$, where $N \in \mathbb{N}$ and χ_N is the characteristic function of the set $S_N \subset S$ of those $\sigma \in S$ for which $w_i(\sigma) \leqslant N$, $i = 0,1,2$. Then, according to step 4,

$$\begin{aligned} A &\geqslant \frac{\|f_{*,N}w_0\|_{L_{p_0}(\Omega)}}{\|f_{*,N}w_1\|^{\theta}_{L_{p_1}(\Omega)}\|f_{*,N}w_2\|^{1-\theta}_{L_{p_2}(\Omega)}} \\ &= \frac{\|f_{*,N}w_{0,N}\|_{L_{p_0}(\Omega)}}{\|f_{*,N}w_{1,N}\|^{\theta}_{L_{p_1}(\Omega)}\|f_{*,N}w_{2,N}\|^{1-\theta}_{L_{p_2}(\Omega)}} \\ &= C\left\|\frac{w_0}{w_1^{\theta}w_2^{1-\theta}}\right\|_{L_q(S_N)}, \end{aligned}$$

where C is independent of N. If $\left\|\frac{w_0}{w_1^{\theta}w_2^{1-\theta}}\right\|_{L_q(S_N)} = \infty$, then, by passing to the limit as $N \to \infty$, we see that for any $A > 0$ inequality (6) cannot be valid. We conclude that also (8) holds.

6. Finally, we prove that also the second condition $d_1 \neq d_2$ in (7) is necessary. Suppose that the first condition (7) and condition (8) are satisfied and $d_1 = d_2$. Hence by the first condition (7) it yields that $d_0 = d_1 = d_2$. Consider the family of functions

$$f_\varepsilon(x) = f_*(x)|x|^{\varepsilon}\chi(x),$$

where $\varepsilon > 0$, χ is the characteristic function of the unit ball in $\mathbb{R}^n$ and f_* is defined by (12). By step 4 it follows that the function k depends on $\sigma = \frac{x}{|x|}$ only, i.e., k is homogeneous of order 0. Consequently,

$$\|f_\varepsilon w_i\|_{L_{p_i}(\Omega)} = \left(\int_\Omega \left(k(x)\frac{w_0^{p_0}(x)}{w_1^{p_1}(x)}\right)^{\frac{p_i}{p_1-p_0}} w_i^{p_i}(x)|x|^{\varepsilon p_i}\chi(x)dx\right)^{\frac{1}{p_i}}$$

$$
\begin{aligned}
&= \left(\int_S \left(k(\sigma)\frac{w_0^{p_0}(\sigma)}{w_1^{p_1}(\sigma)}\right)^{\frac{p_i}{p_1-p_0}} w_i^{p_i}(\sigma)d\sigma\right)^{\frac{1}{p_i}} \times \\
&\qquad \times \left(\int_0^1 \rho^{(\frac{\alpha_0 p_0-\alpha_1 p_1}{p_1-p_0}+\alpha_i+\varepsilon)p_i+n-1} d\rho\right)^{\frac{1}{p_i}} \\
&= \|f_* w_i\|_{L_{p_i}(S)} \left(\int_0^1 \rho^{(-d_i+\alpha_i+\varepsilon)p_i+n-1} d\rho\right)^{\frac{1}{p_i}} \\
&= p_i^{-\frac{1}{p_i}} \|f_* w_i\|_{L_{p_i}(S)} \varepsilon^{-\frac{1}{p_i}} = C_i \varepsilon^{-\frac{1}{p_i}}.
\end{aligned}
$$

We note that $\|f_* w_i\|_{L_{p_i}(S)} < \infty$, $i = 0,1,2$. This follows from (8) and the inequality

$$
(f_* w_i)^{p_i} \leqslant \left(\frac{w_0}{w_1^{\theta} w_2^{1-\theta}}\right)^q, \ i = 0,1,2.
$$

In fact, if $\xi = \frac{\theta q}{r_1}$, then $1-\xi = \frac{(1-\theta)q}{r_2}$ and (13) implies that

$$
\begin{aligned}
(f_* w_0)^{p_0} &= \left(k^{\frac{1}{p_1}} \frac{w_0}{w_1}\right)^{r_1} \\
&= \left(k^{\frac{1}{p_1}} \frac{w_0}{w_1}\right)^{\xi r_1} \left((1-k)^{\frac{1}{p_2}} \frac{w_0}{w_2}\right)^{(1-\xi) r_2} \\
&\leqslant \frac{w_0^{\xi r_1+(1-\xi) r_2}}{w_1^{\xi r_1} w_2^{(1-\xi) r_2}} = \left(\frac{w_0}{w_1^{\theta} w_2^{1-\theta}}\right)^q.
\end{aligned}
$$

(Note that $0 < k < 1$.) Furthermore

$$
(f_* w_1)^{p_1} = \left(k^{\frac{1}{p_0}} \frac{w_0}{w_1}\right)^{r_1} = k\left(k^{\frac{1}{p_1}} \frac{w_0}{w_1}\right)^{r_1} \leqslant (f_* w_0)^{p_0}
$$

and similarly we find that $(f_* w_2)^{p_2} \leqslant (f_* w_0)^{p_0}$.

Now taking $f = f_\varepsilon$ in (6), we get

$$
A \geqslant \frac{\|f_\varepsilon w_0\|_{L_{p_0}(\Omega)}}{\|f_\varepsilon w_1\|^{\theta}_{L_{p_1}(\Omega)} \|f_\varepsilon w_2\|^{1-\theta}_{L_{p_q}(\Omega)}} = C_0 C_1^{-\theta} C_2^{-(1-\theta)} \varepsilon^{-\frac{1}{q}}.
$$

Passing to the limit as $\varepsilon \to 0^+$, we see that for any $A > 0$ inequality (6) cannot be valid, and we have got a contradiction so that we also have $d_1 \neq d_2$. The proof is complete.

4. Concluding remarks and results

Our proof of Theorem 2 shows that the multiplicative inequality (6) can in fact be replaced by some corresponding additive ones. This fact is described in the following remark:

REMARK 2. We note that the inequality

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant A_* \|f w_1\|^{\theta}_{L_{p_1}(\Omega)} \|f w_2\|^{1-\theta}_{L_{p_2}(\Omega)} \tag{33}$$

where A_* is defined by (10), the inequality

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant \left(\theta^{\theta}(1-\theta)^{1-\theta}\right)^{1/p_0} A_* \left(\|f w_1\|^{p_0}_{L_{p_1}(\Omega)} + \|f w_2\|^{p_0}_{L_{p_2}(\Omega)}\right)^{1/p_0} \tag{34}$$

and the inequality

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant \left(\left(\varepsilon^{d_1-d_0} B_{1,*} \|f w_1\|_{L_{p_1}(\Omega)}\right)^{p_0} + \left(\varepsilon^{d_2-d_0} B_{2,*} \|f w_2\|_{L_{p_2}(\Omega)}\right)^{p_0}\right)^{1/p_0} \tag{35}$$

for arbitrary $\varepsilon > 0$, where

$$B_{1,*} = \left(\theta^{\frac{\theta}{p_1'}}(1-\theta)^{-\frac{1-\theta}{p_2}} A_*^{q}\right)^{1/r_1},$$

$$B_{2,*} = \left(\theta^{\frac{\theta}{p_1}}(1-\theta)^{-\frac{1-\theta}{p_2'}} A_*^{q}\right)^{1/r_2}$$

are equivalent. Indeed, taking into consideration (29), (31) and (32), the implication (35) $\Longrightarrow$ (33) was established in the proof of Theorem 2. Moreover, by (25) it follows that (33) implies (34). Finally, applying (34) to $f\left(\frac{x}{\delta}\right)$, $\delta > 0$, using the relation

$$\left\| f\left(\frac{x}{\delta}\right) w_i(x) \right\|_{L_{p_i}(\Omega)} = \delta^{d_i} \|f w_i\|_{L_{p_i}(\Omega)}$$

and choosing appropriate δ one gets (35).

Moreover, similar argument shows that (33) is also equivalent to

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant \theta^{\theta}(1-\theta)^{1-\theta} A_* \left(\|f w_1\|_{L_{p_1}(\Omega)} + \|f w_2\|_{L_{p_2}(\Omega)}\right) \tag{36}$$

and to

$$\|f w_0\|_{L_{p_0}(\Omega)} \leqslant \left(\theta^{\theta}(1-\theta)^{1-\theta}\right)^{1/p_0'} \left(\varepsilon^{d_1-d_0} B_{1,*} \|f w_1\|_{L_{p_1}(\Omega)} + \varepsilon^{d_2-d_0} B_{2,*} \|f w_2\|_{L_{p_2}(\Omega)}\right). \tag{37}$$

Thus all the inequalities (33)–(37) are sharp.

COROLLARY 1. *Let Ω be defined by* (4), $\alpha_i \in \mathbb{R}$, *and let* (7) *and* (9) *be satisfied. Then, for each function f which is measurable on Ω and satisfying* $\|f(x)|x|^{\alpha_i}\|_{L_{p_i}(\Omega)} < \infty$, $i = 1,2$, *it yields that*

$$\|f(x)|x|^{\alpha_0}\|_{L_{p_0}(\Omega)} \leqslant C_1 \|f(x)|x|^{\alpha_1}\|^{\theta}_{L_{p_1}(\Omega)} \|f(x)|x|^{\alpha_2}\|^{1-\theta}_{L_{p_2}(\Omega)}, \tag{38}$$

where

$$C_1 = \theta^{-\frac{\theta}{p_1}}(1-\theta)^{-\frac{1-\theta}{p_2}} \left(\frac{B(\theta\frac{q}{p_1},(1-\theta)\frac{q}{p_2})}{|d_1-d_2|(\theta p_2 + (1-\theta)p_1)} \right)^{1/q} (\mathrm{meas}_{n-1} S)^{1/q}.$$

The constant C_1 is the best possible. Moreover, equality in (38) *holds if and only if, for some $B \geqslant 0$ and $\gamma > 0$,*

$$|f(x)| = B\left(\varphi(\gamma|x|)|x|^{\alpha_0 p_0 - \alpha_1 p_1}\right)^{\frac{1}{p_1 - p_0}},$$

where the function φ is defined by (30).

REMARK 3. Let Ff denote the Fourier transform of a function f on $\mathbb{R}^n$, i.e.,

$$Ff(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} f(x)dx.$$

We apply Corollary 1 with $p_0 = 1$, $p_1 = p_2 = 2$, $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 = l$, where l is an integer. Then $d_0 = n$, $d_1 = \frac{n}{2}$, $d_2 = l + \frac{n}{2}$, $\theta = 1 - \frac{n}{2l}$ and $q = 2$, and, also by using Parseval's relation, we find that

$$\begin{aligned}
\|f\|_{L_\infty(\mathbb{R}^n)} &= \left\|F^{-1}Ff\right\|_{L_\infty(\mathbb{R}^n)} \\
&= (2\pi)^{-n/2} \left\| \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}(Ff)(\xi)d\xi \right\|_{L_\infty(\mathbb{R}^n)} \\
&\leqslant (2\pi)^{-n/2} \|Ff\|_{L_1(\mathbb{R}^n)} \\
&\leqslant C\|Ff\|^{1-\frac{n}{2l}}_{L_2(\mathbb{R}^n)} \left\||\xi|^l (Ff)(\xi)\right\|^{\frac{n}{2l}}_{L_2(\mathbb{R}^n)} \\
&= C\|Ff\|^{1-\frac{n}{2l}}_{L_2(\mathbb{R}^n)} \left\||F(\nabla^l f)|\right\|^{\frac{n}{2l}}_{L_2(\mathbb{R}^n)} \\
&= C\|f\|^{1-\frac{n}{2l}}_{L_2(\mathbb{R}^n)} \left\||\nabla^l f|\right\|^{\frac{n}{2l}}_{L_2(\mathbb{R}^n)},
\end{aligned}$$

where

$$C = \left((2\pi)^{-n/2} \left(1 - \frac{n}{2l}\right)^{-\left(1-\frac{n}{2l}\right)} \left(\frac{n}{2l}\right)^{-\frac{n}{2l}} \frac{B\left(1-\frac{n}{2l}, \frac{n}{2l}\right)}{2l} \sigma_n \right)^{1/2}$$

and σ_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Here $\nabla^l f$ is the weak gradient of order l of the function f :

$$\nabla^l f = \left(\frac{\partial^l f}{\partial x_{i_1} \ldots \partial x_{i_l}} \right)^n_{i_1,\ldots,i_l=1}.$$

Moreover, since $\sigma_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ and $B(1-\frac{n}{2l},\frac{n}{2l}) = \Gamma(1-\frac{n}{2l})\Gamma(\frac{n}{2l}) = \frac{\pi}{\sin\frac{n\pi}{2l}}$ we have

$$C = \left(\frac{2^{-n}\pi^{1-n/2}}{l\Gamma(\frac{n}{2})\sin\frac{n\pi}{2l}} \left(1-\frac{n}{2l}\right)^{-\left(1-\frac{n}{2l}\right)} \left(\frac{n}{2l}\right)^{-\frac{n}{2l}} \right)^{1/2}. \tag{39}$$

Thus Corollary 1 implies the following precise version of Sobolev's inequality: If $l > n/2$, then

$$\|f\|_{L_\infty(\mathbb{R}^n)} \leqslant C\, \|f\|^{1-n/2l}_{L_2(\mathbb{R}^n)}\, \|\,|\nabla^l f|\,\|^{n/2l}_{L_2(\mathbb{R}^n)}, \tag{40}$$

where C is defined by (39).

Moreover, the inequality (40) is sharp and equality occurs if and only if $f = Ff(x)$, where

$$Ff(\xi) = B\frac{1}{b+|\xi|^{2n}},\ \ \xi \in \mathbb{R}^n,$$

for some $B \in \mathbf{C}$ and $b > 0$.

In particular, for the case $n = l$ (40) has the form

$$\|f\|_{L_\infty(\mathbb{R}^n)} \leqslant \left(\frac{2^{-n}\pi^{1-n/2}}{\Gamma(\frac{n}{2}+1)} \right)^{\frac{1}{2}} \|f\|^{1/2}_{L_2(\mathbb{R}^n)}\, \|\nabla^n f\|^{1/2}_{L_2(\mathbb{R}^n)}.$$

REMARK 4. Inequality (6) is a multidimensional generalization of both of the inequalities (2) and (3): In fact, let $p_0 = 1$ and $\alpha_0 = 0$. If $n = 1$ and $\Omega = [0,\infty)$, then $\mathrm{meas}_0 S = 1$ and we get (2). If $n = 1$ and $\Omega = (-\infty,\infty)$, then $\mathrm{meas}_0 S = 2$ and we get (3).

REMARK 5. If in Theorem 2, $p_1 = p_2 = p$, then the optimal constant A_* has the following forms:

$$\begin{aligned} A_* &= \left(\theta^{-\theta}(1-\theta)^{-(1-\theta)}\right)^{1/p} \left(\frac{B(\theta\frac{r}{p},(1-\theta)\frac{r}{p})}{|d_1-d_2|p} \right)^{1/r} \left\| \frac{w_0}{w_1^\theta w_2^{1-\theta}} \right\|_{L_r(S)} \\ &= \left(\theta^{-\theta}(1-\theta)^{-(1-\theta)}\right)^{1/p} \left\| \frac{w_0}{(w_1^p+w_2^p)^{1/p}} \right\|_{L_r(\Omega)}, \end{aligned}$$

where

$$\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p}.$$

In fact, the first equality is just (10) for $p_1 = p_2 = p$, while the second one follows from (29) since in this case (see (13))

$$k = \frac{w_1^p}{w_1^p + w_2^p}.$$

REFERENCES

[1] BEURLING, A: *Sur les intégrales de Fourier absolutement convergentes et leur application á une transformation fonctionelle*, Proceedings from the ninth Nordic Congress of mathematicians, Helsinki, 1938.
[2] CARLSON, F.: *Une inégalité*, Ark. Mat. Astr. Fysik **25B** (1934), 1–5.
[3] LEVIN, V. I.: *Exact constants in inequalities of Carlson type*, Dokl. Akad. Nauk. SSSR **59** (1948), 635-638 (in Russian).
[4] MITRINOVIĆ, D. S.: *Analytic Inequalities*, Springer Verlag, Berlin-Heidelberg-New York, 1979.
[5] MITRINOVIĆ, D. S., PEČARIĆ, J. E. and FINK, A. M.: *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht-Boston-London, 1993.

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