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SOME HILBERT SPACE CHARACTERIZATIONS AND BANACH SPACE INEQUALITIES

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Abstract. It is well known that if X is a normed linear space with dimension not less than three such that the radial projection from X onto the closed unit ball is nonexpansive, then X must be an inner product space. Using this fact, we are able to give a characterization of Hilbert spaces. Two other Hilbert space characterizations and some Banach space inequalities are established via duality maps.

Let X be a real normed linear space. We denote by B the closed unit ball of X and by P the radial projection from X onto B, i.e.,

$$Px = \begin{cases} x & \text{if } x \in B, \\ \dfrac{x}{\|x\|} & \text{if } x \notin B. \end{cases} \tag{1}$$

The following characterization of inner product spaces will be used in order to prove our first result.

LEMMA [4]. *A normed linear space* $(X, \|\cdot\|)$ *with dimension not less than three is an inner product space if and only if the radial projection* $P : X \to B$ *given by* (1) *is nonexpansive, i.e.,*

$$\|Px - Py\| \leqslant \|x - y\|, \quad \textit{for all } x, y \in X.$$

In [7], Kim and Shin proved the following result.

THEOREM. *Let* X *be a uniformly convex Banach space satisfying Condition A (see definition below),* C *a nonempty closed convex subset of* X *and* $f : C \to X$ *a nonexpansive mapping (i.e.,* $\|f(x) - f(x)\| \leqslant \|x - y\|$, $x, y \in C$*) such that* $f(C)$ *is bounded and* $f(\partial C) \subseteq C$*, where* ∂C *is the boundary of* C*. For each* $k \in [0, 1)$*, let* $x_k \in C$ *be the unique fixed point of the contraction* $T_k : C \to X$ *defined by*

$$T_k x = kTx + (1 - k)y, \quad x \in C,$$

where y *is a fixed point in* C*. Then the strong* $\lim_{k\uparrow 1} x_k$ *exists and is a fixed point of* T*.*

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DEFINITION [7]. A normed linear space $(X, \|\cdot\|)$ is said to satisfy Condition A if given $r > 0$, $y \in S_r := \{x \in X : \|x\| = r\}$ and $w \notin B_r := \{x \in X : \|x\| \leqslant r\}$ we have

$$\|y - w\| < \|y - \lambda w\| \quad \text{for all } \lambda > 1. \tag{2}$$

Clearly Condition (A) is equivalent to the validity of the inequality (2) for all $y, w \in X$ such that $\|y\| = 1$ and $\|w\| > 1$. In this note we shall prove that Condition A in fact characterizes inner product spaces with dimension not less than three; hence the above Theorem is reduced precisely to the Theorem of Singh and Watson [10] (see [1], [6], [12] for more and latest results about the strong convergence of $\{x_k\}$). Two more characterizations of Hibert spaces and certain inequalities in Banach spaces will also be obtained via the duality map (see below for definition).

PROPOSITION 1. *A normed linear space $(X, \|\cdot\|)$ satisfies Condition A if and only if $(X, \|\cdot\|)$ is an inner product space.*

Proof. It is easily seen that an inner product space satisfies Condition A. To show the converse assertion, by the Lemma above, it suffices to show that the radial projection P given by (1) is nonexpansive. Towards this end, we distinguish two cases.

Case 1: $\|x\| < 1$, $\|y\| > 1$. It follows from Condition A (with $r := \|x\|$) that

$$\begin{aligned}\|Px - Py\| &= \left\|x - \frac{y}{\|y\|}\right\| < \left\|x - \frac{\lambda}{\|y\|}y\right\| \qquad (\forall \lambda > 1)\\ &= \|x - y\| \qquad (\text{let } \lambda = \|y\|).\end{aligned}$$

Case 2: $\|x\| \geqslant 1$, $\|y\| \geqslant 1$. Without loss of generality we assume $\|y\| \geqslant \|x\|$. For any $\varepsilon > 0$ and $\lambda > 1$, Condition A yields that

$$\begin{aligned}\|Px - (1+\varepsilon)Py\| &= \left\|\frac{x}{\|x\|} - (1+\varepsilon)\frac{y}{\|y\|}\right\|\\ &< \left\|\frac{x}{\|x\|} - \lambda(1+\varepsilon)\frac{y}{\|y\|}\right\|\\ &= \frac{1}{\|x\|}\left\|x - \lambda(1+\varepsilon)\frac{\|x\|}{\|y\|}y\right\|\\ &\leqslant \left\|x - \lambda(1+\varepsilon)\frac{\|x\|}{\|y\|}y\right\| \quad \text{as } \|x\| \geqslant 1.\end{aligned}$$

Taking $\lambda = (1+\varepsilon)\|y\|/\|x\|$ and then letting $\varepsilon \to 0$ we get $\|Px - Py\| \leqslant \|x - y\|$. Therefore P is nonexpansive. □

REMARK. The above proof shows that the conclusion of Proposition 1 remains valid if the strict "$<$" in (2) is replaced by the nonstrict "$\leqslant$", i.e., $\|y - x\| \leqslant \|y - \lambda w\|$ for all $y \in B$, $w \notin B$, and $\lambda > 1$.

Recall that the (normalized) *duality* map $J : X \to X^*$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

It is known (cf. [5]) that X is *smooth* if and only if J is single-valued on X.

In [8] and [9], Park studied the convergence of the Mann iteration for accretive operators by imposing the following assumption on the underlying smooth Banach space X:

$$\langle x-y, J(x)-J(y)\rangle \leqslant \|x-y\|^2 \quad \text{for all } x,y\in X. \tag{3}$$

Our next result shows that the assumption (3) also characterizes Hilbert spaces; thus reducing the main results of [8] and [9] to a Hilbert space setting which have been proved already (cf. Bruck [2]). We shall work in a more general setting; i.e., we do not assume smoothness, instead we use the following assumption:

$$\langle x-y, j_x-j_y\rangle \leqslant \|x-y\|^2 \quad \forall j_x\in J(x),\ \forall j_y\in J(y),\ x,y\in X. \tag{4}$$

It is obvious that if X is smooth, then (4) is reduced to (3).

PROPOSITION 2. *Assume that a Banach space $(X,\|\cdot\|)$ satisfies the condition (4). Then X is a Hilbert space.*

Proof. Let $x,y\in X$ and $t\geqslant 0$ and let $j_{x+ty}\in J(x+ty)$ satisfy

$$\sup\{\langle y,j\rangle : j\in J(x+ty)\} = \langle y, j_{x+ty}\rangle .$$

Then we have for any $j_x\in J(x)$,

$$\begin{aligned}
\frac{d^+}{dt}\frac{1}{2}\|x+ty\|^2 &:= \lim_{s\downarrow 0}\frac{\frac{1}{2}\|x+ty+sy\|^2-\frac{1}{2}\|x+ty\|^2}{s}\\
&= \sup\{\langle y,j\rangle : j\in J(x+ty)\}\\
&= \langle y, j_{x+ty}\rangle\\
&= \frac{1}{t}\langle (x+ty)-x, j_{x+ty}-j_x\rangle + \langle y,j_x\rangle\\
&\leqslant t\|y\|^2+\langle y,j_x\rangle \quad \text{by (4).}
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{2}\|x+y\|^2 &\leqslant \frac{1}{2}\|x\|^2+\int_0^1 t\|y\|^2dt+\langle y,j_x\rangle\\
&= \frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2+\langle y,j_x\rangle,
\end{aligned}$$

or

$$\|x+y\|^2\leqslant \|x\|^2+\|y\|^2+2\langle y,j_x\rangle, \quad x,y\in X. \tag{5}$$

Substitute $-y$ for y into (5) to get

$$\|x-y\|^2\leqslant \|x\|^2+\|y\|^2-2\langle y,j_x\rangle, \quad x,y\in X. \tag{6}$$

Adding (5) and (6) gets

$$\|x+y\|^2+\|x-y\|^2\leqslant 2(\|x\|^2+\|y\|^2), \quad x,y\in X. \tag{7}$$

Replacing x by $\frac{x+y}{2}$ and y by $\frac{x-y}{2}$ in (7) respectively, we obtain

$$\|x+y\|^2+\|x-y\|^2\geqslant 2(\|x\|^2+\|y\|^2), \quad x,y\in X. \tag{8}$$

By (7) and (8) we conclude that the Parallelogram identity

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2) \tag{9}$$

holds for all $x,y\in X$; hence X is an inner product space. □

COROLLARY 1. *A Banach space X is a Hilbert space if and only if the duality map J is Lipschitz with constant one in the sense that*

$$\|j_x-j_y\|\leqslant\|x-y\|,\quad \forall j_x\in J(x),\ \forall j_y\in J(y),\ x,y\in X.$$

Next we consider the dual version to Proposition 2; namely, we consider the case where there exists a selection j of J (i.e., a single-valued map $j:X\to X^*$ such that $j(x)\in J(x)$ for all $x\in X$) satisfying

$$\langle x-y,j(x)-j(y)\rangle\geqslant\|x-y\|^2,\quad x,y\in X. \tag{10}$$

We shall show the inequality (10) also characterizes inner product spaces.

PROPOSITION 3. *If a Banach space X satisfies the inequality* (10)*, then X is a Hilbert space.*

Proof. Since for every $x\in X$, $J(x)=\partial\frac{1}{2}\|x\|^2$, the subdifferential of the function $\frac{1}{2}\|\cdot\|^2$ at x, by the subdifferential inequality, we have for $x,y\in X$ and $t\geqslant 0$,

$$\begin{aligned}\frac{d^+}{dt}\frac{1}{2}\|x+ty\|^2 &:= \lim_{s\to t^+}\frac{\frac{1}{2}\|x+sy\|^2-\frac{1}{2}\|x+ty\|^2}{s-t}\\ &\geqslant\langle y,j(x+ty)\rangle.\end{aligned}$$

This combined with the inequality (10) yields

$$\begin{aligned}\frac{d^+}{dt}\frac{1}{2}\|x+ty\|^2 &\geqslant\frac{1}{t}\langle (x+ty)-x,j(x+ty)-j(x)\rangle+\langle y,j(x)\rangle\\ &\geqslant t\|y\|^2+\langle y,j(x)\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2}\|x+y\|^2 &\geqslant\frac{1}{2}\|x\|^2+\int_0^1(t\|y\|^2+\langle y,j(x)\rangle)dt\\ &=\frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2+\langle y,j(x)\rangle.\end{aligned}$$

Replacing y by $-y$ leads to

$$\frac{1}{2}\|x-y\|^2\geqslant\frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\langle y,j(x)\rangle.$$

Adding these last two inequalities gets

$$\|x+y\|^2+\|x-y\|^2\geqslant 2(\|x\|^2+\|y\|^2)\quad\forall x,y\in X.$$

This is inequality (8). Replacing in (8) x by $\frac{x+y}{2}$ and y by $\frac{x-y}{2}$, respectively yields inequality (7). Hence the Parallelogram indentity (9) holds for all $x, y \in X$ and X is an inner product space. □

REMARK. In Proposition 3, we need that inequality (10) is valid only for some selection j for J; while in Proposition 2 we need the validity of inequality (4) for all $j_x \in J(x)$ and $j_y \in J(y)$. We do not know whether Proposition 2 is still valid if inequality (4) holds only for some selection j of J.

COROLLARY 2 [3]. *A normed linear space $(X, \|\cdot\|)$ is an inner product space if and only if the duality map $J : X \to X^*$ is linear, i.e.,*

$$J(ax+by) = aJ(x) + bJ(y) \quad x, y \in X \ \ a, b \in \mathbb{R}.$$

This means that given any $j_x \in J(x)$ and $j_y \in J(y)$, there exists some $j_{ax+by} \in J(ax+by)$ such that $aj_x + bj_y = j_{ax+by}$.

Proof. Given $x, y \in X$ and any selection j of J, we can find some $j_{x-y} \in J(x-y)$ satisfying $j(x) - j(y) = j_{x-y}$ so that

$$\langle x-y, j(x)-j(y)\rangle = \langle x-y, j_{x-y}\rangle = \|x-y\|^2.$$

So Corollary 2 follows from Proposition 3. □

In the remaining part of this paper, we shall improve some Banach space inequalities established in [11]. Recall that the moduli of convexity and smoothness of a normed linear space X are defined respectively by

$$\delta_X(\varepsilon) = \inf\Big\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = \|y\| = 1, , \|x-y\| = \varepsilon\Big\}, \quad 0 \leqslant \varepsilon \leqslant 2,$$

and

$$\rho_X(\tau) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, , \|y\| = \tau\Big\}, \quad \tau > 0.$$

X is *uniformly convex* if $\delta_X(\varepsilon) > 0$ for $0 < \varepsilon \leqslant 2$ and *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} = 0$. Let $p, q > 1$ be real numbers. Then X is said to be p*-uniformly convex* (resp., q*-uniformly smooth*) if there is a constant $c > 0$ such that $\delta_X(\varepsilon) \geqslant c\varepsilon^p$, $0 \leqslant \varepsilon \leqslant 2$ (resp., $\rho_X(\tau) \leqslant c\tau^q$, $\tau > 0$). It is proved in [11] that X is p-uniformly convex if and only if there is a constant $c > 0$ such that for all $x, y \in X$ and $\lambda \in [0,1]$

$$\|\lambda x + (1-\lambda)y\|^p \leqslant \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)c\|x-y\|^p, \tag{11}$$

where

$$W_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p.$$

It is also proved in [11] that if X is a smooth Banach space, then the dual version to (11), i.e., the inequality

$$\|\lambda x + (1-\lambda)y\|^q \geqslant \lambda\|x\|^q + (1-\lambda)\|y\|^q - W_q(\lambda)c\|x-y\|^q, \quad x, y \in X \text{ and } \lambda \in [0,1] \tag{12}$$

characterizes uniform smoothness. Here we are going to show that the smoothness assumption on X can be removed.

PROPOSITION 4. *Let $q > 1$ be a real number. Then a Banach space X is q-uniformly smooth if and only if there is a constant $c > 0$ such that inequality* (12) *holds.*

Proof. By [11, Theorem 1′], it suffices to show that (12) implies the smoothness of X. By replacing in (12) x by $\frac{x+y}{2}$ and y by $\frac{x-y}{2}$ and setting $\lambda = \frac{1}{2}$ we get

$$\frac{1}{2}(\|x+y\|^q + \|x-y\|^q) \leqslant \|x\|^q + c\|y\|^q, \quad x, y \in X. \tag{13}$$

Assume X is not smooth. Then there exists an $x_0 \in X$, $\|x_0\| = 1$, and $f, g \in X^*$, $f \neq g$, $\|f\| = \|g\| = 1$ such that $f(x_0) = g(x_0) = \|x_0\| = 1$. Take $y_0 \in X$, $\|y_0\| = 1$ such that $(f+g)(y_0) = 0$ and $f(y_0) > 0$. It follows that for any $\tau > 0$,

$$\begin{aligned}
\|x_0 + \tau y_0\|^q + \|x_0 - \tau y_0\|^q &\geqslant [f(x_0 + \tau y_0)]^q + [g(x_0 - \tau y_0)]^q \\
&= [1 + \tau f(y_0)]^q + [1 - \tau g(y_0)]^q \\
&= 2[1 + \tau f(y_0)]^q \qquad \text{as} \quad -g(y_0) = f(y_0) \\
&\geqslant 2[1 + \tau f(y_0)] \qquad \text{as} \quad f(y_0) > 0.
\end{aligned}$$

But inequality (13) then implies for all $\tau > 0$,

$$1 + c\tau^q \geqslant 1 + \tau f(y_0) \quad \text{or} \quad c\tau^{q-1} \geqslant f(y_0),$$

which is impossible since $\tau^{q-1} \to 0$ as $\tau \to 0^+$ and $f(y_0) > 0$. □

We now use the (generalized) *duality* map $J_q : X \to X^*$ defined by

$$J_q(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^q \text{ and } \|x^*\| = \|x\|^{q-1}\}, \quad x \in X$$

to characterize uniform smoothness of Banach spaces. Again we remove the smoothness assumption in [11].

PROPOSITION 5. *Let $q > 1$ be a real number. Then the following are equivalent for a Banach space X.*

(*i*) *X is q-uniformly smooth.*

(*ii*) *There is a constant $c > 0$ such that*

$$\|x+y\|^q \leqslant \|x\|^q + q\langle y, j_q(x)\rangle + c\|y\|^q, \quad \forall x, y \in X,\ \forall j_q(x) \in J_q(x). \tag{14}$$

(*iii*) *There is a constant $\tilde{c} > 0$ such that*

$$\langle x - y, j_q(x) - j_q(y)\rangle \leqslant \tilde{c}\|x-y\|^q, \quad \forall x, y \in X,\ \forall j_q(x) \in J_q(x),\ \forall j_q(y) \in J_q(y). \tag{15}$$

Proof. (*i*) ⇒ (*ii*). Since X is smooth, we have that J_q is single-valued and

$$q\langle y, J_q(x)\rangle = \lim_{\lambda \downarrow 0} \frac{\|x + \lambda y\|^q - \|x\|^q}{\lambda}, \quad x, y \in X.$$

By inequality (12), we have

$$\begin{aligned}\frac{\|x+\lambda y\|^q-\|x\|^q}{\lambda} &= \frac{\|(1-\lambda)x+\lambda(x+y)\|^q-\|x\|^q}{\lambda}\\ &\geqslant \frac{(1-\lambda)\|x\|^q+\lambda\|x+y\|^q-W_q(\lambda)c\|y\|^q-\|x\|^2}{\lambda}\\ &= \|x+y\|^q-\|x\|^q-[(1-\lambda)^q+\lambda^{q-1}(1-\lambda)]c\|y\|^q.\end{aligned}$$

Taking the limit as $\lambda \to 0^+$ we obtain

$$q\langle y, J_q(x)\rangle \geqslant \|x+y\|^q-\|x\|^q-c\|y\|^q,$$

which is (14).

$(ii) \Rightarrow (iii)$. By (14) we have for any $j_q(x) \in J_q(x)$ and $j_q(y) \in J_q(y)$,

$$\|y\|^q \leqslant \|x\|^q + q\langle y-x, j_q(x)\rangle + c\|y-x\|^q \tag{16}$$

and

$$\|x\|^q \leqslant \|y\|^q + q\langle x-y, j_q(y)\rangle + c\|x-y\|^q. \tag{17}$$

Upon adding (17) to (16) we get

$$\langle x-y, j_q(x)-j_q(y)\rangle \leqslant \frac{2c}{q}\|x-y\|^q.$$

Hence (15) is valid with $\tilde{c} = 2c/q$.

$(iii) \Rightarrow (i)$. Let $j_q(x+ty) \in J_q(x+ty)$ satisfy

$$\langle y, j_q(x+ty)\rangle = \sup\{\langle y, j_q\rangle : j_q \in J_q(x+ty)\}.$$

It then follows that

$$\begin{aligned}\frac{d^+}{dt}\frac{1}{q}\|x+ty\|^q &:= \lim_{s\downarrow 0}\frac{\|x+ty+sy\|^q-\|x+ty\|^q}{qs}\\ &= \sup\{\langle y, j_q\rangle : j_q \in J_q(x+ty)\}\\ &= \langle y, j_q(x+ty)\rangle\\ &= \frac{1}{t}\langle (x+ty)-x, j_q(x+ty)-j_q(x)\rangle + \langle y, j_q(x)\rangle\\ &\leqslant \tilde{c}t^{q-1}\|y\|^q + \langle y, j_q(x)\rangle \qquad \text{by (15)}.\end{aligned}$$

Therefore,

$$\|x+y\|^q \leqslant \|x\|^q + q\int_0^1 (\tilde{c}t^{q-1}\|y\|^q + \langle y, j_q(x)\rangle)dt,$$

or

$$\|x+y\|^q \leqslant \|x\|^q + \tilde{c}\|y\|^q + q\langle y, j_q(x)\rangle \quad \forall x,y \in X,\ \forall j_q(x) \in J_q(x). \tag{18}$$

Now for any $\lambda \in [0,1]$, write $z = \lambda x + (1-\lambda)y$. Then by (18) we have

$$\|x\|^q \leqslant \|z\|^q + \tilde{c}\|x-z\|^q + q\langle x-z, j_q(z)\rangle \tag{19}$$

and

$$\|y\|^q \leqslant \|z\|^q + \tilde{c}\|y-z\|^q + q\langle y-z, j_q(z)\rangle . \tag{20}$$

Since $x-z=(1-\lambda)(x-y)$ and $y-z=\lambda(y-x)$, it follows from (19) and (20) that

$$\lambda\|x\|^q + (1-\lambda)\|y\|^q \leqslant \|z\|^q + \tilde{c}[\lambda(1-\lambda)^q + \lambda^q(1-\lambda)]\|x-y\|^q.$$

Hence (12) holds with $c=\tilde{c}$ and X is q-uniformly smooth by Propositon 4. □

Finally we state the local versions of Propositoins 4 and 5 which again delete the smoothness assumption imposed on the space X in [11]. The proofs are omitted here as they are similar to those of Propositions 4 and 5. Recall that $B_r=\{x\in X : \|x\|\leqslant r\}$ is the closed ball centered at the origin with radius $r>0$. Let Γ be the family of continuous, strictly increasing and convex functions $g:[0,\infty)\to[0,\infty)$ such that $\lim_{\tau\to 0^+} g(\tau)/\tau = 0$.

PROPOSITION 6. *Let $q>1$ and $r>0$ be given. Then the following are equivalent for a Banach space X.*

(i) X is uniformly smooth.

(ii) There exists a $g\in\Gamma$ (depending on r) such that

$$\|\lambda x+(1-\lambda)y\|^q \geqslant \lambda\|x\|^q + (1-\lambda)\|y\|^q - W_q(\lambda)g(\|x-y\|) \quad \forall x,y\in B_r.$$

(iii) There exists a $g\in\Gamma$ (depending on r) such that

$$\|x+y\|^q \leqslant \|x\|^q + q\langle y, j_q(x)\rangle + g(\|y\|) \quad \forall x,y\in B_r,\ \forall j_q(x)\in J_q(x).$$

(iv) There exists a $g\in\Gamma$ (depending on r) such that

$$\langle x-y, j_q(x)-j_q(y)\rangle \leqslant g(\|x-y\|) \quad \forall x,y\in B_r,\ \forall j_q(x)\in J_q(x),\ \forall j_q(y)\in J_q(y).$$

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