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OPIAL TYPE INEQUALITIES FOR LINEAR DIFFERENTIAL OPERATORS

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Abstract. Various L_p form Opial type inequalities ([2]) are given for a Linear Differential Operator L, involving its related initial value problem solution y, Ly, the associated Green's function H and initial conditions point $x_0 \in \mathbf{R}$.

1. Background

Here we follow [1], pp. 145–154.

Let I be a closed interval of $\mathbf{R}$. Let $a_i(x)$, $i = 0, 1, \dots, n-1$ $(n \in \mathbf{N})$, $h(x)$ be continuous functions on I and let $L = D^n + a_{n-1}(x)D^{n-1} + \cdots + a_0(x)$ be a fixed linear differential operator on $C^n(I)$. Let $y_1(x), \dots, y_n(x)$ be a set of linear independent solutions to $Ly = 0$. Here the associated Green's function for L is

$$H(x,t) := \begin{vmatrix} y_1(t) \cdots y_n(t) \\ y_1'(t) \dots y_n'(t) \\ \vdots \\ y_1^{(n-2)}(t) \cdots y_n^{(n-2)}(t) \\ y_1(x) \cdots y_n(x) \end{vmatrix} \Bigg/ \begin{vmatrix} y_1(t) \cdots y_n(t) \\ y_1'(t) \cdots y_n'(t) \\ \vdots \\ y_1^{(n-2)}(t) \cdots y_n^{(n-2)}(t) \\ y_1^{(n-1)}(t) \cdots y_n^{(n-1)}(t) \end{vmatrix},$$

which is a continuous function on I^2. Consider fixed $x_0 \in I$, then

$$y(x) = \int_{x_0}^{x} H(x,t)h(t)dt, \quad \text{all } x \in I$$

is the unique solution to the initial value problem

$$Ly = h; \;\; y^{(i)}(x_0) = 0, \;\; i = 0, 1, \dots, n-1.$$

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2. Results

We first present the following

PROPOSITION 1. *Let* $x \geqslant x_0$*;* $x_0, x \in I$ *and* $p, q > 1$*:* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_{x_0}^{x} |y(w)||(Ly)(w)|dw \leqslant 2^{-1/q} \left(\int_{x_0}^{x} \left(\int_{x_0}^{w} |H(w,t)|^p dt \right) dw \right)^{1/p} \times \times \left(\int_{x_0}^{x} |(Ly)(w)|^q dw \right)^{2/q}. \quad (1)$$

Proof. Here take $x \geqslant x_0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. From Hölder's inequality we have

$$|y(x)| \leqslant \left(\int_{x_0}^{x} |H(x,t)|^p dt \right)^{1/p} \left(\int_{x_0}^{x} |h(t)|^q dt \right)^{1/q}. \quad (2)$$

Call

$$z(w) := \int_{x_0}^{w} |h(t)|^q dt, \quad x_0 \leqslant w \leqslant x \ \ (z(x_0) = 0).$$

Thus

$$z'(w) = |h(w)|^q$$

and

$$|h(w)| = (z'(w))^{1/q}.$$

From (2) we obtain

$$|y(w)|\,|h(w)| \leqslant \left(\int_{x_0}^{w} |H(w,t)|^p dt \right)^{1/p} \cdot (z(w) \cdot z'(w))^{1/q}.$$

Integrating the last inequality over $[x_0, x]$ we get

$$\int_{x_0}^{x} |y(w)|\,|h(w)|dw \leqslant \int_{x_0}^{x} \left(\int_{x_0}^{w} |H(w,t)|^p dt \right)^{1/p} \cdot (z(w) \cdot z'(w))^{1/q} dw$$

$$\leqslant \left(\int_{x_0}^{x} \left(\int_{x_0}^{w} |H(w,t)|^p dt \right) dw \right)^{1/p} \cdot \left(\int_{x_0}^{x} z(w) \cdot z'(w) dw \right)^{1/q}$$

$$= \left(\int_{x_0}^{x} \left(\int_{x_0}^{w} |H(w,t)|^p dt \right) dw \right)^{1/p} \cdot \frac{(z(x))^{2/q}}{2^{1/q}},$$

proving the claim of Proposition 1. $\square$

The counterpart of the previous result follows.

PROPOSITION 2. *Let* $x \leqslant x_0$*;* $x_0, x \in I$ *and* $p, q > 1$*:* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_x^{x_0} |y(w)|\,|(Ly)(w)|dw \leqslant 2^{-1/q} \cdot \left(\int_{x_0}^{x}\left(\int_{x_0}^{w} |H(w,t)|^p dt\right) dw\right)^{1/p} \times \times \left(\int_x^{x_0} |(Ly)(w)|^q dw\right)^{2/q}. \tag{3}$$

Proof. Here take $x \leqslant x_0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. From Hölder's inequality we have

$$|y(x)| = \left|\int_x^{x_0} H(x,t)h(t)dt\right| \leqslant \int_x^{x_0} |H(x,t)|\,|h(t)|dt \leqslant \left(\int_x^{x_0} |H(x,t)|^p dt\right)^{1/p} \left(\int_x^{x_0} |h(t)|^q dt\right)^{1/q}. \tag{4}$$

Call

$$z(x) := \int_x^{x_0} |h(t)|^q dt \geqslant 0, \quad z(x_0) = 0.$$

I.e.

$$-z(x) = \int_{x_0}^{x} |h(t)|^q dt \leqslant 0,$$

and

$$-z'(x) = |h(x)|^q \geqslant 0,$$

and

$$|h(x)| = (-z'(x))^{1/q}, \quad x \in I.$$

Hence by (4) $(x \leqslant w \leqslant x_0)$

$$|y(w)| \cdot |h(w)| \leqslant \left(\int_w^{x_0} |H(w,t)|^p dt\right)^{1/p} \cdot (z(w) \cdot (-z'(w)))^{1/q}.$$

Integrating the last inequality over $[x, x_0]$ we get

$$\int_x^{x_0} |y(w)|\,|h(w)|dw \leqslant \int_x^{x_0}\left(\int_w^{x_0} |H(w,t)|^p dt\right)^{1/p} \cdot (z(w) \cdot (-z'(w)))^{1/q} \cdot dw$$
$$\leqslant \left(\int_x^{x_0}\left(\int_w^{x_0} |H(w,t)|^p dt\right) \cdot dw\right)^{1/p} \left(\int_x^{x_0} (-z'(w) \cdot z(w)) \cdot dw\right)^{1/q}$$
$$= 2^{-1/q} \cdot \left(\int_{x_0}^{x}\left(\int_{x_0}^{w} |H(w,t)|^p dt\right) \cdot dw\right)^{1/p} \cdot (z(x))^{2/q},$$

proving the claim of Proposition 2. □

Extreme cases come next.

PROPOSITION 3. *Here* $p = 1$, $q = \infty$ *and* $x \geqslant x_0$. *Then*

$$\int_{x_0}^{x} |y(w)||(Ly)(w)|dw \leqslant \left(\int_{x_0}^{x}\left(\int_{x_0}^{w}|H(w,t)|dt\right)dw\right)\cdot \|Ly\|_{\infty}^{2}. \tag{5}$$

Proof. From

$$y(w) = \int_{x_0}^{w} H(w,t)h(t)dt$$

we obtain

$$|y(w)| \leqslant \left(\int_{x_0}^{w}|H(w,t)|dt\right)\|h\|_{\infty},$$

and

$$|y(w)|\,|(Ly)(w)| \leqslant \left(\int_{x_0}^{w}|H(w,t)|dt\right)\cdot(\|Ly\|_{\infty})^2.$$

Integrating the last inequality we get (5). □

PROPOSITION 4. *Again* $p = 1$, $q = \infty$, *but* $x \leqslant x_0$. *Then*

$$\int_{x}^{x_0} |y(w)|\,|(Ly)(w)|dw \leqslant \left(\int_{x_0}^{x}\left(\int_{x_0}^{w}|H(w,t)|dt\right)dw\right)\cdot(\|(Ly)\|_{\infty}^{2}). \tag{6}$$

Proof. Here $x \leqslant w \leqslant x_0$, and

$$|y(w)| = \left|\int_{x_0}^{w} H(w,t)h(t)dt\right| = \left|\int_{w}^{x_0} H(w,t)h(t)dt\right|$$
$$\leqslant \left(\int_{w}^{x_0}|H(w,t)|dt\right)\|h\|_{\infty}.$$

Thus

$$|y(w)|\,|(Ly)(w)| \leqslant \left(\int_{w}^{x_0}|H(w,t)|dt\right)(\|Ly\|_{\infty})^2,$$

and

$$\int_{x}^{x_0} |y(w)|\,|(Ly)(w)|dw \leqslant \left(\int_{x}^{x_0}\left(\int_{w}^{x_0}|H(w,t)|dt\right)dw\right)\cdot(\|Ly\|_{\infty})^2. \qquad \square$$

COROLLARY 1. *Let* $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left|\int_{x_0}^{x} |y(w)|\,|(Ly)(w)|dw\right| \leqslant 2^{-1/q}\cdot\left(\int_{x_0}^{x}\left(\int_{x_0}^{w}|H(w,t)|^p dt\right)dw\right)^{1/p} \times$$
$$\times\left(\left|\int_{x_0}^{x}|(Ly)(w)|^q dw\right|\right)^{2/q}. \tag{7}$$

Proof. By Propositions 1.2. □

In particular when $p = q = 2$ we get

COROLLARY 2. *It holds*

$$\left|\int_{x_0}^{x}|y(w)|\,|(Ly)(w)|dw\right| \leqslant 2^{-1/2}\cdot\left(\int_{x_0}^{x}\left(\int_{x_0}^{w}(H(w,t))^2dt\right)dw\right)^{1/2}\times$$
$$\times\left|\int_{x_0}^{x}((Ly)(w))^2dw\right|. \tag{8}$$

Furthermore we have

COROLLARY 3. *Let* $p=1$, $q=\infty$. *Then*

$$\left|\int_{x_0}^{x}|y(w)|\,|(Ly)(w)|dw\right| \leqslant \left(\int_{x_0}^{x}\left(\int_{x_0}^{w}|H(w,t)|dt\right)dw\right)\cdot\|Ly\|_\infty^2. \tag{9}$$

Proof. By Propositions 3 and 4. □

To complete our study we present

PROPOSITION 5. *Let* $0<p<1$ *and* q *be such that* $\frac{1}{p}+\frac{1}{q}=1$, *and* $x>x_0$, $x_0,x\in I$. *Assume that*

$$H(w,t)\geqslant 0 \quad \text{for } x_0\leqslant t\leqslant w,\ w\in I.$$

Also assume that $Ly=h$ *is of fixed sign and nowhere zero. Then*

$$\int_{x_0}^{x}|y(w)|\,|(Ly)(w)|dw \geqslant 2^{-1/q}\left(\int_{x_0}^{x}\left(\int_{x_0}^{w}(H(w,t))^pdt\right)dw\right)^{1/p}\times$$
$$\times\left(\int_{x_0}^{x}|(Ly)(w)|^qdw\right)^{2/q} \tag{10}$$

Proof. Here for $x_0\leqslant w\leqslant x$ we have

$$|y(w)|=\int_{x_0}^{w}H(w,t)|h(t)|dt. \tag{11}$$

From (11) by Hölder's inequality we obtain

$$|y(w)|\geqslant\left(\int_{x_0}^{w}(H(w,t))^pdt\right)^{1/p}\cdot\left(\int_{x_0}^{w}|h(t)|^qdt\right)^{1/q}, \tag{12}$$

for $w>x_0$. Consider

$$z(w):=\int_{x_0}^{w}|h(t)|^qdt, \quad z(x_0)=0.$$

So that $z'(w)=|h(w)|^q$ and $|h(w)|=(z'(w))^{1/q}$, all $x_0\leqslant w\leqslant x$. Thus by (12) we get

$$|y(w)|\,|h(w)|\geqslant\left(\int_{x_0}^{w}(H(w,t))^pdt\right)^{1/p}\cdot(z(w)\cdot z'(w))^{1/q},$$

all $x_0 < w \leqslant x$. Let $x_0 < \theta \leqslant w \leqslant x$ and $\theta \downarrow x_0$, then by integration of the last inequality we get

$$\begin{aligned}
\int_{x_0}^{x} |y(w)|\,|h(w)|dw &= \lim_{\theta\downarrow x_0}\int_{\theta}^{x} |y(w)|\,|h(w)|dw \\
&\geqslant \lim_{\theta\downarrow x_0}\left(\int_{\theta}^{x}\left(\int_{x_0}^{w}(H(w,t))^p dt\right)^{1/p}\cdot (z(w)\cdot z'(w))^{1/q}\cdot dw\right) \\
&\geqslant \lim_{\theta\downarrow x_0}\left(\int_{\theta}^{x}\left(\int_{x_0}^{w}(H(w,t))^p dt\right)dw\right)^{1/p}\cdot \lim_{\theta\downarrow x_0}\left(\left(\int_{\theta}^{x} z(w)z'(w)dw\right)^{1/q}\right) \\
&= 2^{-1/q}\cdot\left(\int_{x_0}^{x}\left(\int_{x_0}^{w}(H(w,t))^p dt\right)dw\right)^{1/p}\cdot \lim_{\theta\downarrow x_0}(z^2(x)-z^2(\theta))^{1/q} \\
&= 2^{-1/q}\left(\int_{x_0}^{x}\left(\int_{x_0}^{w}(H(w,t))^p dt\right)dw\right)^{1/p}\cdot (z(x))^{2/q}.
\end{aligned}$$

That is establishing (10). □

PROPOSITION 6. *Let* $0 < p < 1$ *and* q *be such that* $\frac{1}{p}+\frac{1}{q}=1$, *and* $x < x_0$, $x_0, x \in I$. *Assume that*

$$H(w,t) \leqslant 0 \quad \textit{for } w \leqslant t \leqslant x_0,\ w \in I.$$

Also assume that $Ly = h$ *is of fixed sign and nowhere zero. Then*

$$\begin{aligned}
\int_{x}^{x_0} |y(w)|\,|(Ly)(w)|dw \geqslant 2^{-1/q}\cdot\left(\int_{x_0}^{x}\left(\int_{x_0}^{w}|H(w,t)|^p dt\right)dw\right)^{1/p}\times \\
\times\left(\int_{x}^{x_0}|(Ly)(w)|^q dw\right)^{2/q}. \qquad (13)
\end{aligned}$$

Proof. Here for $x \leqslant w \leqslant x_0$ we have

$$\begin{aligned}
|y(w)| &= \left|\int_{x_0}^{w} H(w,t)h(t)dt\right| \\
&= \left|\int_{w}^{x_0} H(w,t)h(t)dt\right| \\
&= \left|\int_{w}^{x_0} (-H(w,t))h(t)dt\right| \\
&= \int_{w}^{x_0} (-H(w,t))|h(t)|dt. \qquad (14)
\end{aligned}$$

From (14) by Hölder's inequality we obtain

$$|y(w)| \geqslant \left(\int_{w}^{x_0}(-H(w,t))^p dt\right)^{1/p}\left(\int_{w}^{x_0}|h(t)|^q dt\right)^{1/q},$$

for $w < x_0$. I.e.

$$|y(w)| \geqslant \left(\int_w^{x_0} |H(w,t)|^p dt\right)^{1/p} \left(\int_w^{x_0} |h(t)|^q dt\right)^{1/q}, \tag{15}$$

for $w < x_0$. Consider

$$z(w) := \int_w^{x_0} |h(t)|^q dt, \quad z(x_0) = 0.$$

So that

$$-z(w) = \int_{x_0}^{w} |h(t)|^q dt$$

and

$$-z'(w) = |h(w)|^q,$$

with

$$|h(w)| = (-z'(w))^{1/q}, \quad \text{all } x \leqslant w \leqslant x_0.$$

Therefore by (15) we get

$$|y(w)|\,|h(w)| \geqslant \left(\int_w^{x_0} |H(w,t)|^p dt\right)^{1/p} \cdot (z(w)\cdot(-z'(w)))^{1/q},$$

all $x \leqslant w < x_0$. Let $x \leqslant w \leqslant \theta < x_0$ and $\theta \uparrow x_0$, then by integration of the last inequality we obtain

$$\begin{aligned}
\int_x^{x_0} |y(w)|\,|h(w)| dw &= \lim_{\theta\uparrow x_0} \int_x^{\theta} |y(w)|\,|h(w)| dw \\
&\geqslant \lim_{\theta\uparrow x_0}\left(\int_x^{\theta}\left(\int_w^{x_0} |H(w,t)|^p dt\right)^{1/p} \cdot (z(w)\cdot(-z'(w)))^{1/q}\cdot dw\right) \\
&\geqslant \lim_{\theta\uparrow x_0}\left(\int_x^{\theta}\left(\int_w^{x_0} |H(w,t)|^p dt\right) dw\right)^{1/p} \cdot \lim_{\theta\uparrow x_0}\left(\int_x^{\theta} (-z(w))(z'(w))\cdot dw\right)^{1/q} \\
&= \left(\int_x^{x_0}\left(\int_w^{x_0} |H(w,t)|^p dt\right) dw\right)^{1/p} \cdot \lim_{\theta\uparrow x_0}\left(-\int_x^{\theta} z(w)dz(w)\right)^{1/q} \\
&= 2^{-1/q}\cdot\left(\int_x^{x_0}\left(\int_w^{x_0} |H(w,t)|^p dt\right) dw\right)^{1/p} \cdot \lim_{\theta\uparrow x_0}(z^2(x)-z^2(\theta))^{1/q} \\
&= 2^{-1/q}\cdot\left(\int_x^{x_0}\left(\int_w^{x_0} |H(w,t)|^p dt\right) dw\right)^{1/p} \cdot (z(x))^{2/q}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_x^{x_0} |y(w)|\,|h(w)| dw \geqslant 2^{-1/q}\cdot\left(\int_{x_0}^{x}\left(\int_{x_0}^{w} |H(w,t)|^p dt\right) dw\right)^{1/p} \times \\
\times\left(\int_x^{x_0} |h(w)|^q dw\right)^{2/q},
\end{aligned}$$

that is proving (13). □

Putting things together we have

COROLLARY 4. *Let* $0 < p < 1$ *and* q *be such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $x_0, x \in I$ *such that* $x \neq x_0$. *Assume that*

$$(w - x_0) \cdot H(w,t) \geqslant 0,$$

for all t *between* $x_0, w \in I$. *Also assume that* $Ly = h$ *is of fixed sign and nowhere zero. Then*

$$\left| \int_{x_0}^{x} |y(w)| \, |(Ly)(w)| dw \right| \geqslant 2^{-1/q} \cdot \left(\int_{x_0}^{x} \left(\int_{x_0}^{w} |H(w,t)|^p dt \right) dw \right)^{1/p} \times \times \left(\left| \int_{x_0}^{x} |(Ly)(w)|^q dw \right| \right)^{2/q}. \tag{16}$$

Proof. By Propositions 5 and 6. □

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