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AN UPPER BOUND FOR THE ZEROS OF THE CYLINDER FUNCTION $C_\nu(x)$

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Abstract. For large values of ν $(\nu > 0)$ the k-th positive zero of the cylinder function $C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha$, $0 \leqslant \alpha < \pi$, has the asymptotic expansion

$$j_{\nu\kappa} = \nu + \gamma_\kappa \nu^{1/3} + \frac{3}{10}\gamma_\kappa^2 \nu^{-1/3} + \mathcal{O}(\nu^{-1})$$

where $\kappa = k - \alpha/\pi$, $\gamma_\kappa = -a_\kappa 2^{-1/3}$ and a_κ is the k-th negative zero of the function $Ai(x)\cos\alpha + Bi(x)\sin\alpha$ and $Ai(x)$, $Bi(x)$ denote the Airy functions of the first and the second kind, respectively [1]. We prove that the sum of the first three terms of the asymptotic expansion gives an upper bound for $j_{\nu\kappa}$, provided $\gamma_\kappa \geqslant \sqrt[3]{35/4} = 2.0606427\ldots$ or $\kappa \geqslant \kappa_0 = 1.13019788\cdots = 2 - \alpha_0/\pi$ where α_0 is determined by the equation $\cos\alpha_0 Ai\left(-\sqrt[3]{35/4}\right) + \sin\alpha_0 Bi\left(-\sqrt[3]{35/4}\right) = 0$. This result covers the cases $j_{\nu 2}, j_{\nu 3}, \ldots$ and $y_{\nu 2}, y_{\nu 3}, \ldots$, for all $\nu > 0$. The main tool used is the well-known Watson formula for $dj_{\nu\kappa}/d\nu$.

1. Introduction and preliminary results

Let $j_{\nu\kappa}$ denote the k-th positive zero of the cylinder function $C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha$, $0 \leqslant \alpha < \pi$ where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and the second kind, respectively, and $\kappa = k - \alpha/\pi$, (see [4]). In [3] we proved that $j_{\nu\kappa}$ has the asymptotic expansion

$$j_{\nu\kappa} = \nu + \gamma_\kappa \nu^{1/3} + \frac{3}{10}\gamma_\kappa^2 \nu^{-1/3} + \frac{5 - \gamma_\kappa^3}{350}\nu^{-1} - \frac{479\gamma_\kappa^4 + 20\gamma_\kappa}{63000}\nu^{-5/3} + \frac{20231\gamma_\kappa^5 - 27550\gamma_\kappa^2}{8085000}\nu^{-7/3} + \mathcal{O}(\nu^{-3}) \quad (1.1)$$

where $\gamma_\kappa = -a_\kappa \cdot 2^{-1/3}$ and a_κ is the k-th negative zero of $Ai(x)\cos\alpha + Bi(x)\sin\alpha$ and $Ai(x)$, $Bi(x)$ denote the Airy functions of the first and the second kind, respectively. Recently, L. Lorch and R. Uberti [6] have proved that the sum of the first three terms in

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(1.1) for $\kappa = 1,2,3$, gives an upper bound for $j_{\nu 1}, j_{\nu 2}, j_{\nu 3}$ at least for $0 < \nu \leqslant 10$. In [5] T. Lang and R. Wong dealt with the case $\nu > 10$ for $\kappa = 1,2$.

In this paper we prove the following general result.

THEOREM. *Let $j_{\nu\kappa}$ be the k-th positive zero of the cylinder function $C_\nu(x)$ and let γ_κ be defined as above. Then for $\gamma_\kappa \geqslant \sqrt[3]{35/4} = 2.0606427\ldots$ or $\kappa \geqslant \kappa_0 = 1.13019788\ldots$ the following upper bound*

$$j_{\nu\kappa} < \nu + \gamma_\kappa \nu^{1/3} + \frac{3}{10}\gamma_\kappa^2 \nu^{-1/3}, \quad \nu > 0 \tag{1.2}$$

holds where $\kappa_0 = 2 - \alpha_0/\pi$, and $\cos\alpha_0 \, Ai\big(-\sqrt[3]{35/4}\big) + \sin\alpha_0 \, Bi\big(-\sqrt[3]{35/4}\big) = 0$.

REMARK. According to the asymptotic formula (1.1) we can not expect an extension of inequality (1.2) to all $\kappa > 0$ because the coefficient of ν^{-1} in (1.1) is positive for $\gamma_\kappa < \sqrt[3]{5} = 1.709976\ldots$. Thus, for example, we can not extend (1.2) to the first zero $y_{\nu 1}$ of $Y_\nu(x)$ for every $\nu > 0$ because in this case $\kappa = 1/2$ and $\gamma_{1/2} = 0.9315768\ldots$, [1, p. 371; 9.5.15].

The proof of our Theorem is based on the well-known Watson formula [7, p. 508]

$$\frac{dj}{d\nu} = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}\,dt \tag{1.3}$$

where $j = j_{\nu\kappa}$ and $K_0(x)$ denotes the modified Bessel function of the third kind.

The upper bound (1.2) can be thought of as complementing the lower bound

$$j_{\nu\kappa} > \nu + \gamma_\kappa\left(\nu + \frac{A_\kappa^3}{\gamma_\kappa^3}\right)^{1/3}, \quad \nu \geqslant 0, \quad \kappa \geqslant 1/2, \tag{1.4}$$

where $A_\kappa = 2\gamma_\kappa\sqrt{2\gamma_\kappa}/3$, (see Remark in [2, p. 185]).

As in [3] we introduce the functions

$$F_i(\vartheta) = \int_0^\infty K_0(u)u^i e^{-u\cos\vartheta}du, \quad i = 0,1,2,\ldots \tag{1.5}$$

where

$$F_0(\vartheta) = \int_0^\infty K_0(u)e^{-u\cos\vartheta}du = \frac{\vartheta}{\sin\vartheta} \tag{1.6}$$

and

$$F_{i+1}(\vartheta) = \frac{\dfrac{d}{d\vartheta}F_i(\vartheta)}{\sin\vartheta}, \quad i = 0,1,2,\ldots. \tag{1.7}$$

In particular we have

$$F_2(\vartheta) = \frac{\vartheta + 2\vartheta\cos^2\vartheta - 3\sin\vartheta\cos\vartheta}{\sin^5\vartheta} = \frac{4}{15} + \frac{6}{35}\vartheta^2 + \ldots. \tag{1.8}$$

Clearly, these functions are analytic in $|\vartheta| < \pi$.

2. Proof of the Theorem

We introduce the notations

$$X = X(\nu) = \gamma\nu^{-2/3} > 0, \quad \gamma = \gamma_\kappa. \tag{2.1}$$

Then (1.2) is equivalent to

$$j < \nu\Big[1 + X + \frac{3}{10}X^2\Big], \quad \nu > 0, \quad j = j_{\nu\kappa}. \tag{2.2}$$

Let $\varphi(\nu)$ and $\psi(\nu) \in \big(0, \frac{\pi}{2}\big)$ be defined by

$$\cos\varphi(\nu) = \frac{\nu}{j}, \quad \cos\psi(\nu) = \frac{1}{1 + X + \frac{3}{10}X^2} \quad \text{for } \nu > 0. \tag{2.3}$$

Concerning the function $\varphi(\nu)$ we have also the restriction $0 < \varphi(\nu) < \pi/2$ because in our case $j_{\nu\kappa} > j_{\nu 1} > \nu > 0$, [7, p. 485]. So by these notations, the inequality in (2.2) is equivalent to

$$0 < \varphi(\nu) < \psi(\nu) < \frac{\pi}{2}. \tag{2.4}$$

By the asymptotic relation (1.1) the inequalities in (2.4) hold at least for sufficiently large ν, because the coefficient of ν^{-1} in (1.1) is negative. Hence the difference $\psi(\nu) - \varphi(\nu)$ is surely positive for sufficiently large ν. We have to prove that this difference remains positive for all $\nu > 0$. Suppose the contrary and define

$$\nu^* = \inf\big\{\overline{\nu} > 0,\ \psi(\nu) - \varphi(\nu) > 0,\ \nu > \overline{\nu}\big\}.$$

Then we have the (indirect) relation

$$\psi(\nu^*) - \varphi(\nu^*) = 0,\ \psi'(\nu^*) - \varphi'(\nu^*) \geqslant 0 \tag{2.5}$$

where $'$ indicates the derivative with respect to ν. We are going to show that (2.5) leads to contradiction. By (2.1), (2.3) we have

$$\nu\frac{d}{d\nu}\frac{1}{\cos\psi(\nu)} = \nu\Big(1 + \frac{3}{5}X\Big)\frac{dX}{d\nu} = -\frac{2}{3}X - \frac{2}{5}X^2. \tag{2.6}$$

On the other hand, by (1.3) and (2.3)

$$\nu\frac{d}{d\nu}\frac{1}{\cos\varphi(\nu)} = \nu\Big(\frac{j}{\nu}\Big)' = \frac{j'\nu - j}{\nu} = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt - \frac{1}{\cos\varphi(\nu)}. \tag{2.7}$$

If we prove the inequality $\varphi'(\nu) > \psi'(\nu)$ under the restriction $\psi(\nu) = \varphi(\nu)$, then clearly we shall have contradiction with (2.5). Due to (2.6) and (2.7) the inequality $\varphi'(\nu) > \psi'(\nu)$ can be written as

$$j' = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt > 1 + \frac{1}{3}X - \frac{1}{10}X^2 \tag{2.8}$$

under the restriction $j = \nu\Big(1 + X + \dfrac{3}{10}X^2\Big)$. By the substitution $u = 2j\sinh t$ in (2.8), we have

$$j' = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt = \int_0^\infty K_0(u)e^{-2\nu\sinh^{-1}\frac{u}{2j}}\frac{du}{\sqrt{1+\dfrac{u^2}{4j^2}}}$$

$$> \int_0^\infty K_0(u)e^{-\frac{\nu}{j}u}\frac{du}{\sqrt{1+\dfrac{u^2}{4j^2}}}.$$

Since $(1+z)^{-1/2} > 1 - z/2$, $z > 0$ and $\nu/j = \cos\varphi$, therefore

$$j' > \int_0^\infty K_0(u)e^{-u\cos\varphi}\Big(1 - \frac{u^2}{8j^2}\Big)du = \int_0^\infty K_0(u)e^{-u\cos\varphi}du$$
$$- \frac{1}{8j^2}\int_0^\infty K_0(u)u^2e^{-u\cos\varphi}du = F_0(\varphi) - \frac{1}{8j^2}F_2(\varphi),$$

where the functions F_0 and F_2 have been defined in (1.6), (1.8), respectively. By (2.1) and (2.3) we find

$$\frac{1}{j^2} = \frac{\nu^2}{j^2}\,\frac{1}{\nu^2} = \cos^2\varphi\cdot\frac{X^3}{\gamma_\kappa^3}$$

hence instead of (2.8) it is sufficient to prove the inequality

$$F_0(\varphi) > 1 + \frac{1}{3}X - \frac{1}{10}X^2 + \frac{1}{8\gamma^3}X^3\cos^2\varphi\cdot F_2(\varphi) \tag{2.9}$$

under the condition that

$$\cos\varphi(\nu) = \frac{1}{1 + X + \dfrac{3}{10}X^2}. \tag{2.10}$$

We shall prove (2.9) in two steps:

$$F_2(\varphi)(\cos\varphi)^{4/3} \leqslant F_2(0) = \frac{4}{15}; \tag{2.11a}$$

$$F_0(\varphi) > 1 + \frac{1}{3}X - \frac{1}{10}X^2 + \frac{2}{525}X^3(\cos\varphi)^{2/3}. \tag{2.11b}$$

Clearly (2.11a) and (2.11b) imply inequality (2.8) for $\gamma^3 \geqslant 35/4$.

For the proof of (2.11a) we use (1.8), and we have to prove that

$$G(\varphi) = \frac{4}{15}\cdot\frac{\sin^5\varphi}{1+2\cos^2\varphi}\cdot\cos^{-4/3}\varphi - \varphi + \frac{3\sin\varphi\cos\varphi}{1+2\cos^2\varphi} > 0, \quad 0 < \varphi < \frac{\pi}{2}.$$

We get $G(0) = 0$, so it is sufficient to show that

$$G'(\varphi) = \frac{4\sin^2\varphi\,(4 + 31\cos^2\varphi + 10\cos^4\varphi - 45\cos^{7/3}\varphi)}{45\cos^{7/3}\varphi\,(1 + 2\cos^2\varphi)^2} > 0.$$

With the substitution $\cos\varphi = z^3$ the expression in parenthesis of the numerator becomes

$$(1-z)(4+4z+4z^2+4z^3+4z^4+4z^5+35z^6-10z^7-10z^8-10z^9-10z^{10}-10z^{11}).$$

Since $0 < z < 1$, then for $i < j$, $z^i > z^j$ and the new expression in parenthesis is larger than

$$4z^6+4z^6+4z^6+4z^6+4z^6+4z^6+35z^6-10z^6 \\ -10z^6-10z^6-10z^6-10z^6 = (59-50)z^6 > 0$$

which proves (2.11a).

For the proof of (2.11b) we need the inequality

$$F_0(\varphi) = \frac{\varphi}{\sin\varphi} > A + B\cos\varphi + C\cos^2\varphi + D\cos^3\varphi + E\cos^4\varphi,\ 0 < \varphi < \frac{\pi}{2} \tag{2.12}$$

where

$$A = \frac{488}{315},\ B = -\frac{55}{63},\ C = \frac{16}{35},\ D = -\frac{10}{63},\ E = \frac{8}{315}.$$

To this end let us consider the function

$$\Phi(\varphi) = \varphi - \sin\varphi[A + B\cos\varphi + C\cos^2\varphi + D\cos^3\varphi + E\cos^4\varphi].$$

Since $\Phi(0) = 0$ and $\frac{d}{d\varphi}\Phi(\varphi) = \frac{8}{63}(1-\cos\varphi)^5 > 0$, this proves (2.12).

Now we define

$$H = 1 + X + \frac{3}{10}X^2 = \frac{1}{\cos\varphi}$$

so we have

$$X^2 = \frac{10}{3}H - \frac{10}{3} - \frac{10}{3}X$$

and

$$X^3 = \frac{10}{3}HX - \frac{100}{9}H + \frac{70}{9}X + \frac{100}{9}.$$

Hence the right-hand side of (2.11b) can be written as

$$\frac{4}{3} - \frac{1}{3}H + X\left[\frac{2}{3} + \frac{4}{315}H^{1/3} + \frac{4}{135}H^{-2/3}\right] + \frac{8}{189}H^{-2/3} - \frac{8}{189}H^{1/3}$$

where the coefficient of X is positive. Therefore we look for an upper bound of X in terms of H, as follows:

$$aH^{2/3} + bH^{1/3} + cH^{-1/3} + d > X, \tag{2.13}$$

where $a = \frac{19}{25}$, $b = \frac{51}{50}$, $c = -\frac{23}{50}$, $d = -\frac{33}{25}$. To show (2.13) we consider the function

$$\Psi(X) = aH^{2/3} + bH^{1/3} + cH^{-1/3} + d - X \tag{2.14}$$

where we recall that $H = H(X) = 1 + X + \frac{3}{10}X^2$. The coefficients a, b, c, d above have been chosen so that

$$\Psi(X) = \frac{79}{16200}X^4 + \mathcal{O}(X^5), \quad X \to 0.$$

Therefore $\Psi(X) > 0$ at least for small values of X.

Suppose now that the inequality $\Psi(X) > 0$ does not hold for all $X > 0$. Then let $X_1 > 0$ be the first positive zero of $\Psi(X)$: $\Psi(X_1) = 0$. Then clearly we have $\Psi'(X_1) = \frac{d}{dX}\Psi(X_1) \leqslant 0$. By direct calculations we get $\frac{d}{dX}H(X) = 1 + \frac{3}{5}X$, hence

$$\begin{aligned}\Psi'(X_1) &= \Big(\frac{2}{3}aH^{-1/3} + \frac{1}{3}bH^{-2/3} - \frac{1}{3}cH^{-4/3}\Big)\Big(1 + \frac{3}{5}X_1\Big) - 1 \\ &= \Big(\frac{2}{3}aH^{-1/3} + \frac{1}{3}bH^{-2/3} - \frac{1}{3}cH^{-4/3}\Big)\Big[1 + \frac{3}{5}(aH^{2/3} + bH^{1/3} + cH^{-1/3} + d)\Big] - 1\end{aligned}$$

where the last equality is a consequence of (2.14) taken at $X = X_1$.

Using the notation $z^3 = H$, we get

$$\Psi'(X_1) = \frac{(z-1)^3}{15z^5} \cdot \frac{8664z^3 + 5934z^2 + 3565z + 1587}{2500}$$

which is clearly positive because $z > 1$. This contradicts the existence of a zero of $\Psi(X)$, thus inequality (2.13) is proved.

To complete the proof of (2.11b), we have still to show the inequality

$$\begin{aligned}L = A + \frac{B}{H} + \frac{C}{H^2} + \frac{D}{H^3} + \frac{E}{H^4} > \frac{4}{3} - \frac{1}{3}H + (aH^{2/3} + bH^{1/3} + cH^{-1/3} + d)\times \\ \times\Big(\frac{2}{3} + \frac{4}{315}H^{1/3} + \frac{28}{945}H^{-2/3}\Big) + \frac{8}{189} - \frac{8}{189}H^{1/3} = M.\end{aligned}$$

Using again the substitution $z^3 = H$, we get:

$$\begin{aligned}L - M = \frac{(z-1)^4}{23625}[600 + 200z + 6000z^2 + 8250z^3 + 6000z^4 - 3900z^5 - 13800z^6 \\ - 16050z^7 - 3000z^8 + 12697z^9 + 18312z^{10} + 7647z^{11}].\end{aligned}$$

Since $z > 1$, $z^i > z^j$ if $i > j$, hence the polynomial in the brackets has the lower bound

$$[-3900 - 13800 - 16050 - 3000 + 12697 + 18312 + 7647]z^8 = 1906z^8 > 0.$$

This completes the proof of inequality (2.11b) and of our Theorem as well.

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