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A LIAPUNOV INEQUALITY FOR LINEAR HAMILTONIAN SYSTEMS

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Abstract. A Liapunov type inequality is proved for a linear Hamiltonian system. This inequality allows estimates of intervals of disconjugacy. The inequality is particularly applicable to equations with oscillatory coefficients. A new criterion of stability is given for a differential equation with periodic coefficients.

0. Introduction

The classical Liapunov inequality states that if a and b are consecutive zeros of a nontrivial solution y of

$$y''(t) + p(t)y(t) = 0, \tag{0.1}$$

where p is a real Lebesgue integrable function, then

$$(b-a)\int_a^b p^+(t)dt > 4. \tag{0.2}$$

The notation p^+ is defined by $p^+(t) = \max\{p(t), 0\}$. There are many proofs of this inequality, e.g., see the survey paper [5]. Liapunov inequalities are quite useful in the study of differential equations. They are used to derive bounds for the distance between zeros of solutions, and to estimate the number of zeros of a solution in an interval [8, p. 346]. In eigenvalue problems, they yield lower bounds on the eigenvalues.

The inequality (0.2) may be stated another way. If $(b-a)\int_a^b p^+(t)dt \leqslant 4$, then the equation (0.1) is disconjugate on $[a,b]$, i.e., no nontrivial solution has more than one zero on $[a,b]$. For the extension to nth order scalar equations, Liapunov's inequality often takes this form, i.e, conditions are given so that no nontrivial solution of the differential equation has n zeros, counting multiplicities, on an interval.

Another extension of Liapunov's inequality for $2n$th order scalar equations is concerned only with the nonexistence of a pair of conjugate points, i.e., the nonexistence of a nontrivial solution with a pair of n-fold zeros. This is also an appropriate extension when considering Hamiltonian systems. Much of the Sturmian theory for the second

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order equation carries over to Hamiltonian systems for conjugate points [6,13]. Further, the disconjugacy of a Hamiltonian system is equivalent to the existence of a solution of the associated Riccati equation [10]. The existence of a solution to the Riccati equation is important in many applications. For example, in optimal control of a linear regulators, the existence of a solution of an associated Riccati equation gives the existence of an optimal control, and an explicit formula for the optimal control can be given in terms of the solution of the Riccati equation [2].

In a series of papers, W. T. Reid [9,11,12] made an extension of (0.2) to Hamiltonian systems. We mention one application here which we will return to in section 3. Suppose y is a nontrivial solution of the vector differential equation

$$(0.3)\qquad y''(t) - C(t)y(t) = H(t)y(t), \quad a \leqslant t \leqslant b, \quad y(a) = y(b) = 0,$$

where C and H satisfy conditions (1.2) and (1.3) below and $H^T(x) = H(x)$. Then

$$(0.4)\qquad \int_a^b \operatorname{trace}[G(s,s)H^+(s)]ds > 1,$$

where $H^+(s) = [H(s)+\sqrt{H^2(s)}]/2$, and $G(t,s)$ is the Green's function for the problem

$$(0.5)\qquad y''(t) - C(t)y(t) = f(t), \quad y(a) = y(b) = 0,$$

so that the unique solution of (0.5) has the representation

$$y(t) = \int_a^b G(t,s)f(s)ds.$$

For $C(t) \equiv 0$ and $[a,b] = [0,1]$, it is readily computed that

$$G(t,s) = \begin{cases} s(1-t)I, & s \leqslant t \\ t(1-s)I, & s > t \end{cases}$$

where I is the identity matrix. Then (0.4) takes the form

$$(0.6)\qquad \int_0^1 s(1-s)\operatorname{trace}[H^+(s)]ds > 1.$$

Note that $s(1-s) \leqslant \frac{1}{4}$ so that (0.2) is recovered for (0.1).

A stronger version of (0.2) has recently been given by Brown and Hinton [3]. It states that if y is a nontrivial solution of (0.1) such that $y(a) = y(b) = 0$, then there exists t_1, t_2 in $[a,b]$ such that

$$(0.7)\qquad (b-a)\left|\int_{t_1}^{t_2} p(t)dt\right| > 4.$$

An advantage of (0.7) is that it shows that disconjugacy of (0.1) can be preserved over a long time interval with a highly oscillatory p. It is the purpose of this paper to extend (0.7) to the Hamiltonian system setting.

In section 1 we state the hypotheses needed for our Hamiltonian system, and state some lemmas needed. The proof of the main result, Theorem 2.1, is given in section

2. In section 3 we give some applications of Theorem 2.1 and compare, for vector differential equations, the results with those of Reid. A new criterion of stability is given for a periodic scalar differential equation.

We use the notation $\mathcal{L}[a,b], \mathcal{L}^2[a,b]$, and $\mathcal{L}^\infty[a,b]$ for the Lebesgue spaces of functions which are respectively Lebesgue integrable, Lebesgue square integrable, and essentially bounded on $[a,b]$. The superscript T indicates a matrix transpose. The matrix inequality $B \geqslant 0$ indicates that B is a positive semi-definite symmetric matrix.

1. Preliminaries

We consider the Hamiltonian differential system on $[a,b]$,

$$\begin{aligned} -v' &= -C(t)u + A^T(t)v - H(t)u \\ u' &= A(t)u + B(t)v, \end{aligned} \tag{1.1}$$

where the real $n \times n$ matrix functions $A(t), B(t), C(t)$, and $H(t)$ satisfy the following conditions:

(1.2) $A(t)$ and $B(t)$ are of class $\mathcal{L}^\infty[a,b]$; $C(t)$ and $H(t)$ are of class $\mathcal{L}[a,b]$; further $B(t) = B^T(t) \geqslant 0$ and $C(t) = C^T(t)$ on $[a,b]$.

(1.3) Define $\mathbb{Q}[a,b]$ to be the set of n-dimensional vector functions η such that: η is absolutely continuous on $[a,b]$, $\eta(a) = \eta(b) = 0$, and for some n-dimensional vector function ξ of class $\mathcal{L}^2[a,b]$,

$$\eta' = A(t)\eta + B(t)\xi \text{ a.e. on } [a,b].$$

Assume that the function

$$J(\eta) = \int_a^b \left[\xi^T(t)B(t)\xi(t) + \eta^T(t)C(t)\eta(t)\right] dt$$

is positive definite on $\mathbb{Q}[a,b]$, i.e., $J(\eta) > 0$ if $\eta \not\equiv 0$.

The assumption (1.3) has an equivalent formulation in terms of conjugate points. A point $c > a$ is said to be *conjugate* to a if there exists a solution of (1.1) with $u(t) \not\equiv 0$ such that $u(a) = u(c) = 0$. For the system (1.1) *under the assumption* $H(t) \equiv 0$, the condition (1.3) is equivalent to there existing no point c in $(a,b]$ which is conjugate to a [9]. The system (0.3) has the formulation (1.1) as can be seen by defining $u = y$, $v = y'$, $A(t) \equiv 0$, and $B(t) \equiv I$. For the equation (0.3) with $H(t) \equiv 0$, the condition (1.3) is equivalent to there existing no nontrivial solution y of (0.3) such that $y(a) = y(c) = 0$ for some $c \in [a,b]$. Note also that (1.3) holds for the special case of (0.3), $y''(t) = 0$.

REMARK 1.1. Note that we do not require $H^T(t) = H(t)$ as in Reid's work.

For our principal result we need three lemmas which we now state. The first is a special case of Lemma 3.1 of [12].

LEMMA 1.1. *Whenever conditions* (1.2) *and* (1.3) *hold there exists a number* $k > 0$ *such that for* $\eta \in \mathbb{Q}[a,b]$ *we have*

$$J(\eta) \geqslant k \int_a^b \eta'(t)^T \eta'(t)dt. \tag{1.4}$$

For the equation (0.3), we have $A(t) \equiv 0$, $B(t) \equiv I$; hence $\xi = \eta'$ and (1.4) will hold with $k = 1$ if $C(t) \geqslant 0$. This explicit k will be used in subsequent calculations.

LEMMA 1.2. *Suppose* f_1, f_2 *are absolutely continuous real functions on* $[a,b]$ *with* $f_i(a) = f_i(b) = 0$ *and* $f_i' \in \mathcal{L}^2[a,b]$, $i = 1,2$. *Then*

$$\int_a^b \Big[|f_1(t) f_2'(t)| + |f_1'(t) f_2(t)| \Big] dt \leqslant \left(\frac{b-a}{2}\right) \left[\int_a^b f_1'(t)^2 dt \int_a^b f_2'(t)^2 dt \right]^{1/2}. \tag{1.5}$$

Further, equality holds only for f_1, f_2 *linear on each of* $[a, (a+b)/2]$, $[(a+b)/2, b]$.

Proof. It has been proven by Calvert [4] or see [1, p. 28] that if g_1, g_2 are absolutely continuous real functions on $[c,d]$ with $g_i(c) = 0$ and $g_i' \in \mathcal{L}^2[c,d]$, $i = 1,2$, then

$$\int_c^d \Big[|g_1(t) g_2'(t)| + |g_1'(t) g_2(t)| \Big] dt \leqslant (d-c) \left[\int_c^d g_1'(t)^2 dt \int_c^d g_2'(t)^2 dt \right]^{1/2} \tag{1.6}$$

with equality only for g_1, g_2 linear on $[c,d]$. The same inequality holds if the condition $g_1(c) = g_2(c) = 0$ is replaced by $g_1(d) = g_2(d) = 0$. Applying (1.6) first on $[a, (a+b)/2]$ and then on $[(a+b)/2, b]$ and adding yields that with $e = (a+b)/2$,

$$\begin{aligned} \int_a^b \Big[|f_1(t) f_2'(t)| + |f_1'(t) f_2(t)| \Big] dt &\leqslant \left(\frac{b-a}{2}\right) \times \\ \times \Bigg\{ \left[\int_a^e f_1'(t)^2 dt \right]^{1/2} \left[\int_a^e f_2'(t)^2 dt \right]^{1/2} &+ \left[\int_e^b f_1'(t)^2 dt \right]^{1/2} \left[\int_e^b f_2'(t)^2 dt \right]^{1/2} \Bigg\}. \end{aligned} \tag{1.7}$$

Application of the Cauchy Schwarz inequality to (1.7) completes the proof of Lemma 1.2.

LEMMA 1.3. *If* f *is a real-valued function of class* $\mathcal{L}[a,b]$, *then there exists* $t_1, t_2 \in [a,b]$ *such that*

$$\inf_{\mu \in (-\infty,\infty)} \left(\max_{a \leqslant x \leqslant b} \left| \int_a^x f(t)dt + \mu \right| \right) = \frac{1}{2} \left| \int_{t_1}^{t_2} f(t)dt \right|. \tag{1.8}$$

Proof. Define M, m by

$$M := \max_{a \leqslant x \leqslant b} \int_a^x f(t)dt, \quad m := \min_{a \leqslant x \leqslant b} \int_a^x f(t)dt.$$

If we choose $\mu_0 = -(M+m)/2$, then the function $\int_a^x f(t)dt + \mu_0$ has a maximum of $(M-m)/2$ and a minimum of $(m-M)/2$. Since a value of $\mu \neq \mu_0$ will either increase the maximum or decrease the minimum of this function, the left hand side of (1.8) is realized for $\mu = \mu_0$ yielding a value of $(M-m)/2$.

2. The Liapunov Inequality

The matrix function Q will denote an antiderivative of the function $H(t)$ of (1.1), i.e., $Q'(t) = H(t)$ a.e. on $[a,b]$. In terms of Q we define a scalar matrix $Q^{\#}$ by

$$Q^{\#}_{ij} = \max_{a \leqslant t \leqslant b} \left| Q_{ij}(t) + Q_{ji}(t) \right|. \tag{2.1}$$

The matrix $Q^{\#}$ will depend on the constant of integration μ in the representation $Q(t) = \int_a^t H(u)du + \mu$. For each i,j, there is by Lemma 1.3, $x_{ij} = x_{ji}$ and $y_{ij} = y_{ji}$ such that for a certain choice of $\mu_{ij} = \mu_{ji}$,

$$Q^{\#}_{ij} = \frac{1}{2} \left| \int_{x_{ij}}^{y_{ij}} \left[H_{ij}(t) + H_{ji}(t) \right] dt \right|. \tag{2.2}$$

We assume $Q^{\#}$ is chosen so that (2.2) holds. Defining the matrix $\tilde{Q}$ by

$$\tilde{Q}_{ij} = \frac{1}{2} \int_a^b \left[\left| H_{ij}(t) \right| + \left| H_{ji}(t) \right| \right] dt,$$

we have then that $Q^{\#}$ and $\tilde{Q}$ are symmetric matrices with nonnegative entries satisfying for all i,j,

$$Q^{\#}_{ij} \leqslant \tilde{Q}_{ij}. \tag{2.3}$$

THEOREM 2.1. *Assume* (1.2) *and* (1.3) *hold and* k *is as in Lemma 1.1. Suppose* u,v *is a nontrivial solution of* (1.1) *such that* $u(a) = 0 = u(b)$. *Then*

$$k \leqslant \left(\frac{b-a}{4} \right) \lambda^{\#} \leqslant \left(\frac{b-a}{4} \right) \tilde{\lambda}, \tag{2.4}$$

where $\lambda^{\#}$ *is the maximum of the set of eigenvalues of* $Q^{\#}$ *and* $\tilde{\lambda}$ *is the maximum of the set of eigenvalues of* $\tilde{Q}$.

Proof. Multiplying the first equation of (1.1) by u^T, the second equation by $-v^T$ and adding yields that

$$-\{u^T v' + v^T u'\} = -u^T Cu - v^T Bv - u^T Hu. \tag{2.5}$$

From $Q' = H$, integration of (2.5) over $[a,b]$ gives

$$-u^T v|_a^b = -\int_a^b [u^T Cu + v^T Bv] dt \; - \int_a^b u^T Q' u dt \tag{2.6}$$

Since $u(a) = 0 = u(b)$ and $[u^T Qu]' = (u')^T Qu + u^T Q'u + u^T Qu'$, simplifying (2.6) we get

$$\begin{aligned}
\int_a^b [u^T Cu + v^T Bv]dt &= \int_a^b [(u')^T Qu + u^T Qu']dt \\
&= \int_a^b \sum_{i,j=1}^n u_i u_j' [Q_{ij} + Q_{ji}]dt \\
&\leqslant \sum_{i,j=1}^n Q_{ij}^\# \int_a^b |u_i|\,|u_j'|dt \\
&= \sum_{i=1}^n Q_{ii}^\# \int_a^b |u_i|\,|u_i'|dt \\
&\quad + \sum_{i>j} \sum_{j=1}^n Q_{ij}^\# \int_a^b \left[|u_i|\,|u_j'| + |u_i'|\,|u_j|\right]dt.
\end{aligned} \tag{2.7}$$

Applying Lemmas 1.1 and 1.2, it follows from (2.7) that

(2.8)
$$\begin{aligned}
k\int_a^b |u'|^2 dt &\leqslant \sum_{i=1}^n \left(\frac{b-a}{4}\right) Q_{ii}^\# \int_a^b (u_i')^2 dt \\
&\quad + \sum_{i>j} \sum_{j=1}^n \left(\frac{b-a}{2}\right) Q_{ij}^\# \left(\int_a^b |u_i'|^2 dt\right)^{1/2} \left(\int_a^b |u_j'|^2 dt\right)^{1/2} \\
&= \left(\frac{b-a}{4}\right) w^T Q^\# w,
\end{aligned}$$

where $w_i = \left(\int_a^b |u_i'|^2 dt\right)^{1/2}$, $i = 1, \ldots, n$. Since $Q^\#$ is symmetric with nonnegative entries, we have from

$$\lambda^\# = \max\{z^T Q^\# z \mid |z| = 1\} \tag{2.9}$$

that

$$w^T Q^\# w \leqslant \lambda^\# |w|^2 = \lambda^\# \int_a^b \sum_{i=1}^n |u_i'|^2 dt. \tag{2.10}$$

Dividing (2.10) by $|w|^2$ gives that $k \leqslant (b-a)\lambda^\#/4$ which is the first inequality in (2.4). Since the entries of $Q^\#$ are nonnegative, equality in (2.9) is achieved for a vector with nonnegative entries. By (2.3) and the characterization of $\tilde{\lambda}$ as in (2.9), the second inequality of (2.4) follows.

REMARK 2.1. We note that for the equation (0.3) with $C(t) \geqslant 0$, the left inequality (2.4) is strict and $k = 1$. Strict inequality follows since equality implies y is linear on

$[a,(a+b)/2]$ and $[(a+b)/2,b]$ with $y(a)=y(b)=0$. Since y is a solution of (0.3), y' is absolutely continuous on $[a,b]$, contrary to y being linear on the two solution intervals.

By choosing different antiderivatives Q_{ij} of H_{ij}, an inequality of the type (2.4) may be obtained for the boundary-conditions $u(a)=v(b)=0$ or $v(a)=u(b)=0$. To this end define matrix functions

$$Q_R(t)=\int_a^t H(u)du, \quad Q_L(t)=\int_t^b H(u)du,$$

and the matrix $Q_R^{\#}$ by

$$(Q_R^{\#})_{ij}=\max_{a\leqslant t\leqslant b}\left|\,(Q_R(t))_{ij}+(Q_R(t))_{ji}\,\right|,$$

and $Q_L^{\#}$ similarly by replacing the subscript R by L. The matrix $\tilde{Q}$ is as before so that

$$(Q_R^{\#})_{ij}\leqslant 2\tilde{Q}_{ij}, \quad (Q_L^{\#})_{ij}\leqslant 2\tilde{Q}_{ij}.$$

COROLLARY 2.1. *Assume* (1.2) *and* (1.3) *hold and k is as in Lemma 1.1. Suppose u,v is a nontrivial solution of* (1.1) *such that* $u(a)=v(b)=0$. *Then*

$$k\leqslant\left(\frac{b-a}{2}\right)\lambda_L^{\#}\leqslant(b-a)\tilde{\lambda} \tag{2.11}$$

where $\lambda_L^{\#}$ is the maximum eigenvalue of $Q_L^{\#}$. Similarly, if u,v is a nontrivial solution of (1.1) *such that* $v(a)=u(b)=0$, *then*

$$k\leqslant\left(\frac{b-a}{2}\right)\lambda_R^{\#}\leqslant(b-a)\tilde{\lambda}, \tag{2.12}$$

where $\lambda_R^{\#}$ is the maximum eigenvalue of $Q_R^{\#}$.

The proof of Corollary 2.1 is the same as that of Theorem 2.1 noting only that $[u^TQu]\,\Big|_a^b=0$ for either $u(a)=v(b)=0$ or $v(a)=u(b)=0$ with the appropriate choice of antiderivative $Q(t)$. The left inequalities (2.11) and (2.12) are strict for equation (0.3) with $C(t)\geqslant 0$ since $v=y'$ in this case and equality implies y is linear which implies y' does not vanish at either a or b.

3. Applications

The results of Theorem 2.1 and earlier work of Reid are independent. Comparisons are difficult as the function H^+ of Reid and the eigenvalues $\lambda^{\#},\tilde{\lambda}$ of Theorem 2.1 are not readily available. A few simple comparisons may be made for (0.3) on $[0,1]$ with $C(t)\equiv 0$ and $H(t)>0$. We have then from (0.6) that

$$\int_0^1 s(1-s)\ \mathrm{trace}\ H(s)ds\leqslant 1 \tag{3.1}$$

implies the equation $y''(t) = H(t)y(t)$ is disconjugate on $[0,1]$.

For $H(t)$ a diagonal matrix (3.1) becomes

$$\int_0^1 s(1-s)\sum_{i=1}^n H_{ii}(s)ds \leqslant 1 \tag{3.2}$$

while Theorem 2.1 gives disconjugacy if (recall Remark 2.1),

$$\tilde{\lambda} = \max_{1\leqslant i\leqslant 4}\left\{\int_0^1 H_{ii}(s)ds\right\} < 4.$$

In the 2×2 case, with $H_{11}(t) = H_{22}(t) > 0$ and $|H_{12}(t)| < H_{11}(t)$, then (3.1) is

$$2\int_0^1 s(1-s)H_{11}(s)ds \leqslant 1$$

while Theorem 2.1 gives disconjugacy if

$$\tilde{\lambda} = \int_0^1 H_{11}(s)ds + \int_0^1 |H_{12}(s)|ds \leqslant 4.$$

As a second application of Theorem 2.1 we consider the scalar equation (0.1) where $p(t) = -H(t)$ is periodic of period T. First we must recall some facts about periodic equations [7]. If $\lambda_0 \leqslant \lambda_1 \leqslant \dots$ are the eigenvalues for an equation of the form $-y''(t)+q(t)y(t) = \lambda y(t), q(t+T) = q(t)$, with periodic boundary conditions $y(0) = y(T)$, $y'(0) = y'(T)$, and if $\mu_0 \leqslant \mu_1 \leqslant \dots$ are the eigenvalues with semi-periodic boundary conditions $y(0) = -y(T)$, $y'(0) = -y'(T)$, then the stability intervals of this equation are given by $[\lambda_0,\mu_0]$, $[\mu_1,\lambda_1]$, $[\lambda_2,\mu_2]$, etc. This means that if λ is in the interior of a stability interval, then the equation $-y''(t)+q(t)y(t) = \lambda y(t)$ has only bounded solutions on $(-\infty,\infty)$.

The condition $\int_0^T p(t)dt \geqslant 0, p \neq$ constant, implies $\lambda_0 < 0$ since an easy argument [7, p. 42] shows that $\lambda_0 < -T^{-1}\int_0^T p(t)dt$. If we also have a criterion that makes $\mu_0 > 0$, then $\lambda = 0 \in (\lambda_0,\mu_0)$ so that all solutions of $y''(t)+p(t)y(t) = 0$ are bounded on $(-\infty,\infty)$. Now ξ_0, the eigenfunction corresponding to μ_0, has exactly one zero in $[0,T)$, and hence has a pair of zeros spaced T units apart since it is semi-periodic. Hence a criterion, that implies the zero spacing of solutions of $y''(t)+p(t)y(t) = 0$ is greater than T, yields $\mu_0 > 0$ when combined with the Sturm comparison theorem.

The criterion of Krein/Borg [14, p. 729] that (0.1) is stable if $p(t) \not\equiv 0$ is periodic of period T and

$$\int_0^T p(t)dt \geqslant 0, \quad T\int_0^T p^+(t)dt \leqslant 4,$$

is based on the above facts. Applying Theorem 2.1 (in the form of (0.7)), we have that (0.1) is stable with $p(t) \not\equiv 0$ periodic of period T if $\int_0^T p(t)dt \geqslant 0$ and for all $t_1,t_2 \in [0,T]$,

$$T\left|\int_{t_1}^{t_2} p(t)dt\right| \leqslant 4.$$

As a final example, suppose a nontrivial $H(t)$ is periodic of period one and $\int_0^1 H(t)dt = 0$. Consider the family of equations,

$$y''(t) = H(kt)y(t), \tag{3.3}$$

for $k = 1, 2, \ldots$. Then $H(kt)$ is also of period one, $\int_0^1 H(kt)dt = 0$, and for $t_1, t_2 \in [0, 1]$,

$$\left| \int_{t_1}^{t_2} H(kt)dt \right| = \frac{1}{k} \left| \int_{kt_1}^{kt_2} H(u)du \right| \leqslant \frac{1}{k} \int_0^1 |H(u)|du.$$

Thus for $k \geqslant (\frac{1}{4}) \int_0^1 |H(u)|du$, the spacing of zeros of solutions of (3.3) is greater than one. By the above remarks, this means (3.3) is stable for such k. Hence, while (3.3) may not be stable for $k = 1$, we can make it stable by increasing the frequency k.

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