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A REFINEMENT OF A THEOREM OF PAUL TURAN CONCERNING POLYNOMIALS

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Abstract. In this paper we establish certain sharp results concerning the maximum modulus of the polar derivative of a polynomial $P(z)$ with restricted zeros. Our results generalize and refine some results of Turán, Malik, Govil and others.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative. It was shown by Turán [12] that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1}$$

Inequality (1) was recently refined by Aziz and Dawood [4] and who under the same hypothesis proved that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2}\left\{\max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)|\right\}. \tag{2}$$

Both the inequalities (1) and (2) are sharp and equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$. As an extension of (1), Malik [7] showed that if $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{3}$$

whereas if $P(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then Govil [6] proved that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{4}$$

Both the estimates (3) and (4) are also sharp, Equality in (3) holds for $P(z) = (z+k)^n$, $k \leqslant 1$ whereas equality in (4) holds for $P(z) = z^n + k^n$, $k \geqslant 1$.

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Let $D_\alpha P(z)$ denote the polar differentiation of the polynomial $P(z)$ of degree n with respect to the point α, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha P(z)}{\alpha}\right] = P'(z). \tag{5}$$

A. Aziz [2] proved several sharp results concerning the maximum modulus of the polar derivative of a polynomial $P(z)$ with restricted zeros. Recently Shah [11] extended (1) to the polar derivative of a polynomial and proved that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then for every real or complex number α with $|\alpha| \geqslant 1$,

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant \frac{n}{2}(|\alpha|-1)\max_{|z|=1}|P(z)|. \tag{6}$$

Here we first prove the following generalization of (6), which extends (3) to the polar derivative of a polynomial.

THEOREM 1. *If all the zeros of the polynomial* $P(z) = c\prod_{j=1}^{n}(z-z_j)$ *of degree* n *lie in* $|z| \leqslant k$ *where* $k \leqslant 1$, *then for every real or complex number* α *with* $|\alpha| \geqslant k$,

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant (|\alpha|-k)\sum_{j=1}^{n}\frac{1}{1+|z_j|}\max_{|z|=1}|P(z)|. \tag{7}$$

The result is best possible and equality holds for $P(z) = (z-k)^n$ *with* $\alpha \geqslant 1$.

The following corollary, which is a generalization of the inequality (6) and which extends (3) to the polar derivative of a polynomial, is an immediate consequence of Theorem 1.

COROLLARY 1. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$ *where* $k \leqslant 1$, *then for every real or complex number* α *with* $|\alpha| \geqslant k$,

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant n\left(\frac{|\alpha|-k}{1+k}\right)\max_{|z|=1}|P(z)|. \tag{8}$$

The result is sharp and equality holds for $P(z) = (z-k)^n$ *with* $\alpha \geqslant 1$.

REMARK 1. For $k=1$, Corollary 1 reduces to (6).

REMARK 2. Dividing the two sides of (8) by $|\alpha|$, letting $|\alpha| \to \infty$, and noting (5), we get the inequality (3).

While seeking the coresponding generalization of the inequality (4) to the polar derivative of a polynomial $P(z)$ with respect to a real or complex number α, here we have been able to prove the following result.

THEOREM 2. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$ *where* $k \geqslant 1$, *then for every real or complex number* α *with* $|\alpha| \geqslant k$,

$$\max_{|z|=1} |D_\alpha P(z)| \geqslant n\left(\frac{|\alpha| - k}{1 + k^n}\right) \max_{|z|=1} |P(z)|. \tag{9}$$

REMARK 3. Dividing the two sides of (9) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the inequality (4).

We next prove the following result which is a generalization of the inequality (2) to the polar derivative of a polynomial.

THEOREM 3. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant 1$, *then for every real or complex number* α *with* $|\alpha| \geqslant 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \geqslant \frac{n}{2}\left\{(|\alpha| - 1) \max_{|z|=1} |P(z)| + (|\alpha| + 1) \min_{|z|=1} |P(z)|\right\}. \tag{10}$$

The result is best possible and equality holds for $P(z) = (z-1)^n$ *with* $\alpha \geqslant 1$.

REMARK 4. Dividing the two sides of (10) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the inequality (2).

If $P(z)$ is a self-reciprocal polynomial, of degree at most n, that is, if $P(z) = z^n P\left(\frac{1}{z}\right)$ for all $z \in \mathbf{C}$, then it is known [3, 5] that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{11}$$

The result is sharp and equality holds for $P(z) = z^n + 1$.

Finally here we extend the inequality (11) for the polar derivative of a polynomial $P(z)$ with respect to a real or complex number α with $|\alpha| \geqslant 1$ by proving the following result.

THEOREM 4. *If* $P(z)$ *is a self-reciprocal polynomial of degree at most* n, *then for every real or complex number* α *with* $|\alpha| \geqslant 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \geqslant \frac{n}{2}(|\alpha| - 1) \max_{|z|=1} |P(z)|. \tag{12}$$

The result is best possible and equality holds for $P(z) = (z-1)^n$, *where* n *is an even positive integer.*

REMARK 5. Dividing the two sides of (12) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the inequality (11).

2. Lemmas

For the proofs of these theorem we need the following lemmas. The first result is due to Malik [7].

LEMMA 1. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$ where $k \geqslant 1$, then*

$$k|P'(z)| \leqslant |Q'(z)| \qquad \text{for} \quad |z| = 1$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

By applying Lemma 1 to the polynomial $z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, we immediately get the following result.

LEMMA 2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then*

$$k|P'(z)| \geqslant |Q'(z)|, \qquad |z| = 1$$

where $Q(z)$ is defined as in Lemma 1.

We also need

LEMMA 3. *If $P(z)$ is a polynomial of degree n which has all its zeros in the disk $|z| \leqslant k$ where $k \geqslant 1$, then*

$$\max_{|z|=k} |P(z)| \geqslant \frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)|.$$

This result is due to A. Aziz [3].

3. Proofs of the theorems

Proof of Theorem 1. Let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then it can be easily verified that

$$|Q'(z)| = |nP(z) - zP'(z)| \quad \text{for} \quad |z| = 1.$$

By hypothesis all the zeros of $P(z)$ lie in $|z| \leqslant k$ where $k \leqslant 1$, therefore, by Lemma 2,

$$\begin{aligned} k|P'(z)| &\geqslant |Q'(z)| \\ &= |nP(z) - zP'(z)| \quad \text{for} \quad |z| = 1. \end{aligned} \tag{13}$$

Now for every real or complex number α with $|\alpha| \geqslant k$, we have

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &\geqslant |\alpha||P'(z)| - |nP(z) - zP'(z)| \quad \text{for} \quad |z| = 1. \end{aligned}$$

This implies with the help of (13) that

$$|D_\alpha P(z)| \geqslant (|\alpha| - k)|P'(z)| \quad \text{for} \quad |z| = 1 \quad \text{and} \quad |\alpha| \geqslant k. \tag{14}$$

Since $P(z) = c\prod_{j=1}^{n}(z - z_j)$, $|z_j| \leqslant k \leqslant 1$, $j = 1, 2, \dots, n$, therefore, for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $P(z)$,

$$\operatorname{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) = \sum_{j=1}^{n}\operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta} - z_j}\right) \geqslant \sum_{j=1}^{n}\frac{1}{1 + |z_j|},$$

which implies

$$|P'(e^{i\theta})| \geqslant \sum_{j=1}^{n}\frac{1}{1 + |z_j|}|P(e^{i\theta})| \tag{15}$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $P(z)$. Since (15) is trivially true for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are the zeros of $P(z)$, it follows that

$$|P'(z)| \geqslant \sum_{j=1}^{n}\frac{1}{1 + |z_j|}|P(z)| \quad \text{for} \quad |z| = 1.$$

This in conjuction with (14) yields

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant (|\alpha| - k)\sum_{j=1}^{n}\frac{1}{1 + |z_j|}\max_{|z|=1}|P(z)|.$$

which is inequality (7) and this completes the proof of Theorem 1.

Proof of Theorem 2. By hypothesis $P(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, therefore, all the zeros of the polynomial $G(z) = P(kz)$ lie in $|z| \leqslant 1$. Applying inequality (6) to the polynomial $G(z)$ and noting that $|\alpha|/k \geqslant 1$, we get

$$\max_{|z|=1}|D_{\alpha/k}G(z)| \geqslant \frac{n}{2}\left(\frac{|\alpha| - k}{k}\right)\max_{|z|=1}|G(z)|.$$

Replacing $G(z)$ by $P(kz)$, we obtain

$$\max_{|z|=1}|D_{\alpha/k}P(kz)| \geqslant \frac{n}{2}\left(\frac{|\alpha| - k}{k}\right)\max_{|z|=1}|P(kz)|.$$

This implies with the help of Lemma 3 that

$$\begin{aligned}\max_{|z|=1}&|nP(kz) + (\alpha/k - z)kP'(kz)| \\ &\geqslant \frac{n}{2}\left(\frac{|\alpha| - k}{k}\right)\max_{|z|=k}|P(z)| \\ &\geqslant \frac{n}{2}\left(\frac{|\alpha| - k}{k}\right)\frac{2k^n}{1 + k^n}\max_{|z|=1}|P(z)|,\end{aligned}$$

which gives

$$\max_{|z|=k} |D_\alpha P(z)| = \max_{|z|=k} |nP(z) + (\alpha - z)P'(z)|$$
$$\geqslant nk^{n-1}\Big(\frac{|\alpha| - k}{1 + k^n}\Big) \max_{|z|=1} |P(z)|.$$

Now if $F(z)$ is a polynomial of degree n, then (see [10, p. 346] or [9, vol. I, p. 137])

$$\max_{|z|=R>1} |F(z)| \leqslant R^n \max_{|z|=1} |F(z)|.$$

Applying this result to the polynomial $nP(z) + (\alpha - z)P'(z) = D_\alpha P(z)$, which is of degree at most $n - 1$, it follows that

$$k^{n-1} \max_{|z|=1} |D_\alpha P(z)| \geqslant nk^{n-1}\Big(\frac{|\alpha| - k}{1 + k^n}\Big) \max_{|z|=1} |P(z)|,$$

which immediately leads to the desired result and this completes the proof of Theorem 2.

Proof of Theorem 3. Let $m = \min_{|z|=1} |P(z)|$. If $P(z)$ has a zero on $|z| = 1$, then $m = 0$ and the result follows from Corollary 1 with $k = 1$. Henceforth we assume that all the zeros of $P(z)$ lie in $|z| < 1$ so that $m > 0$ and $m \leqslant |P(z)|$ for $|z| = 1$. By Rouche's theorem it follows that if β is any complex number such that $|\beta| < 1$, then the polynomial $F(z) = P(z) - \beta m z^n$ of degree n has all its zeros in $|z| < 1$. If $z_1, z_2, \ldots, z_n$ are the zeros of $F(z)$, then $|z_j| < 1$, $j = 1, 2, \ldots, n$. Proceeding similarly as in the proof of Theorem 1 with $k = 1$, we get

$$|D_\alpha F(z)| \geqslant (|\alpha| - 1) \sum_{j=1}^{n} \frac{1}{1 + |z_j|} |F(z)| \quad \text{for} \quad |z| = 1$$
$$. \geqslant \frac{n}{2}(|\alpha| - 1)|F(z)| \quad \text{for} \quad |z| = 1.$$

This gives

$$|D_\alpha P(z) - mn\alpha\beta z^{n-1}| \geqslant \frac{n}{2}(|\alpha| - 1)|P(z) - \beta m z^n| \quad \text{for} \quad |z| = 1. \tag{16}$$

It is a simple consequence of Laguerre Theorem (see [1] or [8, p. 52]) on the polar derivative of a polynomial that for every α with $|\alpha| \geqslant 1$, the polynomial

$$D_\alpha F(z) = D_\alpha P(z) - mn\alpha\beta z^{n-1}$$

has all its zeros in $|z| < 1$. This clearly implies that

$$|D_\alpha P(z)| \geqslant nm|\alpha||z|^{n-1} \quad \text{for} \quad |z| \geqslant 1. \tag{17}$$

Now choosing argument of β in the left hand side of (16) such that

$$|D_\alpha P(z) - mn\alpha\beta z^{n-1}| = |D_\alpha P(z)| - mn|\alpha||\beta| \quad \text{for} \quad |z| = 1$$

(which is possible by (17)), we get

$$|D_\alpha P(z)| - mn|\alpha||\beta| \geqslant \frac{n}{2}(|\alpha|-1)\Big\{|P(z)| - |\beta|m\Big\} \quad \text{for} \quad |z|=1. \tag{18}$$

Finally letting $|\beta| \to 1$ in (18), we easily obtain

$$\max_{|z|=1} |D_\alpha P(z)| \geqslant \frac{n}{2}\Big\{(|\alpha|-1)\max_{|z|=1}|P(z)| + (|\alpha|+1)\min_{|z|=1}|P(z)|\Big\},$$

which is inequality (10) and this completes the proof of Theorem 3.

Proof of Theorem 4. Since $P(z)$ is self-reciprocal polynomial of degree at most n, we have

$$P(z) = z^n P\Big(\frac{1}{z}\Big) \qquad \text{for all} \quad z \in \mathbf{C}.$$

This implies

$$z^{n-1}P'\Big(\frac{1}{z}\Big) = nP(z) - zP'(z),$$

which in particular gives

$$\begin{aligned}\max_{|z|=1}|P'(z)| &= \max_{|z|=1}\Big|z^{n-1}P'\Big(\frac{1}{z}\Big)\Big| \\ &= \max_{|z|=1}|nP(z) - zP'(z)|. \end{aligned} \tag{19}$$

Now for $|z| = 1$,

$$\begin{aligned}|D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &\geqslant |\alpha||P'(z)| - |nP(z) - zP'(z)| \end{aligned} \tag{20}$$

If $\max\limits_{|z|=1}|P'(z)| = |P'(z_0)|$, then with the help of (19) it follows from (20) that

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| &\geqslant \Big\{|D_\alpha P(z)|\Big\}_{z=z_0} \\ &\geqslant |\alpha||P'(z_0)| - |nP(z_0) - z_0P'(z_0)| \\ &\geqslant |\alpha||P'(z_0)| - \max_{|z|=1}|nP(z) - zP'(z)| \\ &= (|\alpha|-1)\max_{|z|=1}|P'(z)|. \end{aligned}$$

Combining this with inequality (11), we conclude that

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant \frac{n}{2}(|\alpha|-1)\max_{|z|=1}|P(z)|.$$

This proves the desired result.

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